ON THE APPLICATION OF SUCCESSIVE APPROXIMATIONS TO MOTION STARTED IMPULSIVELY FROM REST IN COMPRESSIBLE MEDIA

by

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SUMMARY — The author proposes the application of the successive approximation procedure to the calculation of the boundary layer growth on a cylinder started impulsively from rest in compressible media. The first two approximations are discussed more in detail. The effect of the temperature is taken into account which results in the necessity of treating the partial differential system consisting of equations of momentum, continuity, state and energy.

INTRODUCTION

Blasius [1] was the first to treat analytically the boundary layer growth on a body started impulsively from rest in an incompressible medium. He calculated the first two approximations to the velocity distribution. The third approximation to the velocity distribution in an incompressible medium was calculated by Goldstein and Rosenhead [4]. Thorough representations of the results of some attacks on this problem can be found in [3, pp. 55—60, 163, 181—187] and in [6, Chapter 11, pp. 167—180].

This note represents an extension of this technique to the more general case where compressibility and heat phenomena and effects are considered. The most general forms of equations of momentum, continuity, state and energy are used. Fundamentally, the used technique is a generalization of the method proposed by Goldstein [2;3], Goldstein and Rosenhead [4] and applied in the past by the author to several problems [5].

The flow of a compressible fluid about a body of cylindrical shape is considered. The radius of the cylinder is large, therefore, the curvature „k" is very small. The cylinder extends to ∞ in the z direction so that the flow is a two dimensional one.

A system of cylindrical polar coordinates is assumed where i_2 denotes the y - direction - this direction is perpendicular to the surface of the cylinder. The x - coordinate is measured along the y equal to a constant — the direction is tangent to this family of lines (circumferential) and i_1 denotes this x direction.

The velocities in the i_1 and i_2 direction are called u and v, respectively.

The surface of the cylinder is assumed to be $y = 0$. At time $t = t_0$ it is assumed that the cylinder begins to move. It moves impulsively from rest, attains the velocity U_0 and thereafter the velocity does not change. The problem is to obtain a method to determine the growth of the boundary layer in any time interval $(t - t_0)$.

1. *BASIC EQUATIONS*

The coefficient of viscocity „μ" and the coefficient of heat conductivity „K" are assumed to be variable functions[1]. The equations of motion, continuity, state, and energy become respectively:

$$(1.1)\qquad \rho\left[\vec{V}_{,t} + \operatorname{grad}\left(\frac{1}{2}\vec{V}^2\right) - \vec{V}\times\vec{\omega}\right] = \rho\vec{F} - \operatorname{grad} p - \frac{2}{3}\operatorname{grad}\mu \operatorname{div}\vec{V}$$

$$+ 2(\operatorname{grad}\mu\cdot\nabla)\vec{V} + \operatorname{grad}\mu\times\vec{\omega} + \frac{1}{3}\mu\operatorname{grad}(\operatorname{div}\vec{V}) + \mu(\nabla^2\vec{V});$$

where $\vec{\omega} = \operatorname{curl}\vec{V}$;

$$(1.2)\qquad \rho_{,t} + \operatorname{div}(\rho\vec{V}) = 0;$$

$$(1.3)\qquad p = R\rho T;$$

$$(1.4)\qquad J c_v \rho(T_{,t} + \vec{V}\cdot\operatorname{grad} T) + p(\operatorname{div}\vec{V}) =$$

$$= J[(\operatorname{grad} K)\cdot(\operatorname{grad} T) + K\operatorname{div}(\operatorname{grad} T)] + \Phi;$$

where Φ, the dissipation function, was written as:

$$(1.5)\qquad \Phi = \mu\left\{2\nabla[(\vec{V}\cdot\nabla)\vec{V}] + \vec{\omega}^2 - 2\vec{V}\cdot\operatorname{grad}(\operatorname{div}\vec{V}) - \frac{2}{3}(\operatorname{div}\vec{V})^2\right\}.$$

