SHEARING STRESS IN BENDING OF T BEAMS

by

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SUMMARY. In the case of bending of a cantilever of a constant cross-section of any shape by a force P applied at the end parallel to one of the principal axis of the cross-section, in addition to normal stresses proportional in each cross-section to the bending moment, there will act also shearing stresses proportional to the shearing force.

The calculation of shearing stress depends on the form of cross-section and must be found in all special cases of the boundaries. This paper presents the solution of the problem in the case of the T cross-section.

NOTATIONS

The following notations are used in this paper:

P = force applied at the end,

J = moment of inertia of the cross-section,

μ = Poisson's ratio,

n = normal to the boundary,

x, y = principal axis of the cross-section.

The solution of the problem of bending of prismatic cantilever can be reduced to the determination of the stress function $\Phi(x, y)$ which satisfies the differential equation in the region of the cross-section

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \tag{1}$$

and the condition

$$\left(\frac{\partial \Phi}{\partial y} - \frac{P}{2J} x^2 + \frac{\mu}{1+\mu} \frac{P}{2J} y^2\right) \cos(nx) - \frac{\partial \Phi}{\partial x} \cos(ny) = 0 \tag{2}$$

on the boundary.

We denote by Φ_1 the value of the function $\Phi(x, y)$ in the region 1, 2, 3, 6 (Fig. 1) and by Φ_2 its value in the region 4, 5, 7, 8. These functions must be harmonic and must satisfy the boundary condition (2) and the condition of continuity on the line 4 – 5.

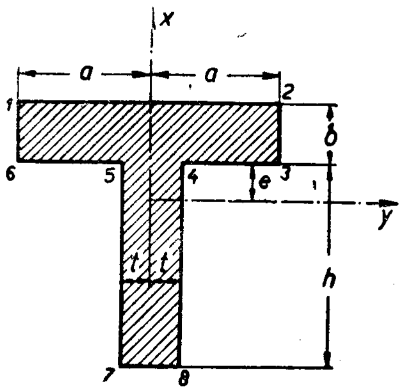

Fig. 1

If we take Φ_1 in the form

$$\Phi_1 = \frac{P}{2J}\left[(e+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{3}\right]y + \sum_{n=1,2..}^{\infty}\left[A_n Ch\frac{n\pi(x-e)}{a} + B_n Sh\frac{n\pi(x-e)}{a}\right]\sin\frac{n\pi y}{a} =$$

$$= \sum_{n=1,2...}^{\infty}\left\{-\frac{a}{\pi}\frac{P}{J}\left[(e+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{3}\right]\frac{(-1)^n}{n} + A_n Ch\frac{n\pi(x-e)}{a} + B_n Sh\frac{n\pi(x-e)}{a}\right\}\sin\frac{n\pi y}{a}, \tag{3}$$

this function satisfies the differential equation (1) and the boundary conditions along the edges 2 – 3 and 1 – 6.

Taking Φ_2 in the form

$$\Phi_2 = \frac{P}{2J}\left[e^2 - \frac{\mu}{1+\mu}\frac{t^2}{3} + \left(2eb + b^2\right)\frac{a}{t}\right]y + \sum_{m=1,2...}^{\infty}\left[C_m Ch\frac{m\pi(x-e)}{t} + D_m Sh\frac{m\pi(x-e)}{t}\right]\sin\frac{m\pi y}{t} =$$

$$= \sum_{m=1,2...}^{\infty}\left\{-\frac{1}{\pi}\frac{P}{J}\left[e^2 t - \frac{\mu}{1+\mu}\frac{t^3}{3} + (2eb + b^2)a\right]\frac{(-1)^m}{m} + C_m Ch\frac{m\pi(x-e)}{t} + D_m Sh\frac{m\pi(x-e)}{t}\right\}\sin\frac{m\pi y}{t}, \tag{4}$$

this function satisfies the differential equation (1) and the boundary conditions along the edges 4 – 8 and 5 – 7.

