THE SMALL DEFLECTION OF A NORMALLY LOADED SQUARE PLATE, ELASTICALLY SUPPORTED ALONG ITS EDGES

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SUMMARY — The problem considered in this paper is the behaviour of a uniformly loaded square plate attached to flexible beams along its edges. A solution is obtained by constructing a form for the deflection which is general — and hence involves an infinite number of parameters — and which satisfies the appropriate boundary conditions. The parameters are found by minimising the total potential energy.

In the general case the accuracy of the method is indicated by the rate at which the values for the deflection and bending moment converge as the number of parameters taken into account is increased. For the two special cases in which the edge beams are non existent or completely rigid a direct comparison is possible with accurately known results. In general, agreement is good.

1. *INTRODUCTION*

The behaviour of a normally loaded flat plate with various types of edge support has been investigated theoretically and experimentally for a long time. Summaries of the work done and the results obtained are given in Refs. 1 (loading mainly hydrostatic) and 2.

For investigating such problems theoretically, two methods are available. The first is to solve the equations of equilibrium; the second is to evaluate the total potential energy of the system and express the fact that it is stationary if there is equilibrium. When an exact solution can be found, there is little to choose between the two methods, either as regards difficulty or amount of work. But when an exact solution is unobtainable, the strain energy method has the merit of invariably providing an answer, albeit only an approximate one.

2. *DESCRIPTION OF PROBLEM*

The particular problem considered in this paper originated from road bridge design, and is the behaviour of an initially flat panel, supported by rows of equidistant columns and acted on by a uniform normal pressure P. The heads of the columns form a square mesh of side a, and are bridged by identical elastic reinforcing beams such as OA, AB, BC, CO, which are assumed rigidly and continuously attached to the panel (Fig. 1). It is also assumed (i) that the overall dimensions of the panel are large compared with a, (ii) that a is large compared with the cross-sectional dimensions of the supporting columns or reinforcing beams, (iii) that the weight of the panel and reinforcing beams is negligible compared with the total applied load, and (iv) that the neutral axes of the reinforcing beams lie in the middle surface of the panel.

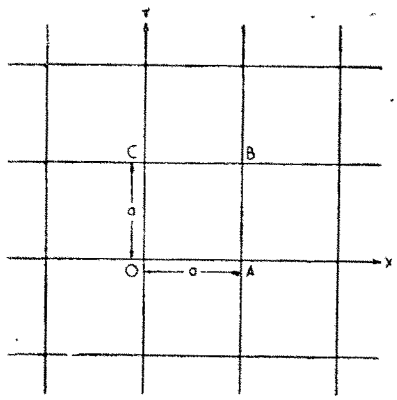

Fig. 1

From these assumptions it follows that except near the boundary of the panel the square elements into which the reinforcing beams divide it will all deflect in an identical manner. In practice we may therefore confine our attention to any one such element, e. g. $OABC$. For convenience we shall call this element the plate $OABC$, and shall refer to O, A, B, C as its vertices, and to OA, AB, BC, CO as its edge beams which, owing to the symmetry, may bend but will not twist. Denoting the flexural rigidity of the edge beams by EI, it follows that when EI is infinite the problem reduces to that of a square plate with edges clamped, and that when EI is zero it becomes a square plate supported only at its corners. As solutions for these two special cases are known, the purpose of the work described in this paper is to link them together.

Investigating what happens to the square elements near the boundary of the panel is a more difficult problem because symmetry considerations no longer preclude twisting of the edge beams. The related problem in which the edge beams can twist but not bend is considered in Ref. 3.

3. *METHOD OF SOLUTION*

Obtaining the deflection of the square plate, $OABC$, is made up of three stages. Stage 1 consists of finding an expression for the normal deflection w which is general — and hence involves the presence of an infinite number of unknown parameters — and which satisfies the appropriate conditions on the boundary. Stage 2 consists of writing down a formal expression for the total potential energy of the system in terms of w, and then evaluating it in terms of the unknown parameters. Stage 3 consists of finding the unknown parameters by expressing the fact that in an equilibrium configuration the potential energy is stationary.

Stage 1 — Expression for w.

Taking OA, OC as coordinate axes, the edges of the plate are $X=0, a$; $Y=0, a$. Since, however, it is more convenient to use coordinates which are non-dimensional, we introduce x, y defined by $x=\pi X/a$, $y=\pi Y/a$, so that the edges of the plate are $x=0, \pi$; $y=0, \pi$. (Fig. 2).