It is necessary to express these equations in the system used in this problem. The general transformation formulae to be used are:

$$(1.6)\qquad \operatorname{div}\vec{V} = (e_1 e_2 e_3)^{-1}[(u e_2 e_3)_{,x} + (v e_1 e_3)_{,y} + (w e_1 e_2)_{,z}];$$

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[1] The coefficients of viscosity and heat conducitivity are assumed to be power series expansions in T with properly selected coefficients for convergent series.

$$(1.7)\qquad \operatorname{grad}\Phi = e_1^{-1}\,\Phi_{,x}\,\vec{i}_1 + e_2^{-1}\,\Phi_{,y}\,\vec{i}_2 + e_3^{-1}\,\Phi_{,z}\,\vec{i}_3\,;$$

$$(1.8)\qquad \vec{\omega} = \operatorname{curl}\vec{V} = (e_2\,e_3)^{-1}\,[(e_3\,w)_{,y} - (e_2\,v)_{,z}]\,\vec{i}_1 +$$

$$+ (e_3\,e_1)^{-1}\,[(e_1\,u)_{,z} - (e_3\,w)_{,x}]\,\vec{i}_2 + (e_1\,e_2)^{-1}\,[(e_2\,v)_{,x} - (e_1\,u)_{,y}]\,\vec{i}_3\,;$$

$$(1.9)\qquad \nabla^2\Phi = (e_1\,e_2\,e_3)^{-1}\,[(e_2\,e_3\,e_1^{-1}\,\Phi_{,x})_{,x} + (e_1\,e_3\,e_2^{-1}\,\Phi_{,y})_{,y} + (e_1\,e_2\,e_3^{-1}\,\Phi_{,z})_{,z}].$$

In the coordinate system adopted e_1, e_2, e_3 have the following values:

(1.10) $e_1 = 1 + k\,y$, where k is the curvature of the cylinder,

(1.11) $e_2 = 1$,

and

(1.12) $e_3 = 1$.

In the transformation of these equations to represent the flow in the boundary layer it is assumed that k, k^2, $k\,y$, were of such magnitude that they could be neglected.

2. *TRANSFORMATIONS*

Applying the transformation formulae to the above equations the following results are obtained for the various terms in these equations.

For the equation of motion,

$$(2.1)\qquad \rho\,\vec{V}_{,t} = \rho\,u_{,t}\,\vec{i}_1 + \rho\,v_{,t}\,\vec{i}_2\,;$$

$$(2.2)\qquad \rho\operatorname{grad}\frac{1}{2}\,\vec{V}^2 = \frac{1}{2}\,\rho\operatorname{grad}(u^2 + v^2) =$$

$$= \rho\,(e_1^{-1}\,u\,u_{,x} + e_1^{-1}\,v\,v_{,x})\,\vec{i}_1 + \rho\,(u\,u_{,y} + v\,v_{,y})\,\vec{i}_2\,;$$

$$(2.3)\qquad \vec{V}\times\vec{\omega} = \vec{i}_1\,(v\,\zeta) + \vec{i}_2\,(-\,u\,\zeta)\,,$$

where $\vec{\omega} = \xi\,\vec{i}_1 + \eta\,\vec{i}_2 + \zeta\,\vec{i}_3$ and $\zeta = e_1^{-1}\,v_{,x} - u_{,y} - e_1^{-1}\,k\,u$, thus

$$(2.4)\qquad \vec{V}\times\vec{\omega} = (e_1^{-1}\,v\,v_{,x} - v\,u_{,y} - e_1^{-1}\,k\,u\,v)\,\vec{i}_1 +$$

$$+ (u\,u_{,y} - e_1^{-1}\,u\,v_{,x} + e_1^{-1}\,k\,u^2)\,\vec{i}_2\,;$$

$$(2.5)\qquad \operatorname{grad}p = e_1^{-1}\,p_{,x}\,\vec{i}_1 + p_{,y}\,\vec{i}_2;$$

