SOME THEOREMS ON INTRANSITIVE GROUPS OF MOTIONS

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SUMMARY: Some theorems and their consequences are given for the intransitive groups of motions which are the groups of stability of more then one point in Riemannian spaces of constant curvature.

Starting from a theorem of L. Bianchi ([1], p. 541) (here Theorem B), an da theorem of G. Fubini ([2] or [3] p. 221) (here Theorem F) on intransitive groups, the intention of this paper is to demonstrate some theorems on geometric properties of intransitive groups of motions of a given order in Riemannian spaces V_n.

The starting theorems are:

THEOREM B: *If a Riemannian space V_n admits an intransitive group of motions G_r, where $r = {}^1/_2\, n\,(n-1)$ is the order of the group, the minimal invariant varieties of G_r are geodesically parallel hypersurfaces V_{n-1} of constant curvature.*

THEOREM F: *If a Riemannian space V_n admits an intransitive group of motions G_r, the group induced on the minimum invariant variety V_q, where q is the generic rank of the matrix $\| \xi^i_\alpha \|$ of the group, is an r-parameter group and by a suitable choice of coordinates the finite equations of the group G_r are reduced to equations of a transitive group in V_q on q variables.*

We are going first to show that the inverse of the Theorem B also holds:

THEOREM 1: *If a Riemannian space V_n admits a family of geodesically parallel hypersurfaces V_{n-1} of constant curvature, then V_n admits an intransitive group G_r of motions, $r = {}^1/_2\, n\,(n-1)$, for which the hypersurfaces V_{n-1} are the minimum invariant varieties.*

We can choose in V_n a system of coordinates such that hypersurfaces V_{n-1} are determined by $x^n = \text{const.}$, where the coordinate lines of parameter x^n are geodesics in V_n and are orthogonal trajectories of the hypersurfaces V_{n-1}. The fundamental form in V_n with respect to the so chosen system of coordinates is

$$ds^2 = g_{ij}\, dx^i\, dx^j + e_n (dx^n)^2, \quad (i, j = 1, 2, \ldots, n-1)$$

where is $e_n = \pm 1$.

On each of the hypersurfaces V_{n-1} there exists a complete transitive group of motions of order $r = {}^1/_2\, n\,(n-1)$, since the V_{n-1}'s are of constant curvature. The finite equations of the group on V_{n-1} are

$$(1) \qquad x^i = x^i\,(\overline{x}', \ldots, \overline{x}^{n-1};\ p^1, \ldots, p^r), \quad (i = 1, \ldots, n-1)$$

where p's are parameters of the group. Since the hypersurfaces V_{n-1} are in V_n determined by $x^n = \text{const}$, we can add to the transformations (1) the identity transformation

$$(2) \qquad x^n = \overline{x}^n.$$

Equations (1) and (2) are the finite equations of the group G_r, $r = \frac{n\,(n-1)}{2}$, in V_n which is intransitive and the minimum invariant varieties are the hypersurfaces V_{n-1}, what we had to prove.

THEOREM 2: *If a Riemannian space of constant curvature V_n admits an intransitive group of motions G_{r_k}, where $r_k = {}^1/_2\,(n-k+1)\,(n-k)$ is the order of the group, the minimum invariant varieties of the group G_{r_k} are subspaces V_{n-k} of constant curvature and geodesically parallel with respect to some enveloping subspaces V_{n-k+1} of V_n, which are also of constant curvature.*

The case $k = 0$ is excluded since G_{r_0}, $r_0 = {}^1/_2\, n\,(n+1)$, is the complete group of motions in a V_n of constant curvature.

The subgroup of G_{r_0} which is the group of stability of certain point A_1 in the V_n is the group of order $r_0 - n = r_1 = {}^1/_2\, n\,(n-1)$. Accordingly to the Theorem B the minimum invariant varieties of G_r, are geodesically parallel hypersurfaces V_{n-1} of constant curvature. Hence for $k = 1$, Theorem 2 is identical with the Theorem B.

Now we shall show that if Theorem 2 is valid for any k, $o < k \leqslant n-1$, it is valid for $k+1$.

Let $V_{n_k}, n_k = n-k+1$, be an n_k-dimensional space of constant curvature. If V_{n_k} admits an intransitive group of motions $G_{r_k}, r_k = {}^1/_2\, n_k(n_k-1) \equiv \equiv {}^1/_2\,(n-k+1)(n-k)$, accordingly to the Theorem B the minimum invariant varieties for G_{r_k} are hypersurfaces $V_{n_{k-1}}$, geodesically parallel with respect to V_{n_k} and of constant curvature.

The subgroup of stability of a certain point A_{k+1} in $V_{n_{k-1}}$ is the group of order $r_{k+1} = r_k-(n_k-1) \equiv {}^1/_2\,(n_k-1)(n_k-2)$. Since now $G_{r_{k+1}}$ is intransitive in $V_{n_{k-1}}$, accordingly to the Theorem B the minimum invariant varieties for $G_{r_{k+1}}$ are hypersurfaces $V_{n_{k-2}}$ of constant curvature and geodesically parallel with respect to $V_{n_{k-1}}$. Since

$$r_{k+1} = {}^1/_2\,(n_k-1)(n_k-2) \equiv {}^1/_2\,[n-(k+1)+1]\,[n-(k+1)],$$

we see that if Theorem 2 is valid for any k, it is valid for $k+1$, too. Since it is shown that the Theorem 2 holds for $k=1$, it follows that it will hold for any $k = 1, 2, \ldots, n-1$.

NOTE: In case $k=n$, $r_k = o$ and G_{r_k} reduces to the group of identity transformations.

From the way in which the Theorem 2 is proved we immediately have:

COROLLARY 1: *If a V_n of constant curvature admits an intransitive group of motions G_{r_k}, then G_{r_k} is the group of stability of k points in V_n, not all belonging to the same V_{n-k+1} of constant curvature.*

COROLLARY 2: *The group of stability of n points in V_n, not all belonging to the same V_{n-1} of constant curvature is the group of stability of the V_n itself.*

THEOREM 3: *The group G_{r_k} from the Theorem 2 is the group of stability of a $(k-1)$-dimensional totally geodesic subspace V_{k-1} of V_n.*

To prove this statement we have first to show that:

LEMMA: *If a group G_r of motions in V_n is a group of stability of two points in V_n, than G_r is the group of stability of the geodesic line passing through these two points.*

Since by the transformations of the groups of motions geodesics go into geodesics and since the two points on the geodesic are by hypothesis stabile with respect to G_r, it follows that the geodesic through these two points is stabile for the group G_r.

Accordingly to the Corollary 1, the group G_{r_k} is the group of stability of k points in V_n not belonging to the same V_{n-k+1}, and accordingly to the Lemma all of the geodesics passing through these k points are stabile with respect to G_{r_k}. $k-1$ of vectors tangential to the geodesics passing through these k points are linearly independent and, accordingly to a theorem of E. Cartan ([4], p. 294), they define a $(k-1)$-dimensional totally geodesic subspace of V_n stabile for the group G_{r_k}.

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