LINEAR FUNCTIONALS ON BANACH SPACE AND THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

By

FRANCE KRIŽANIČ (Ljubljana)

The aim of this paper is to prove the following theorems which comprise all the generalizations of the fundamental lemma of the calculus of variations. In the theorems $\mathfrak{X}$ and $\mathfrak{Y}$ denote Banach spaces, $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ their adjoint spaces of linear functionals.

THEOREM I. *Let A be a linear operator with domain $\mathfrak{D}$, dense in $\mathfrak{X}$ and with range $\mathfrak{R} \subset \mathfrak{X}$. Let $x_1^*, x_2^*, \ldots, x_n^*$ denote linearly independent functionals on $\mathfrak{X}$. If x^* is a linear functional such that*

$$x^*(Ax) = 0 \tag{1}$$

for every $x \in \mathfrak{D}$, orthogonal to $x_1^*, x_2^*, \ldots, x_n^*$:

$$x_k^*(x) = 0, \quad k = 1, 2, \ldots, n, \tag{2}$$

then $x^* \in \mathfrak{D}^*$ and there exist n scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that:

$$A^* x^* = \sum_{k=1}^{k=n} \lambda_k x_k^* . \tag{3}$$

THEOREM II. *Let A be a linear operator with domain $\mathfrak{D}$, dense in $\mathfrak{X}$ ond with range $\mathfrak{R} \subset \mathfrak{Y}$. Let $y_1^*, y_2^*, \ldots, y_n^*$ denote linearly independent functionals on $\mathfrak{X}$. If $y^* \in \mathfrak{Y}^*$ is a linear functional such that*

$$y^*(Ax) = 0, \tag{4}$$

for every $x \in \mathfrak{D}$, satisfying

$$y_k^*(Ax) = 0, \quad k = 1, 2, \ldots, n, \tag{5}$$

then there exist a solution of $A^* z^* = 0$ and n scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$y^* = z^* + \sum_{k=1}^{k=n} \lambda_k x_k^* . \tag{6}$$

THEOREM III. *Let* $A_1, A_2, \ldots, A_n$ *denote linear operators with domains* $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_n$ *dense in* $\mathfrak{X}$ *and with ranges* $\mathfrak{R}_k \subset \mathfrak{X}$, *with bounded and comutative inverses* $A_1^{-1}, A_2^{-1}, \ldots, A_n^{-1}$. *Let* $x_1^*, x_2^*, \ldots, x_n^*$ *be linear functionals such that*

$$\sum_{k=1}^{k=n} x_k^*(A_k x) = 0, \tag{7}$$

for every $x \in \bigcap_k \mathfrak{D}_k$. *Then there exist n functionals* $y_1^*, y_2^*, \ldots, y_n^*$ *such that:*

$$\sum_{k=1}^{k=n} y_k^* = 0, \qquad y_k^* \in \mathfrak{D}_k^* \cap \mathfrak{D}_{1,2,\ldots k-1,k+1,\ldots,n}^*, \qquad k=1,2,\ldots,n \tag{8}$$

and

$$(A_1^* \ldots A_{k-1}^* A_{k+1}^* \ldots A_n^*)\, y_k^* = x_k^*, \qquad k=1,2,\ldots,n, \tag{9}$$

($\mathfrak{D}_{1,2,\ldots,n}^*$ *denotes the domain of* $A_1^* A_2^* \ldots A_n^*$).

Theorems I and II contain the lemma of du Bois-Reymond [1] and its generalizations [3, 5, 6, 7, 10]. They contain also the theorems of Hilbert [2], Mason [4] and Kubota [8] on double integrals. Theorem III contains Razmadzé's formulation of the fundamental lemma [9] and Haar's lemma for double integrals [11].

The second part of the paper gives another generalization of the fundamental lemma.

PART I.

LEMMA 1,1. *Let* $x_1^*, x_2^*, \ldots, x_n^*$ *denote linearly independent functionals. Then n elements* $x_1, x_2, \ldots x_n$ *exist such that*

$$\operatorname{Det} x_i^*(x_k) \neq 0. \tag{1,1}$$

Proof. It can easily be shown that if the lemma were not true the functionals would be linearly dependent, which contradicts our assumption.