The condition (2) along the edge 1 – 2 becomes

$$\left(\frac{\partial\Phi}{\partial y}\right)_{x=e+b} = \frac{P}{2J}\left[(e+b)^2 - \frac{\mu}{1+\mu}y^2\right], \tag{5}$$

which is satisfied, if

$$\left(\frac{\partial \Phi_1}{\partial y}\right)_{x=e+b} = \frac{P}{2J}\left[(e+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{3}\right] +$$

$$+ \sum_{n=1,2\ldots}^{\infty} \frac{n\pi}{a}\left(A_n Ch\frac{n\pi b}{a} + B_n Sh\frac{n\pi b}{a}\right)\cos\frac{n\pi y}{a} =$$

$$= \left(\frac{\partial \Phi}{\partial y}\right)_{x=e+b} = \frac{P}{2J}\left[(e+b)^2 - \frac{\mu}{1+\mu}y^2\right]. \qquad (6)$$

Substituting in the equation (6) the known development y^2 in Fourier's series, we obtain

$$\sum_{n=1,2\ldots}^{\infty} \frac{n\pi}{a}\left(A_n Ch\frac{n\pi b}{a} + B_n Sh\frac{n\pi b}{a}\right)\cos\frac{n\pi y}{a} =$$

$$= -\frac{P}{J}\frac{\mu}{1+\mu}\frac{2a^2}{\pi^2}\sum_{n=1,2\ldots}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi y}{a}. \qquad (7)$$

From the equation (7) it follows that the coefficients B_n and A_n must satisfy the condition

$$B_n = -\frac{P}{J}\frac{\mu}{1+\mu}\frac{2a^3(-1)^n}{\pi^3 n^3 Sh\frac{n\pi b}{a}} - A_n Cth\frac{n\pi b}{a}. \qquad (8)$$

The boundary condition along the edges 7 – 8 from (2)

$$(\Phi)_{x=-(h-e)} = \frac{P}{2J}\left[(h-e)^2 y - \frac{\mu}{1+\mu}\frac{y^3}{3}\right], \qquad (9)$$

can be developed in the following Fourier's series

$$(\Phi)_{x=-(h-e)} = -\frac{P}{J}\frac{t}{\pi}\sum_{m=1,2\ldots}^{\infty}\frac{(-1)^m}{m}\left[(h-e)^2 + \right.$$

$$\left. + \frac{\mu}{1+\mu}t^2\left(\frac{2}{m^2\pi^2} - \frac{1}{3}\right)\right]\sin\frac{m\pi y}{t}, \qquad (10)$$

which is satisfied, if

$$(\Phi_2)_{x=-(h-e)} = \sum_{m=1,2\ldots}^{\infty}\left\{-\frac{1}{\pi}\frac{P}{J}\left[e^2 t - \frac{\mu}{1+\mu}\frac{t^3}{3} + \right.\right.$$

$$\left.\left. + (2eb+b^2)a\right]\frac{(-1)^m}{m} + C_m Ch\frac{m\pi h}{t} - D_m Sh\frac{m\pi h}{t}\right\}\sin\frac{m\pi y}{t}.$$

$$= (\Phi)_{x=-(h-e)} = -\frac{P}{J}\frac{t}{\pi}\sum_{m=1,2\ldots}^{\infty}\frac{(-1)^m}{m}\left[(h-e)^2 + \right.$$

$$\left. + \frac{\mu}{1+\mu}t^2\left(\frac{2}{m^2\pi^2} - \frac{1}{3}\right)\right]\sin\frac{m\pi y}{t}. \tag{11}$$

From the equation (11) it follows that the coefficients D_m and C_m must satisfy the condition

$$D_m = C_m\, Cth\frac{m\pi h}{t} + \frac{P(-1)^m}{J\pi m}\left[\left(h^2 - 2he + \right.\right.$$

$$\left.\left. + \frac{\mu}{1+\mu}\frac{2t^2}{m^2\pi^2}\right)t - (2eb + b^2)a\right]\frac{1}{Sh\frac{m\pi h}{t}} =$$

$$= C_m\, Cth\frac{m\pi h}{t} + \frac{P}{J}\frac{R_m}{Sh\frac{m\pi h}{t}}. \tag{12}$$

Along the edges 3 – 4 and 5 – 6 the function Φ_1 must satisfy the condition (2). Its value along the line 4 – 5 must coincide with the value of function Φ_2.

