

ON UNIFORM CONVERGENCE OF FOURIER SERIES

By

R. BOJANIĆ (Beograd)

SUMMARY. — A theorem on uniform convergence of Fourier series containing the well-known Dini-Lipschitz test for uniform convergence and a theorem of M. Tomić, is given here.

1. (i) Let C denote the class of continuous and periodic functions of period 2π and C_F the class of functions $f(x) \in C$ having uniformly convergent Fourier series.

The modulus of continuity of $f(x) \in C$ is as usually defined by

$$\omega(t) = \text{Max}_{|x-y| \leq t} |f(x) - f(y)|.$$

Let $\Omega(t)$ be a continuous function decreasing monotonically to 0 as $t \rightarrow +0$, and

$$\Omega(0) = 0, \quad \Omega(x+y) \leq \Omega(x) + \Omega(y).$$

Then there exists a function $f(x) \in C$ having the modulus of continuity $\Omega(t)$ ([9], p. 486).

A function $f(x) \in C$ belongs to the class C^Ω if its modulus of continuity $\omega(t)$ satisfies the condition

$$\omega(t) = O\{\Omega(t)\}, \quad t \rightarrow 0.$$

Given two classes P and Q of integrable functions, we shall denote, following Zygmund ([3], p. 100), by (P, Q) the class of sequences $\{\lambda_n\}$ such that

$$(1) \quad \frac{1}{2} \lambda_0 a_0 + \sum_{\nu=1}^{\infty} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is the Fourier series of a function $f^*(x) \in Q$ whenever

$$(2) \quad \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is the Fourier series of a function $f(x) \in P$.

(ii) Recently J. Karamata ([1]; [2], p. 127) proved the following theorem:

A necessary and sufficient condition that $\{\lambda_n\}$ should belong to the class (C, C_F) is

$$(3) \quad \int_0^{2\pi} |\Lambda_n(t)| dt = O(1),$$

where

$$\Lambda_n(t) = \frac{1}{2} \lambda_0 + \sum_{\nu=1}^n \lambda_\nu \cos \nu t.$$

In other words, the condition (3) is necessary and sufficient in order that the series (1) should be uniformly convergent for an arbitrary $f(x) \in C$.

This theorem contains in particular the theorem ([3], p. 58—59, [6], p. 23)

If $\{\lambda_n\}$ is quasi-convex, i. e. if

$$\sum_{\nu=0}^{\infty} (\nu+1) |\Delta^2 \lambda_\nu| < \infty, \text{ where } \Delta^2 \lambda_\nu = \lambda_\nu - 2\lambda_{\nu+1} + \lambda_{\nu+2},$$

and $\lambda_n \lg n = O(1)$, then $\{\lambda_n\}$ belongs to the class (C, C_F) .

The condition (3) imposes a restriction on the behavior of λ_n which is very severe. H. Helson [7] has shown that from (3) it follows $\lambda_n = o(1)$.

However, if we consider the class C^Ω , instead of the class C , the condition (3) can be essentially enlarged. Here we shall prove the following

THEOREM. If

$$(4) \quad \int_0^{2\pi} \left| \sum_{\nu=0}^n \Lambda_\nu(t) \right| dt = O(n)$$

and

$$(5) \quad \Omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)| dt = o(1),$$

then $\{\lambda_n\}$ belongs to the class (C^Ω, C_F) .

In other words, if $\{\lambda_n\}$ satisfies conditions (4) and (5), then the series (1) is uniformly convergent for an arbitrary $f(x) \in C^\Omega$.

Our theorem may be regarded as a generalization of the well known Dini—Lipschitz test for uniform convergence ([3], p. 30), to which it

reduces when $\lambda_n = 1$, $n = 0, 1, 2, \dots$. It contains also the following theorem proved first by M. Tomić ([6], p. 24, [8])

If $\{\lambda_n\}$ is quasi-convex and $\Omega(1/n) \lambda_n \lg n = o(1)$, then $\{\lambda_n\}$ belongs to the class (C^Ω, C_F) .

2. (i) The proof of the just cited theorem, which will be given here, depends on Jackson's fundamental theorem on the approximation of a continuous function by trigonometrical polynomials ([4], p. 7). We shall use Jackson's theorem in the following form ([5], p. 319):

If $f(x) \in C$ and

$$(6) \quad T_n(x) = \sum_{v=0}^{2n-1} D(v/n) (a_v \cos vx + b_v \sin vx),$$

where

$$(7) \quad D(x) = \begin{cases} 1 - \frac{3}{2}x^2 + \frac{3}{4}x^3, & 0 \leq x \leq 1, \\ \frac{1}{4}(2-x)^3, & 1 \leq x \leq 2, \\ 0, & x \geq 2, \end{cases}$$

then

$$(8) \quad |f(x) - T_n(x)| \leq A\omega(1/n).$$

This problem was suggested by M. Tomić to which we are very indebted for his constant advice and criticism.

