A REMARK CONCERNING THE BEHAVIOUR OF A POWER-SERIES ON THE PERIPHERY OF ITS CONVERGENCE-CIRCLE

P. TURÁN (Budapest)

1. Let $f(z)$ be regular for $|z|<1$ and maps it onto a domain A. If $|z_0|<1$, then the map of $|z|<1$ by the functions $f\left(\frac{z-z_0}{1-\bar{z}_0 z}\right)$ is obviously again A. This gives a natural classification of the functions $f(z)$ regular in $|z|<1$; $f_1(z)$ and $f_2(z)$ belong to the same class if $f_1(z)$ is regular in $|z|<1$ and with a suitable $|z_0|<1$

$$f_2(z)=f_1\left(\frac{z-z_0}{1-\bar{z}_0 z}\right). \tag{1.1}$$

The functions of a class are thus „functiontheoretically equivalent". It is quite natural to raise the general question, whether or not the functions of a class „series-theoretically equivalent" are. This somewhat vaguely raised question can clarified on many ways. One of the most natural forms is to ask that if the power-series

$$f_1(z)=\sum_{\nu=1}^{\infty} a_\nu z^\nu \tag{1.2}$$

is convergent for $z=1$, does it follow that the power-series

$$f_2(z)=f_1\left(\frac{z-z_0}{1+\bar{z}_0 z}\right)=\sum_{\nu=0}^{\infty} b_\nu(z_0) z^\nu \tag{1.3}$$

converges for the corresponding

$$z=\frac{1+z_0}{1+\bar{z}_0}\ (=e^{i\gamma},\ \gamma \text{ real})$$

point? In this note we are going to establish the fact that this is generally *not* the case[1]. More exactly we assert the following

THEOREM. *Given any* ζ_0 *with* $0 < |\zeta_0| < 1$ *there is an*

$$f_1^*(z) = \sum_{\nu=0}^{\infty} a_\nu^* z^\nu \tag{1.4}$$

regular for $|z| < 1$ *with convergent* $\sum_0^\infty a_\nu^*$ *such that the series*

$$f_2^*(z) = f_1^*\left(\frac{z-\zeta_0}{1-\bar{\zeta}_0 z}\right) = \sum_{\nu=0}^{\infty} b_\nu^*(\zeta_0) z^\nu \tag{1.5}$$

diverges for the corresponding $z = \dfrac{1+\zeta_0}{1+\bar{\zeta}_0}$.

This phenomenon is rather surprising, since the structure of values of $f_1^*(z)$ in the neighbourhood of an *arbitrary* peripherypoint $z = z^*$ is „nearly the same" as that of $f_2^*(z)$ for $z = \dfrac{z^*+\zeta_0}{1+\bar{\zeta}_0 z^*}$ and still the MacLaurin series of $f_1^*(z)$ converges for $z=1$ and that of $f_2^*(z)$ diverges for $z = \dfrac{1+\zeta_0}{1+\bar{\zeta}_0}$. One can discuss along similar lines e. g. the interesting question of $(C, 1)$-summability of the series (1.3) for $z = \dfrac{1+z_0}{1+\bar{z}_0}$ if the series (1.2) is $(C, 1)$-summable for $z=1$; in **4.** we shall show in a few lines that the phenomenon described in the theorem does *not* occur for Abel-summability. It would be more difficult to decide whether or not the function $f_1^*(z)$ in our theorem can be continuous for the whole circle $|z| \leq 1$; it would be interesting to decide whether or not the convergence of $\sum_0^\infty |a_\nu|$ that of $\sum_0^\infty |b_\nu(z_0)|$ implies. To give a final sample of the many interesting problems of this type I mention one due to Vera T. Sós; this asks

---

[1]) In the paper „On some connections of the theories of functions and series". *Annuary vol. of the Eötvös Loránd Univ. Budapest,* 1952—1953, p. 5—13 (in Hungarian), I stated without proof the opposite theorem. Reviewed in *Math. Rev.* Vol. 17, № 6 (1956), p. 598.

whether or not there is a function $f_1^{**}(z) = \sum_0^\infty a_\nu^{**} z^\nu$ with convergent $\sum_0^\infty a_\nu^{**}$ such that for *all* ζ_0 with $0 < |\zeta_0| < 1$ the series

$$f_2^{**}(z) = f_1^{*}\left(\frac{z-\zeta_0}{1-\bar{\zeta}_0 z}\right) = \sum_0^\infty b_\nu^{**}(\zeta_0) z^\nu$$

diverges for $z = \dfrac{1+\zeta_0}{1+\bar{\zeta}_0}$. It is easy to prove that this holds for any denombrably infinite set of ζ_0-values in $0 < |\zeta_0| < 1$.[2] I intend to return to some of these questions elsewhere.