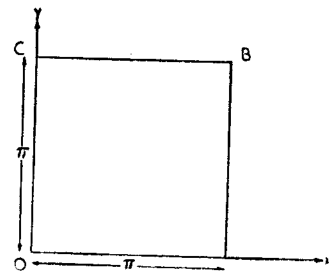

Fig. 2

The conditions which w must satisfy on the boundary are

$$\left.\begin{aligned} &w=0 \text{ at } (0,0),\ (0,\pi),\ (\pi,0),\ (\pi,\pi), \\ &\frac{\partial w}{\partial x}=0, \text{ when } x=0,\pi, \text{ and } 0\leqslant y\leqslant\pi, \\ &\frac{\partial w}{\partial y}=0, \text{ when } y=0,\pi, \text{ and } 0\leqslant x\leqslant\pi. \end{aligned}\right\} \tag{1}$$

Further, w must be general, and, having regard to stage 2, must, if an infinite series, be differentiable term by term.

We shall begin by finding w for the edge beam OA. It will be a function of x, and must, if represented by an infinite series, be general, differentiable term by term, and such that

$$w=\frac{dw}{dx}=0, \qquad \text{when } x=0, \pi. \tag{2}$$

Since on physical grounds $\frac{d^4w}{dx^4}$ is a well behaved function of x throughout the range $0<x<\pi$, it can be expressed quite generally as a sine series valid throughout this range in the form

$$\frac{d^4w}{dx^4}=\sum_{m=1}^{\infty} C_m \sin mx$$

where $|C_m| < K/m$, (Ref. 4) and K is a constant. Integrating this series term by term four times, and using the boundary conditions to determine the arbitrary constants of integration, a legitimate procedure since the integrated series is valid in the range $0 \leqslant x \leqslant \pi$, we have

$$w = \sum_{m=1}^{\infty} \frac{C_m}{m^4} \left[\sin mx - mx + \frac{m}{\pi} \{2 + (-)^m\} x^2 - \frac{m}{\pi^2} \{1 + (-)^m\} x^3 \right]. \tag{3}$$

Since by symmetry $w(x) = w(\pi - x)$, it follows that m is odd, and hence that (3) can be put in the form

$$w = \sum_{\substack{1 \\ (m \text{ odd})}}^{\infty} B_m \, \alpha_m(x)$$

where

$$\alpha_m(t) = \frac{\sin mt}{m^2} - \frac{t(\pi - t)}{\pi m}, \quad \text{and} \quad |B_m| < K/m^3.$$

Since the plate is square and the loading uniform, it follows that the deflection of corresponding points on the four edge beams are equal, and hence that for *any* of the edge beams

$$w = \sum_{\substack{1 \\ (m \text{ odd})}}^{\infty} B_m \{\alpha_m(x) + \alpha_m(y)\} \tag{4}$$

The next step is to derive a comparable expression for the deflection of the plate. To do this it will be found convenient to express w in the form

$$w = \sum_{\substack{1 \\ (m \text{ odd})}}^{\infty} B_m \{\alpha_m(x) + \alpha_m(y)\} + w' \tag{5}$$

and find w'.

From (1) and (2) it follows that the conditions which w' must satisfy on the boundary are

$$w' = \frac{\partial w'}{\partial x} = 0, \qquad x = 0, \pi,$$

$$w' = \frac{\partial w'}{\partial y} = 0, \qquad y = 0, \pi.$$

In addition, w' must be general and, if expressed in the form of an infinite series, must be differentiable term by term. We now proceed in a very similar manner to that by which we found w for the edge beams. On physical grounds $\frac{\partial^8 w'}{\partial x^4 \partial y^4}$ is a well behaved function of x and y throughout the domain $0 < x < \pi$, $0 < y < \pi$, and hence can be expressed as a double sine series valid throughout this domain in the form

$$\frac{\partial^8 w'}{\partial x^4 \partial y^4} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin mx \sin ny$$

where $|C_{mn}| < K'/mn$, (Ref. 4), and K' is a constant. Integrating this series four times with respect to x and four times with respect to y, and using the boundary conditions to determine the arbitrary constants of integration, we have

$$w' = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{m^4 n^4} \left[\sin mx - mx + \frac{m}{\pi} \{2 + (-)^m\} x^2 - \frac{m}{\pi^2} \{1 + (-)^m\} x^3 \right] \times$$
$$\times \left[\sin ny - ny + \frac{n}{\pi} \{2 + (-)^n\} y^2 - \frac{n}{\pi^2} \{1 + (-)^n\} y^3 \right]. \qquad (6)$$