The boundary condition along the edges 3 - 4 and 5 – 6 from (2)

$$f_1(y) = \begin{cases} \frac{P}{2J}\left[e^2 y - \frac{\mu}{1+\mu}\frac{y^3}{3} + (2eb + b^2)a\right] & \text{for } -a < y < -t \\ 0 & \text{for } -t < y < +t \\ \frac{P}{2J}\left[e^2 y - \frac{\mu}{1+\mu}\frac{y^3}{3} - (2eb + b^2)a\right] & \text{for } +t < y < +a \end{cases} \tag{13}$$

can be developed in the following Fourier's series

$$f_1(y) = \sum_{n=1,2\ldots}^{\infty}\delta_n \sin\frac{n\pi y}{a} = \sum_{n=1,2\ldots}^{\infty}\frac{P}{J}\left[\frac{e^2 a}{n\pi}\left(\frac{t}{a}\cos\frac{n\pi t}{a} - \right.\right.$$

$$\left. - \frac{1}{n\pi}\sin\frac{n\pi t}{a} - (-1)^n\right] - \frac{\mu}{1+\mu}\frac{1}{3}\left[\frac{t}{n\pi}\left(t^2 - \frac{6a^2}{n^2\pi^2}\right)\cos\frac{n\pi t}{a} - \right.$$

$$\left. - \frac{3a}{n^2\pi^2}\left(t^2 - \frac{2a^2}{n^2\pi^2}\right)\sin\frac{n\pi t}{a} + \frac{a^3}{n\pi}(-1)^n\left(\frac{6}{n^2\pi^2} - 1\right)\right] -$$

$$\left. - (2eb + b^2)\frac{a}{n\pi}\cdot\left[(-1)^n - \cos\frac{n\pi t}{a}\right]\right\}\sin\frac{n\pi y}{a}. \tag{14}$$

The multiplicator of the function Φ_2 from (4)

$$f_2(y) = \begin{cases} 0, & \text{for } -a < y < -t, \\ \sin \dfrac{m\pi y}{t}, & \text{for } -t < y < +t, \\ 0, & \text{for } +t < y < +a, \end{cases} \tag{15}$$

can be developed in Fourier's series

$$f_2(y) = -\frac{(-1)^m 2mt}{a\pi} \sum_{n=1,2\ldots}^{\infty} \frac{\sin \dfrac{n\pi t}{a}}{m^2 - \left(\dfrac{nt}{a}\right)^2} \sin \frac{n\pi y}{a}. \tag{16}$$

Conditions at the boundary 3 – 4 and 5 – 6 from (14) and the coincidence of the functions $(\Phi_1)_{x=e}$ and $(\Phi_2)_{x=e}$ from (16) on the line 4 – 5 gives the following equation

$$(\Phi_1)_{x=e} = \sum_{n=1,2\ldots}^{\infty} \left\{-\frac{a}{\pi}\frac{P}{J}\left[(e+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{n}\right]\frac{(-1)^n}{n} + A_n\right\} \sin \frac{n\pi y}{a} =$$

$$= (\Phi_2)_{x=e} = \sum_{m=1,2\ldots}^{\infty} \left\{-\frac{1}{\pi}\frac{P}{J}\left[e^2 t - \frac{\mu}{1+\mu}\frac{t^3}{3} + (2eb + b^2)\,a\right]\frac{(-1)^m}{m} + C_m\right\} \times$$

$$\times \left[-(-1)^m\right]\frac{2mt}{a\pi} \sum_{n=1,2\ldots}^{\infty} \frac{\sin \dfrac{n\pi t}{a}}{m^2 - \left(\dfrac{nt}{a}\right)^2} \sin \frac{n\pi y}{a} + \sum_{n=1,2\ldots}^{\infty} \delta_n \sin \frac{n\pi y}{a}. \tag{17}$$