$$\text{div}\,\vec{V} = e_1^{-1}\,u_{,x} + e_1^{-1}\,e_{1,y}\,v + v_{,y}; \tag{2.6}$$

finally the result:

$$-\frac{2}{3}(e_1^{-2}\,\mu_{,x}\,u_{,x} + e_1^{-2}\mu_{,x}\,k\,v + e_1^{-1}\,\mu_{,x}\,v_{,y} + e_1^{-2}\,\mu\,u_{,xx} + e_1^{-2}\,\mu\,k\,v_{,x} + e_1^{-1}\,\mu\,v_{,xy})\,\vec{i_1} - \frac{2}{3}(e_1^{-1}\,\mu_{,y}\,u_{,x} + e_1^{-1}\,\mu_{,y}\,k\,v + \mu_{,y}\,v_{,y} + e_1^{-1}\,\mu\,u_{,xy} - e_1^{-2}\,\mu\,k\,u_{,x} + e_1^{-1}\,\mu\,k\,v_{,y} - e_1^{-2}\,\mu\,k^2\,v + \mu\,v_{,yy})\,\vec{i_2}; \tag{2.7}$$

$$(2\,\text{grad}\,\mu\cdot\nabla)\,\vec{V} = (2\,e_1^{-1}\,\mu_{,x}\,u_{,x} + 2\,\mu_{,y}\,u_{,y})\,\vec{i_1} + (2\,e_1^{-1}\,\mu_{,x}\,v_{,x} + 2\,\mu_{,y}\,v_{,y})\,\vec{i_2}; \tag{2.8}$$

$$\text{grad}\,\mu\times\vec{\omega} = \mu_{,y}\,(e_1^{-1}\,v_{,x} - u_{,y} - e_1^{-1}\,k\,u)\,\vec{i_1} - \mu_{,x}\,e_1^{-1}\,(e_1^{-1}\,v_{,x} - u_{,y} - e^{-1}\,k\,u)\,\vec{i_2}; \tag{2.9}$$

$$\frac{1}{3}\,\mu\,\text{grad div}\,\vec{V} = \frac{1}{3}\,\mu\,(e_1^{-2}\,u_{,xx} + e_1^{-2}\,k\,v_{,y} + e_1^{-1}\,v_{,xy})\,\vec{i_1} + \frac{1}{3}\,\mu\,(e_1^{-1}\,u_{,xy} - e_1^{-2}\,k\,u_{,x} + e_1^{-1}\,k\,v_{,y} - e_1^{-2}\,k^2\,v + v_{,yy})\,\vec{i_2}; \tag{2.10}$$

$$\mu\,\nabla^2\,\vec{V} = \mu\,(e_1^{-2}\,u_{,xx} + e_1^{-1}\,k\,u_{,y} + u_{,yy})\,\vec{i_1} + \mu\,(e_1^{-2}\,v_{,xx} + e_1^{-1}\,k\,v_{,y} + v_{,yy})\,\vec{i_2}. \tag{2.11}$$

As previously stated it is assumed that k is small since the radius of the cylinder is very large. Furthemore when y is small, the term $k\,y$ becomes very small and is neglected.

Collecting terms in the $\vec{i_1}$ direction, the first equation of motion becomes:

$$\rho\,u_{,t} - \mu\,u_{,yy} = -\,p_{,x} + \frac{4}{3}\,\mu_{,x}\,u_{,x} + \frac{2}{3}\,\mu\,u_{,xx} - \rho\,u\,u_{,x} - \rho\,v\,u_{,y} - \frac{1}{3}\,\mu\,v_{,xy} + \mu_{,y}\,u_{,y} - \frac{2}{3}\,\mu_{,x}\,v_{,y} + \mu_{,y}\,v_{,x}. \tag{2.12}$$

Equation (2.12) was derived with the assumption that k is small and $k\,y \cong 0$.