2. Our theorem is in connection with certain results of Hardy-Littlewood[3] and Carleman[4]. The former authors proved that if for

$$f_1(z) = \sum_{\nu=0}^\infty a_\nu z^\nu \tag{2.1}$$

the series $\sum_0^\infty a_\nu$ converges and for an $0 < \alpha < 1$

$$w = z + \alpha,$$

then for the function

$$f_1(w) = f_1(z+\alpha) = f_2(z) = \sum_0^\infty c_\nu(\alpha) z^\nu$$

which is regular for $|z| < 1-\alpha$, the series

$$\sum_0^\infty c_\nu(\alpha)(1-\alpha)^\nu$$

necessarily converges. Carleman proved more generally that if $w = \varphi(z)$ is regular for $|z| < x_0 + \varepsilon\,(>0)$ with $\varepsilon > 0$ and $\varphi(x_0) = 1$ and maps

[2]) Generally given a denombrably infinite set of „bad" Toeplitz-summation-matrices, one can easily construct a convergent sequence not summable by any of the given summation-methods. Is this true for the case of a higher power of summation-methods?

[3]) „Theorems concerning the summability of a series by Borel's exponential method". *Rend. del Circolo Mat. di Palermo* **41** (1916), p. 36—53, esp. p. 49—50.

[4]) „Some theorems concerning the convergence of power-series on the circle of convergence". *Arkiv för Math. Astr. och Fys.* **15** (1920), p. 1—13.

$|z| \leqq x_0$ onto a domain B lying in $|w| \leqq 1$ and touching it *of the first order* at $w=1$ and having no other common point with $|w|=1$, then if the series (2.1) of $f_1(z)$ converges for $z=1$ then the series

$$f_1(w) = f_1(\varphi(z)) = f_2(z) = \sum_{\nu=0}^{\infty} d_\nu z^\nu$$

converges for $z=x_0$ too. In our case is $\varphi(z) = \frac{z-z_0}{1-\bar{z}_0 z}$ and the condition of first-order touching is obviously violated.

3. For the proof of the announced theorem we have to find the connection between the partial-sums

$$A_n = \sum_{\nu=0}^{n} a_\nu \tag{3.1}$$

and

$$B_n = \sum_{\nu=0}^{n} b_\nu(\zeta_0) \left(\frac{1+\zeta_0}{1+\bar{\zeta}_0}\right)^\nu \tag{3.2}$$

for the series (1.2) and (1.3). We consider the function

$$G(s) = \sum_{\nu=0}^{\infty} B_n s^n, \tag{3.3}$$

which is regular for $|s| < 1$ owing to (1.3).

We have for $|s| < 1$

$$\begin{aligned} G(s) &= \frac{1}{1-s} \sum_{\nu=0}^{\infty} b_\nu(\zeta_0) \left(\frac{1+\zeta_0}{1+\bar{\zeta}_0}\right)^\nu s^\nu = \frac{1}{1-s} f_2\left(\frac{1+\zeta_0}{1+\bar{\zeta}_0} s\right) = \\ &= \frac{1}{1-s} f_1\left(\frac{\frac{1+\zeta_0}{1+\bar{\zeta}_0} s - \zeta_0}{1-\bar{\zeta}_0 \frac{1+\zeta_0}{1+\bar{\zeta}_0} s}\right) = \frac{1}{1-s} f_1\left(\frac{(1+\zeta_0)s - \zeta_0(1+\bar{\zeta}_0)}{(1+\bar{\zeta}_0) - \bar{\zeta}_0(1+\zeta_0)s}\right). \end{aligned} \tag{3.4}$$