It follows from the equation (17) that the coefficients A_n and C_m must satisfy the condition

$$A_n = \frac{P}{J}\left\{\frac{a}{\pi}\left[(e+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{3}\right]\frac{(-1)^n}{n} + \frac{2t}{a\pi^2}\left[e^2 t - \frac{\mu}{1+\mu}\frac{t^3}{3} + \right.\right.$$

$$\left.\left. + (2eb + b^2)\,a\right] \sin \frac{n\pi t}{a} \sum_{m=1,2\ldots}^{\infty} \frac{1}{m^2 - \left(\dfrac{nt}{a}\right)^2}\right\} + \delta_n -$$

$$- \frac{2t}{a\pi} \sin \frac{n\pi t}{a} \sum_{m=1,2\ldots}^{\infty} \frac{(-1)^m m}{m^2 - \left(\dfrac{nt}{a}\right)^2} C_m. \tag{18}$$

Equally on the line 4 – 5, the value of the derivative of Φ_1 must coincide with the value of the derivative of Φ_2. The multiplicator of the function Φ_1 from (3)

$$f_3(y) = \sin\frac{n\pi y}{a}, \quad \text{for } -t < y < +t, \tag{19}$$

can be developed in Fourier's series

$$f_3(y) = -\frac{2\sin\frac{n\pi t}{a}}{\pi} \sum_{m=1,2\ldots}^{\infty} \frac{(-1)^m m}{m^2 - \left(\frac{nt}{a}\right)^2} \sin\frac{m\pi y}{t}. \tag{20}$$

This condition, according to (20), gives the following equation

$$\left(\frac{\partial\Phi_1}{\partial x}\right)_{x=e} = -\sum_{n=1,2\ldots}^{\infty} \frac{n\pi}{a} B_n \frac{2\sin\frac{n\pi t}{a}}{\pi} \sum_{m=1,2\ldots}^{\infty} \frac{(-1)^m m}{m^2 - \left(\frac{nt}{a}\right)^2} \sin\frac{m\pi y}{t} =$$

$$= \left(\frac{\partial\Phi_2}{\partial x}\right)_{x=e} = \sum_{m=1,2\ldots}^{\infty} \frac{m\pi}{t} D_m \sin\frac{m\pi y}{t}. \tag{21}$$

It follows from the equations (12) and (21) that the coefficients C_m and B_n must satisfy the condition

$$C_m = -\frac{P}{J}\frac{R_m}{Ch\frac{m\pi h}{t}} - \frac{2t}{\pi a}(-1)^m Th\frac{m\pi h}{t} \sum_{n=1,2\ldots}^{\infty} \frac{n\sin\frac{n\pi t}{a}}{m^2 - \left(\frac{nt}{a}\right)^2} B_n. \tag{22}$$

Replacing in the expression (22) the index n by i and vice versa, and introducing (22) in (15) and (8), we obtain

$$B_n + \frac{4t^2}{a^2\pi^2}\sin\frac{n\pi t}{a} Cth\frac{n\pi b}{a} \sum_{i=1,2\ldots}^{\infty} i\sin\frac{i\pi t}{a} B_i \times$$

$$\times \sum_{m=1,2\ldots}^{\infty} \frac{m\, Th\frac{m\pi h}{t}}{\left[m^2 - \left(\frac{nt}{a}\right)^2\right]\left[m^2 - \left(\frac{it}{a}\right)^2\right]} =$$

$$= -\frac{P}{J}\left\{\frac{\mu}{1+\mu}\frac{2a^3(-1)^n}{\pi^3 n^3 Sh\frac{n\pi b}{a}} + \frac{a}{\pi}\left[(e+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{3}\right]\frac{(-1)^n}{n}\right.$$