LEMMA 1,2. *Let* $x_1^*, x_2^*, \ldots, x_n^*$ *denote linearly independent functionals. Let* $x^* \in \mathfrak{X}^*$ *be a functional such that*

$$x^*(x) = 0, \tag{1,2}$$

for every element $x \in \mathfrak{X}$ *such that*

$$x_k^*(x) = 0, \qquad k = 1, 2, \ldots, n. \tag{1,3}$$

Then x^* *is a linear combination of* x_k^*

$$x^* = \sum_{k=1}^{k=u} \lambda_k x_k^* . \tag{1,4}$$

Proof. First we consider a functional $y^* \in \mathfrak{X}^*$ which satisfies the conditions of the lemma and is orthogonal to elements with property (1,1). Let y denote an arbitrary element of the Banach space $\mathfrak{X}$ and let $\mu_1, \mu_2, \ldots, \mu_n$ be the solution of the system

$$\sum_{k=1}^{k=n} \mu_k x_i^* (x_k) = x_i^* (y), \quad i = 1, 2, \ldots, n, \tag{1,5}$$

where Det $x_i^* (x_k) \neq 0$. Equations (1,5) can be written

$$x_i^* (y - \sum_{k=1}^{k=n} \mu_k x_k) = 0 . \tag{1,6}$$

Now (1,2) implies $y^* (y - \Sigma \mu_k x_k) = 0$. Since $y^* (x_k) = 0$ it follows that $y^* \equiv 0$.

Now let x^* denote a functional satisfying the conditions of the lemma. Consider

$$y^* = x^* - \sum_{k=1}^{k=n} \lambda_k x_k^* .$$

We determine the coefficients λ_k so that $y^* (x_k) = 0$, i. e. as the solutions of the system

$$\sum_{k=1}^{k=n} \lambda_k x_k^* (x_i) = x^* (x_i) \quad i = 1, 2, \ldots, n .$$

Owing to (1,1) the solution exists. Thus $y^* \equiv 0$ and (1,4) holds.

LEMMA 1,3. *Let* $x_1^*\ x_2^*, \ldots, x_n^*$ *denote linearly independent functionals. The set of elements with proprety* (1,3) *forms a subspace* $\mathfrak{X}' \subset \mathfrak{X}$. *The dimension of the factor space* $\mathfrak{X}/\mathfrak{X}'$ *is* n. *Every element* $y \in \mathfrak{X}$ *can be written as*

$$y = x + \sum_{k=1}^{k=n} \mu_k x'_k , \tag{1,7}$$

where $x \in \mathfrak{X}'$ *and* x'_k *belong to a given linear manifold, dense in* $\mathfrak{X}$.

P r o o f. If $x \in \mathfrak{X}$ is an element of a coset modulo $\mathfrak{X}'$, we denote the corresponding coset by X'_x. Let x_k be elements from 1,1. The corresponding coseta X'_{x_k} are linearly independent. Otherwise there are numbers $\nu_1, \nu_2, \ldots, \nu_n$ some at least of them different from zero such that

$$\sum_{k=1}^{k=n} \nu_k x_k \in \mathfrak{X}',$$

or by definition of $\mathfrak{X}'$

$$\sum_{k=1}^{k=n} \nu_k x_i^* (x_k) = 0, \quad i = 1, 2, \ldots, n.$$

Because of (1,1) this system has only the trivial solution $\nu_k = 0$ in contradiction with our assumption.

On the other hand, every coset X'_y linearly depends on cosets X'_{x_k}. Given an arbitrary $y \in \mathfrak{X}$ we can obtain the corresponding coefficients $\mu_1, \mu_2, \ldots, \mu_n$ from (1,5). Writing (1,5) in the form of (1,6) we see that

$$y - \sum_{k=1}^{k=n} \mu_k x_k \in \mathfrak{X}',$$

or

$$y = x + \sum_{k=1}^{k=n} \mu_k x_k, \quad x \in \mathfrak{X}',$$

which proves the first part of the lemma.

Now let us denote by $\mathfrak{D}$ a given lienear manifold, dense in $\mathfrak{X}$. Choose the elements $x'_k \in \mathfrak{D}$ in the neighbourhoods of x_k so that Det $x_i^*(x_k') \neq 0$. Thus the cosets $X'_{x'_k}$ form a basis of $\mathfrak{X}/\mathfrak{X}'$ and this completes the proof.