For

$$F(s) = \sum_{\nu=0}^{\infty} A_\nu s^\nu \tag{3.5}$$

we have obviously for $|s|<1$

$$F(s)=\frac{1}{1-s}f_1(s), \quad f_1(s)=(1-s)F(s),$$

i. e. from this and (3.4)

$$G(s)=\frac{1}{1-s}\left(1-\frac{(1+\zeta_0)s-\zeta_0(1+\bar\zeta_0)}{(1+\bar\zeta_0)-\bar\zeta_0(1+\zeta_0)s}\right)F\left(\frac{(1+\zeta_0)s-\zeta_0(1+\bar\zeta_0)}{(1+\bar\zeta_0)-\bar\zeta_0(1+\zeta_0)s}\right)=$$

(3.6)
$$=\frac{|1+\zeta_0|^2}{(1+\bar\zeta_0)-\bar\zeta_0(1+\zeta_0)s}F\left(\frac{(1+\zeta_0)s-\zeta_0(1+\bar\zeta_0)}{(1+\bar\zeta_0)-\bar\zeta_0(1+\zeta_0)s}\right).$$

Hence for $n=0, 1, \ldots$ from (3.3) and (3.6)

$$B_n=\frac{|1+\xi_0|^2}{2\pi i}\int_{(l_1)}\frac{F\left(\frac{(1+\zeta_0)s-\zeta_0(1+\bar\zeta_0)}{(1+\bar\zeta_0)-\bar\zeta_0(1+\zeta_0)s}\right)}{(1+\bar\zeta_0)-\bar\zeta_0(1+\zeta_0)s}\,\frac{ds}{s^{n+1}}$$

where l_1 runs in $|s|<1$ around the origin.

Putting

$$\frac{1+\zeta_0}{1+\bar\zeta_0}s=w$$

we get

$$B_n=(1+\zeta_0)\frac{1}{2\pi i}\left(\frac{1+\zeta_0}{1+\bar\zeta_0}\right)^n\int_{(l_1)}\frac{F\left(\frac{w-\zeta_0}{1-\bar\zeta_0 w}\right)}{1-\bar\zeta_0 w}\,\frac{dw}{w^{n+1}}.$$

Putting

$$\frac{w-\zeta_0}{1-\bar\zeta_0 w}=\omega \qquad \text{i. e.} \qquad w=\frac{\omega+\zeta_0}{1+\bar\zeta_0\omega}$$

we obtain

(3.7)
$$B_n=(1+\zeta_0)\left(\frac{1+\zeta_0}{1+\bar\zeta_0}\right)^n\frac{1}{2\pi i}\int_{(l_2)}\frac{F(\omega)}{1+\bar\zeta_0\omega}\cdot\left(\frac{1+\bar\zeta_0\omega}{\omega+\zeta_0}\right)^{n+1}d\omega$$

where l_2 runs in $|\omega|<1$ enclosing the point $\omega=-\zeta_0$. We may obviously insert the series (3.5) in (3.7) and integrate termwise; this gives for $n=0,1,\ldots$

$$B_n=\sum_{\nu=0}^{\infty} A_\nu\, e_{n\nu}(\zeta_0) \tag{3.8}$$

where for $\nu=0, 1, 2, \ldots$ the representation

$$e_{n\nu}(\zeta_0)=\frac{1}{2\pi i}(1+\zeta_0)\left(\frac{1+\zeta_0}{1+\bar{\zeta}_0}\right)^n \int_{(l_2)} \frac{\omega^\nu}{1+\bar{\zeta}_0\,\omega}\left(\frac{1+\bar{\zeta}_0\,\omega}{\omega+\zeta_0}\right)^{n+1} d\omega \tag{3.9}$$

holds. (3.8) defines obviously a summation-process. Our problem can be formulated in terms of this summation-process simply; we have to decide whether or not this process permanent is. For the permanence is according to the classical theorem of Toeplitz – I. Schur necessary that

$$\sum_{\nu=0}^{\infty} |e_{n\nu}(\zeta_0)| < C \tag{3.10}$$

independently upon n. But one can verify that this is *not* the case if $0<|\xi_0|<1$. Namely as G. Szegö remarked in a letter[5], writing

$$\zeta_0=|\zeta_0|e^{i\alpha}$$

one has from (3.9) at once for $|x|<1$

$$\begin{aligned} H(x)&=\sum_{\nu=}^{\infty}(-1)^\nu e_{n\nu}(\zeta_0)\,e^{-\nu i\alpha}x^\nu= \\ &=(-e^{-i\alpha})^n\frac{(1+\zeta_0)^{n+1}}{(1+\bar{\zeta}_0)^n}\left(\frac{x-|\zeta_0|}{1-|\zeta_0|x}\right)^n\frac{1}{1-|\zeta_0|x}. \end{aligned} \tag{3.11}$$