$$Cth\frac{n\pi b}{a}+\frac{2t}{a\pi^2}\left[e^2t-\frac{\mu}{1+\mu}\frac{t^3}{3}+(2eb+b^2)a\right]\sin\frac{n\pi t}{a}\times$$

$$\times Cth\frac{n\pi b}{a}\sum_{m=1,2\ldots}^{\infty}\frac{1}{m^2-\left(\frac{nt}{a}\right)^2}+\frac{2t}{a\pi}\sin\frac{n\pi t}{a}Cth\frac{n\pi b}{a}\times$$

$$\times\sum_{m=1,2\ldots}^{\infty}\frac{(-1)^m m R_m}{\left[m^2-\left(\frac{nt}{a}\right)^2\right]Ch\frac{m\pi h}{t}}\Bigg\}-\delta_n Cth\frac{n\pi b}{a}. \qquad (23)$$

The coefficients B_n can be calculated from this system of linear equations, after wich lequations (8), (22) and (12) determine explicitely the values of A_n, C_m and D_m.

Let us take

$$a=4{,}5\text{ cm};\quad b=2\text{ cm};\quad h=4{,}85\text{ cm}\quad t=1\text{ cm}.$$

The system of linear equations becomes

$$+1{,}01B_1+0{,}04B_2+0{,}08B_3+0{,}10B_4+0{,}11B_5=-7{,}84\frac{P}{J},$$

$$+0{,}02B_1+1{,}07B_2+0{,}13B_3+0{,}17B_4+0{,}17B_5=-2{,}14\frac{P}{J},$$

$$+0{,}02B_1+0{,}08B_2+1{,}16B_3+0{,}21B_4+0{,}22B_5=-0{,}87\frac{P}{J},$$

$$+0{,}02B_1+0{,}08B_2+0{,}16B_3+1{,}22B_4+0{,}23B_5=-0{,}13\frac{P}{J},$$

$$+0{,}02B_1+0{,}07B_2+0{,}13B_3+0{,}19B_4+1{,}22B_5=-0{,}07\frac{P}{J}.$$

Solving it we obtain the values of the coefficients

$$A_1=+7{,}43\frac{P}{J};\qquad B_1=-7{,}66\frac{P}{J};\qquad C_1=-1{,}27\frac{P}{J};\qquad D_1=-1{,}27\frac{P}{J};$$

$$A_2=+1{,}85\frac{P}{J};\qquad B_2=-1{,}89\frac{P}{J};\qquad C_2=+0{,}39\frac{P}{J};\qquad D_2=+0{,}39\frac{P}{J};$$

$$A_3=+0{,}53\frac{P}{J};\qquad B_3=-0{,}53\frac{P}{J};\qquad C_3=-0{,}17\frac{P}{J};\qquad D_3=-0{,}17\frac{P}{J};$$