LEMMA 1,4. *Let $\mathfrak{D}$ be a given linear manifold, dense in $\mathfrak{X}$ and let $\mathfrak{X}$ denote the subspace from lemma* 1,3. *Then $\mathfrak{D} \cap \mathfrak{X}'$ is dense in $\mathfrak{X}'$.*

P r o o f. We start by proving that the component x in (1,7) can be estimated by

$$\| x \| \leq \rho \| y \|, \tag{1,8}$$

where ρ is a number which does not depend on y. The coefficients μ_k in (1,7) are solutions of the system

$$\sum_{k=1}^{k=n} \mu_k x_i^* (x_k') = x_i^* (y), \quad i = 1, 2, \ldots, n$$

Solving this system we get

$$|\mu_k| \leqq \sum_{i=1}^{i=n} |x_i^*(y)|\,|D_{ki}/D| \leqq \|y\| \sum_{i=1}^{i=n} \|x_i^*\|\,|D_{ki}/D|,$$

where D is the determinant $\operatorname{Det} x_i^*(x'_k)$ and D_{ik} is the subdeterminant of $x_i^*(x'_k)$. Since $D \neq 0$ there is a number σ such that

$$|\mu_k| \leqq \sigma \|y\|.$$

Now let us consider

$$\|x\| \leqq \|y\| + \sum_{k=1}^{k=n} |\mu_k|\,\|x'_k\| \leqq \|y\| \left(1 + \sigma \sum_{k=1}^{k=n} \|x'_k\|\right).$$

Thus (1,8) holds.

Let now x be an arbitrary element of $\mathfrak{X}'$. Choose an $y' \in \mathfrak{D}$ such that

$$\|x - y'\| \leqq \varepsilon/\rho. \tag{1.9}$$

Write $y' = x' + \Sigma \mu_k x'_k$, where $x' \in \mathfrak{X}'$. Obviously $x' \in \mathfrak{D}$, therefore $x' \in \mathfrak{X}' \cap \mathfrak{D}$. Since

$$x - y' = x - x' - \sum_{k=1}^{k=n} \mu_k x'_k,$$

we obtain from (1,8) and (1,9)

$$\|x - x'\| \leqq \rho \|x - y'\| \leqq \varepsilon.$$

This completes the proof.

P r o o f of theorem I. We must show that there exists a functional $y^* \in \mathfrak{X}^*$ such that

$$y^*(Ax) = y^*(x), \quad x \in \mathfrak{D}. \tag{1,10}$$

Suppose that y^* exists. Then $y^*(x) = 0$ whenever $x \in \mathfrak{D} \cap \mathfrak{X}'$ in virtue of (1) and (2). But $\mathfrak{D} \cap \mathfrak{X}'$ is dense in $\mathfrak{X}'$ therefore $y^*(x) = 0$ for every $x \in \mathfrak{X}'$. By lemma 1,2 we have

$$y^* = \sum_{k=1}^{k=n} \lambda_k x_k^*.$$

The coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ must be determined in such a manner that (1,10) is valid for every $x \in \mathfrak{D}$. Every $x \in \mathfrak{D}$ can be represented by

$$x = x' + \sum_{i=1}^{i=n} \mu_i x_i'; \quad x_i' \in \mathfrak{D}; \quad x' \in \mathfrak{D} \cap \mathfrak{X}'.$$

It follows that

$$\sum_{i=1}^{i=n} \mu_i \sum_{k=1}^{k=n} \lambda_k x_k^* (x_i') = \sum_{i=1}^{i=n} \mu_i x^* (A x_i')$$

must hold identically in μ_i. So λ_k must be solution of the system

$$\sum_{k=1}^{k=n} \lambda_k x_k^* (x_i') = x^* (A x_i'); \quad i = 1, 2, \ldots, n.$$

This system has an unique solution because of $\operatorname{Det} x_k^* (x_i') \neq 0$. The corresponding $y^* = \sum \lambda_k x_k^*$ is therefore uniquely determined. This y^* has all the properties required. By definition of the adjoint operator it is $y^* = A^* x^*$ and the theorem follows.

P r o o f of theorem II. First we note that $y^* (Ax)$ is a distributive functional defined on $\mathfrak{D}$. Assuming that $y_k^* (Ax)$ are linearly independent we let

$$z^* = y^* - \sum_{k=1}^{k=n} \lambda_k y_k^* .$$

Since lemma 1,2 evidently holds for distributive functionals on linear spaces and $\mathfrak{D}$ is such a space, the coefficients λ_k can be determined so that

$$z^* (Ax) = 0,$$

for every $x \in \mathfrak{D}$. The theorem follows by definition of A^*.