From symmetry considerations

$$w(x, y) = w(\pi - x, y) = w(x, \pi - y) = w(\pi - x, \pi - y),$$

so that m and n are both odd, and (6) can be expressed in the form

$$w' = \sum_{\substack{1 \\ (m \text{ odd})}}^{\infty} \sum_{\substack{1 \\ (n \text{ odd})}}^{\infty} A_{mn} \alpha_m(x) \alpha_n(y) \qquad (7)$$

where $|A_{mn}| < K'/m^5 n^5$.

From (5) and (7) it follows that a form for w satisfying all the required conditions is

$$w = \sum_{\substack{1 \\ (r \text{ odd})}}^{\infty} B_r \{\alpha_r(x) + \alpha_r(y)\} + \sum_{\substack{1 \\ (m \text{ odd})}}^{\infty} \sum_{\substack{1 \\ (n \text{ odd})}}^{\infty} A_{mn} \alpha_m(x) \alpha_n(y). \qquad (8)$$

Stage 2 — Expression for Total Potential Energy

Since the deflection of the plate is small, stretching of its middle surface is negligible. The plate's strain energy may therefore be considered as entirely due to bending, and hence is

$$\frac{D}{2} \int_0^a \int_0^a \left[\left(\frac{\partial^2 w}{\partial X^2} + \frac{\partial^2 w}{\partial Y^2} \right)^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial X^2} \frac{\partial^2 w}{\partial Y^2} - \left(\frac{\partial^2 w}{\partial X \partial Y} \right)^2 \right\} \right] dX \, dY$$

where D is the flexural rigidity and ν is Poisson's ratio. The strain energy of the edge member OA is also due only to bending, and hence is

$$\frac{EI}{2}\int_0^a \left(\frac{\partial^2 w}{\partial X^2}\right)^2_{Y=0} dX$$

Taking account of symmetry and noting that each edge beam is an edge member of *two* plates, the strain energy of the four edge beams OA, AB, BC, CO is

$$EI\int_0^a \left(\frac{\partial^2 w}{\partial X^2}\right)^2_{Y=0} dX.$$

The potential energy of the externally applied pressure P is

$$-P\int_0^a\int_0^a w\, dX\, dY.$$

Denoting by U the total potential energy of a plate and its edge beams

$$U=\frac{D\pi^2}{2a^2}\int_0^\pi\int_0^\pi\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\}\right]dx\,dy$$

$$-\frac{Pa^2}{\pi^2}\int_0^\pi\int_0^\pi w\,dx\,dy+\frac{EI\pi^3}{a^3}\int_0^\pi\left(\frac{\partial^2 w}{\partial x^2}\right)^2_{y=0}dx.$$

After substituting for w from (1) and performing the integration

$$U=\sum_m^\infty\sum_n^\infty\sum_p^\infty\sum_q^\infty A_{mn}A_{pq}E_1(m,p,n,q)+\sum_r^\infty\sum_s^\infty B_r B_s E_2(r,s)$$

$$+\sum_m^\infty\sum_n^\infty\sum_r^\infty A_{mn}B_r E_3(n,m,r)+P\left[\sum_m^\infty\sum_n^\infty A_{mn}E_4(m,n)+\sum_r^\infty B_r E_5(r)\right],$$

where the summations extend over all odd positive integers and

$$E_1(m,p,n,q)=\frac{D\pi^2}{2a^2}\left(I^{mp}_{\alpha\alpha}I^{nq}_{\gamma\gamma}+I^{mp}_{\gamma\gamma}I^{nq}_{\alpha\alpha}+2I^{mp}_{\beta\beta}I^{nq}_{\beta\beta}\right),$$

$$E_2(r,s)=\frac{D\pi^3}{a^2}I^{rs}_{\gamma\gamma}\left(1+\frac{EI}{aD}\right)\equiv\frac{D\pi^3}{a^2}I^{rs}_{\gamma\gamma}\,\varepsilon,\quad \text{where } \varepsilon=1+\frac{EI}{aD},$$