He called in his letter my attention to the interesting paper of B. M. Bajšanski[6], where it is proved among others (as a special case of his theorem III) when

$$\left(\frac{x-|\zeta_0|}{1-|\zeta_0|x}\right)^n=\sum_{\nu=0}^{\infty} e^{*}_{n\nu}(|\zeta_0|)\,x^\nu,$$

[5]) From 4. Nov. 1957.

[6]) Sur une classe générale de procédés de sommations du type d'Euler-Borel. *Publ. Inst. Math. Acad. Serbe Sci.* **X** (1956), p. 131—153.

then

$$\lim_{n\to\infty}\sum_{\nu=0}^{\infty}|e^{*}_{n\nu}(|\zeta_0|)|=+\infty. \tag{3.12}$$

This cannot be applied directly to (3.11); however writing it in the form

$$e_{n0}(\zeta_0)+\sum_{\nu=0}^{\infty}(-e^{-i\alpha})^{\nu}(e_{n\nu}(\zeta_0)+|\zeta_0|e_{n,\nu-1}(\zeta_0)e^{i\alpha})x^{\nu}=$$

$$=(-e^{-i\alpha})^{n}\frac{(1+\zeta_0)^{n+1}}{(1+\bar{\zeta}_0)^{n}}\cdot\left(\frac{x-|\zeta_0|}{1-|\zeta_0|x}\right)^{n}$$

it follows from (3.12)

$$\lim_{n\to\infty}\left\{|e_{n0}(\zeta_0)|+\sum_{\nu=1}^{\infty}|e_{n\nu}(\zeta_0)+|\zeta_0|e^{i\alpha}e_{n,\nu-1}(\zeta_0)|\right\}=+\infty. \tag{3.13}$$

But this contradicts already to (3.10), since from it one could derive

$$\left|e_{n0}(\zeta_0)+\sum_{\nu=1}^{\infty}|e_{n\nu}(\zeta_0)+|\zeta_0|e^{i\alpha}e_{n,\nu-1}(\zeta_0)\right|\leqq$$

$$\leqq(1+|\xi_0|)\sum_{\nu=0}^{\infty}|e_{n\nu}(\zeta_0)|<2C$$

qu. e. d.

4. Finally we show that if $f_1(z)$ from (1.2) is Abel-summable for $z=1$, then $f_2(z)$ from (1.3) is also Abel-summable for $z=\frac{1+z_0}{1+\bar{z}_0}$. The hypothesis means that

$$\lim_{r\to 1}f_1(r)=A \tag{4.1}$$

exists and is finite; we have to investigate

$$\lim_{r\to 1}\sum_{\nu=0}^{\infty}b_{\nu}(z_0)\left(\frac{1+z_0}{1+\bar{z}_0}\right)^{\nu}r^{\nu}. \tag{4.2}$$

But this is owing to (1.3) the same as

$$\lim_{r\to 1}f_1\left(\frac{\frac{1+z_0}{1+\bar{z}_0}r-z_0}{1-\bar{z}_0\frac{1+z_0}{1+\bar{z}_0}r}\right)=\lim_{r\to 1}f_1\left(\frac{1-\frac{1+z_0}{1-|z_0|^2}(1-r)}{1+\frac{\bar{z}_0(1+z_0)}{1-|z_0|^2}(1-r)}\right). \tag{4.3}$$

As easy to see, for $r \to 1$ we have

$$\frac{1-\dfrac{1+z_0}{1-|z_0|^2}(1-r)}{1+\dfrac{\bar{z}_0(1+z_0)}{1-|z_0|^2}(1-r)} = 1-\frac{|1+z_0|^2}{1-|z_0|^2}(1-r)+O(1-r)^2.$$

This shows owing to Stolz's generalisation of Abel's theorem that the limes in (4.2) exists and equals A indeed.

(Received 25 december 1957)