Evident generalizations of theorems I and II are the following theorems which we get by replacing the space $\mathfrak{X}$ by a cartesian product $\mathfrak{X} \times \mathfrak{X} \times \ldots \times \mathfrak{X}$.

THEOREM I'. *Let* $A_{i,k}$, $i, k = 1, 2, \ldots, n$ *be linear operators with domains* $\mathfrak{D}_{ik}$ *and let the sets* $\bigcap_k \mathfrak{D}_{ik}$ *be dense in* $\mathfrak{X}$. *Let* $x_{jk}^*, j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$ *be given linear functionals on* $\mathfrak{X}$. *If a n-tuple of functionals* $x_k^* \in \mathfrak{X}^*$ *satisfies the condition*

$$\sum_{i=1}^{i=n} \sum_{k=1}^{k=n} x_k^* (A_{ik} x_i) = 0, \tag{11,1}$$

for every n-tuple of elements $x_i \in \bigcap_k \mathfrak{D}_{ki}$ *which possesses the property*

$$\sum_{k=1}^{k=n} x_{jk}^*(x_k) = 0, \quad j = 1, 2, \ldots, m,$$

then $x_k^* \in \bigcap \mathfrak{D}_{ik}^*$ *and there exist m scalars* λ_j *so that*

$$\sum_{k=1}^{k=n} A_{ik}^* x_k^* = \sum_{j=1}^{j=m} \lambda_j x_{ji}^*; \quad i = 1, 2, \ldots, n.$$

THEOREM II'. *Let* A_{ik} *and* x_{jk}^* *denote operators and functionals defined in* I'. *If an n-tuple of functionals* x_k^* *satisfies the condition* (1,11) *for every n-tuple of elements* $x_i \in \bigcap_k \mathfrak{D}_{ik}$ *obeying*

$$\sum_{k=1}^{k=n} \sum_{l=1}^{l=n} x_{jk}^*(A_{kl} x_l) = 0; \quad j = 1, 2, \ldots, m,$$

then there exist m scalars λ_j *and a solution* y_k^* *of the system*

$$\sum_{k=1}^{k=n} A_{jk}^* y_k^* = 0; \quad i = 1, 2, \ldots, m,$$

so that

$$x_k^* = y_k^* + \sum_{j=1}^{j=m} \lambda_j x_{jk}^*, \quad k = 1, 2, \ldots, n.$$

Proof of theorem III. The condition (7) must obviously hold for every element $x \in \mathfrak{X}$ of the form

$$x = \prod_i A_i^{-1} y,$$

where y denotes an arbitrary element of the space $\mathfrak{X}$. Replacing this into the k-th therm on the left side of (7) and remembering that the inverse operators A_i^{-1} are comutative, we obtain the linear functional

$$y_k^*(y) = x_k^* \left(\prod_{l \neq k} A_l^{-1} y \right),$$

which is obviously defined for each element $y \in \mathfrak{X}$. Let $y \in \mathfrak{D}_m, m \neq k$. Replacing in the last relation y by $A_m y$ we see that

$$y_k^*(A_m y) = x_k^* \left(\prod_{l \neq k, m} A_l^{-1} y \right),$$

for every $y \in \mathfrak{D}_m$. Therefore $y_k^* \in \mathfrak{D}_m^*$ and

$$A_m^* y_k^* (y) = x_k^* \left(\prod_{i \neq k, m} A_i^{-1} y \right).$$

Thus we see that $y_k^* \in \mathfrak{D}_{1,2,\ldots,k-1,k+1,\ldots,n}^*$ and that (9) holds. From the definition of y_k^* and (7) we have $\Sigma y_k^* = 0$. Therefore y_k^* can be expressed in terms of $y_i^*, i \neq k$ which belong all to $\mathfrak{D}_k^*$. It follows $y_k^* \in \mathfrak{D}_k^*$ too. The theorem is proved. As an special case we mention the case $n = 2$.