$$E_3(m, n, r) = \frac{D\pi^2}{a^2}\left(I_\alpha^m\, I_{\gamma\gamma}^{nr} + I_\alpha^n\, I_{\gamma\gamma}^{mr}\right),$$

$$E_4(m, n) = -\frac{a^2}{\pi^2} I_\alpha^m\, I_\alpha^n\,,$$

$$E_5(r) = -\frac{2a^2}{\pi^3} I_\alpha^r\,,$$

and the I's are given by

$$I_{\alpha\alpha}^{mn} = \int_0^\pi \alpha_m(\varphi)\,\alpha_n(\varphi)\,d\varphi = \frac{\delta_{mn}\pi}{2m^2n^2} - \frac{4}{\pi mn}\left(\frac{1}{m^4}+\frac{1}{n^4}\right) + \frac{\pi^3}{30\,mn}\,,$$

$$I_{\gamma\gamma}^{mn} = \int_0^\pi \gamma_m(\varphi)\,\gamma_n(\varphi)\,d\varphi = \delta_{mn}\frac{\pi}{2} - \frac{4}{\pi\, mn}\,,$$

$$I_{\beta\beta}^{mn} = \int_0^\pi \beta_m(\varphi)\,\beta_n(\varphi)\,d\varphi = \frac{\delta_{mn}\pi}{2\,mn} - \frac{4}{\pi\, mn}\left(\frac{1}{m^2}+\frac{1}{n^2}\right) + \frac{\pi}{3\,mn}\,,$$

$$I_\alpha^m = \int_0^\pi \alpha_m(\varphi)\,d\varphi = \frac{2}{m^3} - \frac{\pi^2}{6\,m}\,,$$

where

$$\beta_m(\varphi) = \frac{d}{d\varphi}\alpha_m(\varphi), \qquad \gamma_m(\varphi) = \frac{d}{d\varphi}\beta_m(\varphi),$$

and

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n. \end{cases}$$

Stage 3 — Evaluation of A_{mn}, B_r

Since U is stationary in an equilibrium configuration

$$\frac{\partial U}{\partial A_{pq}} = 0, \qquad \frac{\partial U}{\partial B_s} = 0,$$

whence

$$\sum_m^\infty \sum_n^\infty 2\,A_{mn}\,E_1(m, p, n, q) + \sum_r^\infty B_r\,E_3(p, q, r) + P E_4(p, q) = 0, \quad (p, q \text{ odd})$$

$$\sum_m^\infty \sum_n^\infty A_{mn}\,E_3(m, n, s) + \sum_r^\infty 2\,B_r\,E_2(s, r) + P E_5(s) = 0. \qquad (s \text{ odd}) \tag{9}$$

From symmetry we note that $A_{mn} = A_{nm}$.

Since it is not practicable to solve for the A_{mn}'s and B_r's directly, we proceed by a method of successive approximation, that is, we solve the finite set of equations given by retaining only a finite number of A_{mn}'s and B_r's. In the general case the accuracy of this procedure is indicated by the rate at which the values for the deflection and bending moment converge to limiting values as the number of A's and B's taken into account is increased. But for the two special cases in which the edge beams are non existent ($\varepsilon = 1$) or completely rigid ($\varepsilon = \infty$) a direct comparison is possible with accurately known results.

4. *DISCUSSION OF RESULTS*

The equations (9) have been solved for $\varepsilon = 1$, 1.5, 2, 5, 10. The A_{mn}'s and B_r's retained have been limited to those involving the suffices 1, 3, 5, 7 and give rise to a set of 11 successive approximations for each value of ε. The particular A's and B's included in each approximation are in accordance with the following scheme.

	A_{11}	A_{11}, A_{13}, A_{33}	A_{11}, A_{13}, A_{15} A_{33}, A_{35}, A_{55}	$A_{11}, A_{13}, A_{15}, A_{17}$ $A_{33}, A_{35}, A_{37}, A_{55}$ A_{57} A_{77}
B_1	1st Approx.	3rd Approx.	5th Approx.	8th Approx.
B_1, B_3	2nd Approx.	4th Approx.	6th Approx.	9th Approx.
B_1, B_3, B_5			7th Approx.	10th Approx.
B_1, B_3, B_5, B_7				11th Approx.