(ii) Proof of the Theorem. We note first that (4) is a necessary and sufficient condition that the series

$$(9) \quad \frac{1}{2} \lambda_0 + \sum_{v=1}^{\infty} \lambda_v \cos vt$$

should be a Fourier-Stieltjes series ([3], p. 79). On the other hand, a necessary and sufficient condition for $\{\lambda_n\}$ to belong to the class (C, C) is that the series (9) should be a (Fourier-Stieltjes series ([3], p. 101).

From these facts it follows that (4) is a necessary and sufficient condition for $\{\lambda_n\}$ to belong to the class (C, C) .

Since $f(x) \in C$, it follows therefore from (4) that (1) is the Fourier series of a continuous function $f^*(x) \in C$.

Let $s_n^*(x)$ be the n -th partial sum of the series (1) and let $\sigma_n^*(x)$ be the first arithmetic mean of $s_n^*(x)$. We have

$$|f^*(x) - s_n^*(x)| \leq |f^*(x) - \sigma_n^*(x)| + |\sigma_n^*(x) - s_n^*(x)|.$$

Now, since $f^*(x)$ is continuous, $\sigma_n^*(x)$ converges uniformly to $f^*(x)$ in virtue of Fejér's theorem ([3], p. 45), and it remains to be proved that

$$\sigma_n^*(x) - s_n^*(x) = \frac{1}{n+1} \sum_{\nu=1}^n \nu \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

converges uniformly to 0 as $n \rightarrow \infty$.

Let $T_n(x)$ be the Jackson's polynomial corresponding to the function $f(x) \in C^\Omega$. Then

$$\begin{aligned} s_n^*(x) &= \frac{1}{\pi} \int_0^{2\pi} f(x+t) \Lambda_n(t) dt = \\ &= \frac{1}{\pi} \int_0^{2\pi} T_n(x+t) \Lambda_n(t) dt + \frac{1}{\pi} \int_0^{2\pi} \{f(x+t) - T_n(x+t)\} \Lambda_n(t) dt. \end{aligned}$$

Let us denote the last term on the right by $A_n(x)$. Then from (6) it follows that

$$(10) \quad s_n^*(x) = \frac{1}{2} \lambda_0 a_0 + \sum_{\nu=1}^n D\left(\frac{\nu}{n}\right) \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x) + A_n(x).$$

Since, according to (7),

$$D\left(\frac{\nu}{n}\right) = 1 - \frac{3}{2} \left(\frac{\nu}{n}\right)^2 + \frac{3}{4} \left(\frac{\nu}{n}\right)^3, \quad 0 \leq \nu \leq n.$$

we obtain from (10)

$$(11) \quad 2n \sum_{\nu=1}^n \nu^2 \alpha_\nu(x) - \sum_{\nu=1}^n \nu^3 \alpha_\nu(x) = \frac{4}{3} n^3 A_n(x),$$

where $\alpha_\nu(x) = \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$.

Now, if $\Delta^2 w_n = w_n - 2w_{n+1} + w_{n+2}$, then

$$\Delta^2 \left\{ n \sum_{\nu=1}^n u_\nu \right\} = (n+2) u_{n+2} - n u_{n+1}$$

and

$$\Delta^2 \left\{ \sum_{\nu=1}^n v_\nu \right\} = v_{n+2} - v_{n+1}.$$

Applying Δ^2 to the equation (11) we obtain easily

$$(n+2)^3 \alpha_{n+2}(x) - (n+1)^2 (n-1) \alpha_{n+1}(x) = \frac{4}{3} \Delta^2 \{n^3 A_n(x)\}.$$