THEOREM III'. *Let A_1 and A_2 denote two linear operators with domains $\mathfrak{D}_1$ and $\mathfrak{D}_2$, dense in $\mathfrak{X}$, and let the inverse operators A_1^{-1} and A_2^{-1} be bounded and comutative. If the functionals x_1^* and x_2^* satisfy the condition*

$$x_1^* (A_1 x) + x_2^* (A_2 x) = 0,$$

for every $x \in \mathfrak{D}_1 \cap \mathfrak{D}_2$, a functional $x^* \in \mathfrak{D}_1^* \cap \mathfrak{D}_2^*$ exists such that

$$x_1^* = -A_2^* x^*, \qquad x_2^* = A_1^* x^*.$$

PART II.

From now on let $\mathfrak{X}$ denote a reflexive Banach space. Let $\{x^*\} \subset \mathfrak{X}^*$ be a given set of functionals. Linear combinations of the functionals from $\{x^*\}$ with positive coefficients and their accumulation points form a closed semigroup in $\mathfrak{X}^*$. Let us denote it by $\mathfrak{K}\{x^*\}$. The set of elements $x \in \mathfrak{X}$ satisfying

$$x^* (x) = 0, \tag{2,1}$$

for every $x^* \in \{x^*\}$ forms a subspace $\mathfrak{X}' \subset \mathfrak{X}$. The set of functionals $x^* \in \mathfrak{X}^*$ fulfilling (2.1) for every element $x \in \mathfrak{X}'$ forms a subspace $\mathfrak{X}^{*\prime} \subset \mathfrak{X}^*$. Obviously $\{x^*\} \subset \mathfrak{X}^{*\prime}$ and $\mathfrak{K}\{x^*\} \subset \mathfrak{X}^{*\prime}$. Let us prove

LEMMA 2,1. *The spaces $\mathfrak{X}^{*\prime}$ and $\mathfrak{X}/\mathfrak{X}'$ are mutually adjoint.*

Proof. Every $x^* \in \mathfrak{X}^{*\prime}$ is a functional on $\mathfrak{X}/\mathfrak{X}'$, it has on all elements of the coset $\mathfrak{X} \in \mathfrak{X}/\mathfrak{X}'$ the same value which we call the value of the functional on the coset $\mathfrak{X}$. Conversely every functional $y^* \in (\mathfrak{X}/\mathfrak{X}')^*$ is also a functional on the primary space, the value of the y^* is the same on all elements of the coset $\mathfrak{X}$. It follows that y^* obeys (2,1) for every $x \in \mathfrak{X}'$.

Thus y^* belongs to $\mathfrak{X}^{*\prime}$ and

$$(\mathfrak{X}/\mathfrak{X}')^* = \mathfrak{X}^{*\prime}.$$

On the other hand, every functional on $\mathfrak{X}^{*\prime}$ can be extended into a functional on $\mathfrak{X}^*$. Since $\mathfrak{X}^*$ is reflexive, it follows that $\mathfrak{X}$ is homomorphic to $\mathfrak{X}^{*\prime *}$. Obviously $\mathfrak{X}'$ is the set of elements which are mapped into $D \in \mathfrak{X}^{*\prime *}$ by the homomorphism. Thus

$$\mathfrak{X}^{*\prime *} = \mathfrak{X}/\mathfrak{X}'$$

and lemma is therefore established.

The semigroup $\mathfrak{K}\{x^*\}$ has in $\mathfrak{X}/\mathfrak{X}'$ adjoined a semigroup $\mathfrak{K}'$. $\mathfrak{K}'$ contains all the cosets of $\mathfrak{X}/\mathfrak{X}'$ which satisfy the condition

$$x^*(X) \geqq 0,$$

for every $x^* \in \{x^*\}$ (also for every $x^* \in \mathfrak{K}\{x^*\}$). It can easily be shown that the semigroup $\mathfrak{K}'$ is a cone. Because the space $\mathfrak{X}$ is reflexive the adjointness of $\mathfrak{K}\{x^*\}$ and $\mathfrak{K}'$ is mutual [13].

THEOREM IV. *Let x_τ^* be a continuous function of the parameter τ on the interval $0 \leqslant \tau \leqslant 1$ and let at least one element x_0 and a positive number k exist such that*

$$x_\tau^*(x_0) \geqslant k, \tag{2,2}$$

for every value of the parameter τ. Every functional x^* such that

$$x^*(x) \geqslant 0, \tag{2,3}$$

for every element x with the property

$$x_\tau^*(x) \geqslant 0, \quad 0 \leqslant \tau \leqslant 1, \tag{2,4}$$

can be expreseed by a Stieltjes' integral

$$x^* = \int_0^1 x_\tau^* \, d\gamma(\tau), \tag{2,5}$$

where the integrator $\gamma(\tau)$ is a bounded and increasing scalar function.