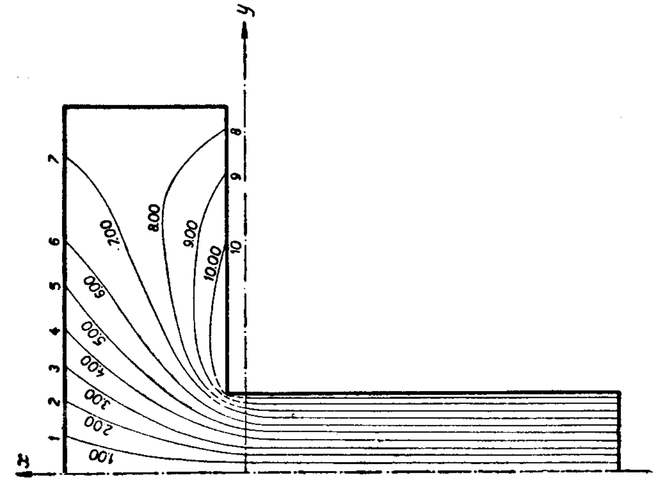

Fig. 2

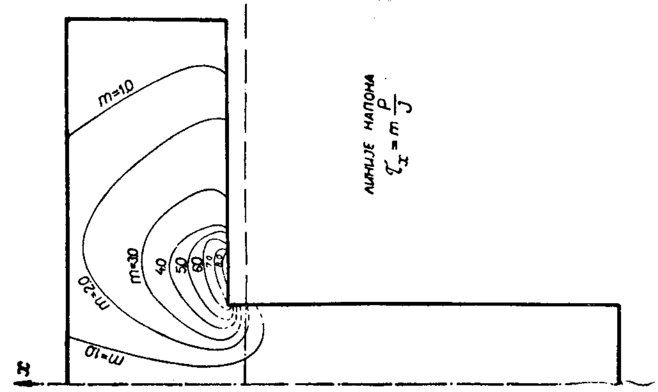

Fig. 3

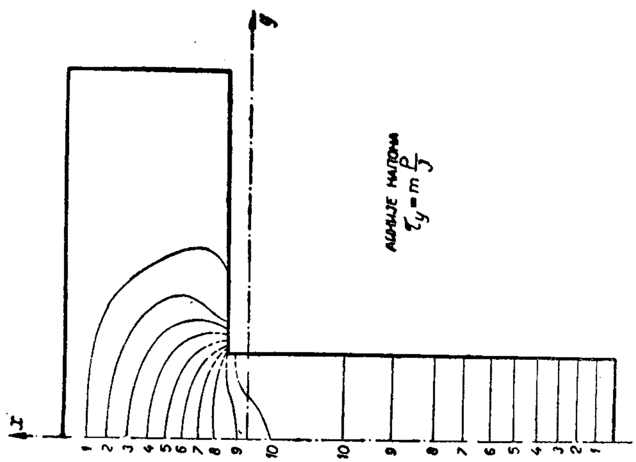

Fig. 4

$$A_4 = -\,0{,}18\,\frac{P}{J}\,; \qquad B_4 = +\,0{,}18\,\frac{P}{J}\,; \qquad C_4 = +\,0{,}09\,\frac{P}{J}\,; \qquad D_4 = +\,0{,}09\,\frac{P}{J}\,;$$

$$A_5 = -\,0{,}30\,\frac{P}{J}\,; \qquad B_5 = +\,0{,}30\,\frac{P}{J}\,; \qquad C_5 = -\,0{,}06\,\frac{P}{J}\,; \qquad D_5 = -\,0{,}06\,\frac{P}{J}\,.$$

The Fig. 2 gives the functions Φ_1 and Φ_2; the Fig. 3 gives the lines of equal shear stress τ_x and the Fig. 4 the lines of equal shear stress τ_y.

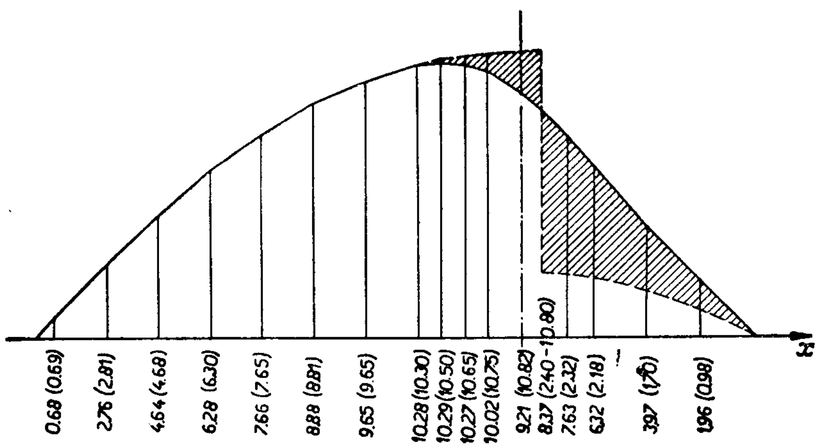

Fig. 5

In Fig. 5 stress values τ_y are compared to values calculated by the usual elementary formula from the Resistence of materials.

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