Multiplying this equation by $n(n+1)$ and summing we obtain

$$\begin{aligned} \frac{3}{4} n^3 (n-1)(n-2) \alpha_n(x) &= \sum_{\nu=1}^{n-2} \nu(\nu+1) \Delta^2 \{ \nu^3 A_\nu(x) \} = \\ &= n^3 (n-1)(n-2) A_n(x) - (n+1)(n-1)^3 (n-2) A_{n+1}(x) + 2 \sum_{\nu=1}^{n-2} \nu^3 A_\nu(x), \end{aligned}$$

or

$$\begin{aligned} \frac{3}{4} n \alpha_n(x) &= n A_n(x) - (n-1) A_{n-1}(x) + \frac{n-1}{n^2} A_{n-1}(x) + \\ &+ \frac{2}{n^2 (n-1)(n-2)} \sum_{\nu=1}^{n-2} \nu^3 A_\nu(x). \end{aligned}$$

Summing once again we get

$$\begin{aligned} \frac{3}{4} \sum_{\nu=1}^n \nu \alpha_\nu(x) &= n A_n(x) + \sum_{\nu=1}^{n-1} \frac{\nu}{(\nu+1)^2} A_\nu(x) + \\ &+ 2 \sum_{\nu=1}^{n-2} \frac{1}{(\nu+2)^2 (\nu+1) \nu} \sum_{k=1}^n k^3 A_k(x), \end{aligned}$$

and so

$$(12) \quad \frac{3}{4} \left| \sum_{\nu=1}^n \nu \alpha_\nu(x) \right| \leq n |A_n(x)| + \sum_{\nu=1}^n |A_\nu(x)| + 2 \sum_{\nu=1}^n \nu^{-4} \sum_{k=1}^{\nu} k^3 |A_k(x)|.$$

Since, for $k \geq 1$

$$\sum_{\nu=k}^{\infty} \nu^{-4} = k^{-4} + \int_k^{\infty} x^{-4} dx < 2 k^{-3}$$

we have

$$\begin{aligned} \sum_{\nu=1}^n \nu^{-4} \sum_{k=1}^{\nu} k^3 |A_k(x)| &= \sum_{k=1}^n k^3 |A_k(x)| \sum_{\nu=k}^{\infty} \nu^{-4} \leq \\ &\leq 2 \sum_{k=1}^n |A_k(x)|, \end{aligned}$$

so that the inequality (12) reduces to

$$\frac{1}{n+1} \left| \sum_{\nu=1}^n \nu \alpha_\nu(x) \right| \leq 2 |A_n(x)| + \frac{10}{n+1} \sum_{\nu=1}^n |A_\nu(x)|.$$

Hence, it remains to be proved that $A_n(x)$ converges uniformly to 0 as $n \rightarrow \infty$. For $n \geq 1$ we have, according to (9)

$$\begin{aligned} |A_n(x)| &= \frac{1}{\pi} \int_0^{2\pi} \left| \{f(x+t) - T_n(x+t)\} \Lambda_n(t) dt \right| \leq \\ &\leq \frac{1}{\pi} \int_0^{2\pi} |f(x+t) - T_n(x+t)| |\Lambda_n(t)| dt \leq \\ &\leq A \omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)| dt, \end{aligned}$$

and so for an arbitrary $f(x) \in C^{\Omega}$, according to (5),

$$|A_n(x)| \leq B \Omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)| dt \rightarrow 0, \quad n \rightarrow \infty.$$

(Received September 5, 1956)

REFERENCES

- [1] J. Karamata — Suite de fonctionelles linéaires et facteurs de convergence des séries de Fourier, *Journal de Math. P. et Appl.* **35** (1956), 87–95.
- [2] J. Karamata et M. Tomić — Sur la sommation des séries de Fourier des fonctions continues, *Publ. Inst. Math. Acad. Serbe Sci.* **VIII** (1955), 123–138.
- [3] A. Zygmund — Trigonometrical Series, 2 ed., New York, 1952.
- [4] D. Jackson — The Theory of Approximation, *Am. Math. Soc. Coll. Publ.*, **XI**, New York, 1930.
- [5] Н. И. Ахиезер — Лекции по теории аппроксимации, Москва–Ленинград, 1947.
- [6] М. Томич — Sur les facteurs de convergence des séries de Fourier des fonctions continues, *Publ. Inst. Math. Acad. Serbe Sci.* **VIII** (1955), 23–32.
- [7] H. Helson — Proof of a conjecture of Steinhaus, *Proc. Nat. Acad. Sci. USA*, **40** (1954), 205–206.
- [8] М. Томич — Примедба о једном поступку збирљивости, *Билтен на друштво на маш. и физ. од НР Македонија*, **VI** (1955), 35–43.
- [9] Н. К. Бари и С. Б. Стечкин — Найлучшие приближения и дифференциальные свойства двух сопряженных функций, *Труды Московского маш. общества*, **5** (1956), 483–522.