When $\varepsilon = \infty$, all the B_r's are zero so that the number of approximations reduces to four. In this case the first, second, third and fourth approximations involve A_{11}; A_{11}, A_{13}, A_{33}; $A_{11}, A_{13}, A_{15}, A_{33}, A_{35}, A_{55}$; and $A_{11}, A_{13}, A_{15}, A_{17}, A_{33}, A_{35}, A_{37}, A_{55}, A_{57}, A_{77}$ respectively.

The results of the calculations are summarised in Tables 1—4, and give the values of the plate deflection and bending moment (i) at its centre, and (ii) at the mid-point of an edge.

Approx. \ ε	1	1·5	2	5	10	∞
1	·5270	·3214	·2561	·1714	·1495	·1300
2	·5219	·3179	·2535	·1703	·1490	·1222
3	·5451	·3229	·2538	·1652	·1425	·1234
4	·5566	·3226	·2535	·1651	·1425	·1232
5	·5436	·3226	·2541	·1661	·1435	
6	·5568	·3226	·2537	·1660	·1435	
7	·5604	·3226	·2537	·1660	·1435	
8	·5440	·3227	·2541	·1659	·1434	
9	·5569	·3226	·2537	·1659	·1434	
10	·5611	·3226	2537	·1659	·1434	
11	·5626	·3226	·2537	·1659	·1434	
Timoshenko	·5655					1230

Table 1 — Values of b, giving central deflection $\frac{Pa^4}{D\pi^4}b$.

Approx. \ ε	1	1·5	2	5	10	∞
1	·4033	·1949	·1280	·0420	·0198	Zero
2	·4007	·1926	·1267	·0415	·0196	
3	·4103	·1947	·1276	·0416	·0196	
4	·4112	·1929	·1267	·0416	·0196	
5	·4107	·1947	·1276	·0416	·0196	
6	·4119	·1929	·1267	·0415	·0196	
7	·4205	·1934	·1269	·0415	·0196	
8	·4107	·1947	·1276	·0416	·0196	
9	·4120	·1929	·1267	·0415	·0196	
10	·4210	·1934	·1269	·0415	·0196	
11	·4210	·1933	·1268	·0415	·0196	
Timoshenko	4242					Zero

Table 2 — Values of c, giving mid edge deflection $\frac{Pa^4}{D\pi^4}c$.

Approx. \ ε	1	1·5	2	5	10	∞
1	·0276	·0283	·0286	·0289	·0229	·0290
2	·0240	·0259	·0268	·0281	·0286	·0215
3	·0386	·0299	·0270	·0233	·0224	·0233
4	·0356	0293	·0268	·0233	·0224	·0227
5	·0358	·0293	0272	·0246	·0240	
6	·0362	·0293	·0272	·0246	·0240	
7	·0349	·0291	·0272	·0246	·0240	
8	·0370	·0295	·0271	·0241	·0234	
9	·0357	·0292	·0270	·0242	·0234	
10	·0363	·0293	·0270	·0241	·0233	
11	·0356	·0292	·0270	·0241	·0234	
Timoshenko	·0359					·0230

Table 3 — Values of d, giving central bending moment Pa^2d.

Approx. \ ε	1	1·5	2	5	10	∞
1	·0162	·0282	·0319	·0367	·0379	·0391
2	·0217	·0317	·0346	·0378	·0385	·0468
3	·0036	·0263	·0334	·0424	·0447	·0493
4	·0222	·0294	·0346	·0424	·0447	·0502
5	·0000	·0260	·0340	·0443	·0470	
6	·0205	·0293	·0352	·0444	·0470	
7	·0089	·0282	·0349	·0444	·0470	
8	— ·0020	·0255	·0338	·0450	·0478	
9	·0215	·0292	·0352	·0450	·0478	
10	·0091	·0282	·0349	·0450	·0478	
11	·0173	·0286	·0352	·0450	·0478	
Timoshenko	·0128					·0513

Table 4 — Values of e, giving mid edge bending moment $-Pa^2 e$.

A comparison with the accurate results given by Timoshenko (Ref. 2) when ε is 1 or ∞ shows satisfactory agreement except for the particular case of the bending moment at the mid point of an edge when ε is 1. In this one case eleven approximations are clearly insufficient. In all other cases the approximations converge to values which can be estimated to within an error of not more than 5%.

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