Proof. As the space $\mathfrak{X}$ is reflexive it follows from (2,3), (2,4) and and lemma 2,1 that $x^* \in \mathfrak{K}\{x_\tau^*, 0 \leqq \tau \leqq 1\}$. Thus there exists a sequence of functionals x_1^*, x_2^*, x_3^*, ... converging to x^* and such that

$$x_n^* = \sum_{k=1}^{k=m} \lambda_{nk} x_{\tau_{nk}}^*,$$

where the coeficients λ_{nk} are all positive. Let $0 \leqq \tau_{n1} < \tau_{n2} < \ldots < \tau_{nm} \leqq 1$ and let us define a step function $\gamma_n(\tau)$

$$\gamma_n(0) = 0,$$

$$\gamma_n(\tau) = \sum_{k=1}^{k=i} \lambda_{nk}, \quad \tau_{ni} < \tau \leqq \tau_{n,i+1}.$$

Then we can put

$$x_n^* = \int_0^1 x_\tau^* \, d\gamma_n(\tau).$$

The total variations of functions $\gamma_n(\tau)$

$$V \gamma_n(\tau) = \sum_{k=1}^{k=m} \lambda_{nk}$$

are uniformly bounded. Indeed, we have

$$\sum_{k=1}^{k=m} \lambda_{nk} x_{\tau_{nk}}^*(x_0) = x_n^*(x_0).$$

Because of (2,2) we can write

$$k \sum_{h=1}^{h=m} \lambda_{nk} \leqq \| x_n^* \| \cdot \| x_0 \|.$$

Hence

$$V \gamma_n(\tau) \leqq \| x_n^* \| \cdot \| x_0 \| / k \leqq (\| x^* \| + \varepsilon) \| x_0 \| / k,$$

for all sufficiently large n. The increasing functions $\gamma_n(\tau)$ being uniformly bounded it follows, according to Helly's principle, that we can select from the sequence $\gamma_1(\tau)$, $\gamma_2(\tau)$, ... a subsequence

$$\gamma_{n_1}(\tau), \gamma_{n_2}(\tau), \gamma_{n_3}(\tau), \ldots,$$

converging towards a monotonically increasing function $\gamma(\tau)$. Then we have

$$x_{n_k}^* \to \int_0^1 x_\tau^* \, d\gamma(\tau).$$

Hence (2,4) holds and the theorem is proved.

THEOREM V. *Let* x_τ^* *be a continuous function of the parameter* τ *on the interval* $0 \leqq \tau \leqq 1$. *Let there be an element* x_0 *and a positive number* k *such that the condition* (2,2) *is fulfilled. Further let there be a functional* x_0^* *and a positive number* ρ *so that*

$$x_0^* (X) \geqq \rho \| X \|, \qquad (2,6)$$

for every $X \in \mathfrak{K}'$.

If a functional x^* *satisfies condition* (2,1) *for all* $x \in X$ *such that*

$$x_\tau^* (x) = 0, \qquad (2,7)$$

for every τ, $0 \leqq \tau \leqq 1$, *then* x^* *can be represented by Stieltjs' integral* (2,5), *where* $\gamma(\tau)$ *is a scalar function of bounded variation.*

Proof. (2,1) and (2,7) imply $x^* \in \mathfrak{X}^{*\prime}$. Owing to (2,6) two functionals $x_1^* \in \mathfrak{K}\{x_\tau^*, 0 \leqq \tau \leqq 1\}$ and $x_2^* \in \mathfrak{K}\{x_\tau^*, 0 \leqq \tau \leqq 1\}$ exist to that

$$x^* = x_1^* - x_2^*$$

[13]. Hence by theorem IV we get

$$x_1^* = \int_0^1 x_\tau^* \, d\gamma_1(\tau), \qquad x_2^* = \int_0^1 x_\tau^* \, d\gamma_2(\tau),$$

where integrators $\gamma_1(\tau)$ and $\gamma_2(\tau)$ are bounded and increasing functions. So (2,4) holds and $\gamma(\tau) = \gamma_1(\tau) - \gamma_2(\tau)$ is a function of bounded total variation. This proves the theorem.

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