By a similar procedure the second equation of motion in the $\vec{i}_2$ direction:

$$\rho v_{,t} - \mu v_{,yy} = - p_{,y} \rho v v_{,y} - \rho u v_{,x} - u v_{,xx} + \mu_{,x} v_{,x} \tag{2.13}$$

$$- \frac{1}{3} \mu u_{xy} - \frac{2}{3} \mu_{,y} u_{,x} + \mu_{,x} u_{,y} + \frac{4}{3} \mu_{,y} v_{,y} - \frac{1}{3} \mu v_{,yy}.$$

The energy equation is transformed in a similar manner.

For the energy equation the terms are as follows:

$$T_{,t} = T_{,t}; \tag{2.14}$$

$$\vec{V} \cdot \text{grad } T = (u \vec{i}_1 + v \vec{i}_2) \cdot (e_1^{-1} T_{,x} \vec{i}_1 + T_{,y} \vec{i}_2) = \tag{2.15}$$

$$= e_1^{-1} u T_{,x} + v T_{,y};$$

$$p \text{ div } \vec{V} = e_1^{-1} p (u_{,x} + k v) + p v_{,y}; \tag{2.16}$$

$$\text{grad } K \cdot \text{grad } T = e_1^{-2} K_{,x} T_{,x} + K_{,y} T_{,y}; \tag{2.17}$$

$$\text{div grad } T = e_1^{-2} T_{,xx} + e_1^{-1} k T_{,y} + T_{,yy}; \tag{2.18}$$

$$\nabla [(\vec{V} \cdot \nabla) \vec{V}] = (u_{,x})^2 + u u_{,xx} + v_{,x} u_{,y} + v u_{,xy} \tag{2.19}$$

$$+ u_{,y} v_{,x} + u v_{,xy} + (v_{,y})^2 + v v_{,yy};$$

$$\vec{V} \text{ grad div } \vec{V} = u u_{,xx} + u v_{,xy} + v u_{,xy} + v v_{,yy}; \tag{2.20}$$

$$\vec{\omega}^2 = (v_{,x})^2 - 2 u_{,y} v_{,x} + (u_{,y})^2; \tag{2.21}$$

$$(\text{div } \vec{V})^2 = (u_{,x})^2 + 2 u_{,x} v_{,y} + (v_{,y})^2. \tag{2.22}$$

Collecting terms and making the usual assumptions concerning k, the energy equation becomes:

$$J c_v \rho T_{,t} - J K T_{,yy} = - J c_v \rho (u T_{,x} + v T_{,y}) - R \rho T (u_{,x} + v_{,y}) \tag{2.23}$$

$$+ J (K_{,x} T_{,x} + K_{,y} T_{,y} + K T_{,xx}) + \Phi,$$

where $\Phi = \mu \left\{ \frac{4}{3} [(u_{,x})^2 - u_{,x} v_{,y} + (v_{,y})^2] + (v_{,x} + u_{,y})^2 \right\}.$

For incompressible flow, the dissipation function, reduces to:

$$\Phi = \mu \{2 [(u_{,x})^2 + (v_{,y})^2] + (u_{,y} + v_{,x})^2. \tag{2.24}$$

For this type of flow the term $v_{,x}$ is of O (δ) and therefore is neglected. The final equation is

$$\Phi = \mu \left\{ \frac{4}{3} [u_{,x} (u_{,x} - v_{,y}) + (v_{,y})^2] + 2 u_{,y} v_{,x} + (u_{,y})^2 \right\}. \tag{2.25}$$

3. *FIRST APPROXIMATIONS*

The following method will be used to solve the set of differential equations. The coefficients of the left hand side of equations of motion and the equation of energy will be assumed constant so as to obtain ordinary differential equations of the second order. On the right hand side the results from preceding approximations will be used to obtain higher approximations.

3.1. *First Approximation to u*

In the first approximation to velocity component in the $\vec{i}_1$ direction the following assumptions are made concerning the flow:

(3.1.1) $$p = const.; \quad \mu = \mu_\infty; \quad v = 0; \quad p_{,x} = 0; \quad u_{,x} = u_{,xx} = 0;$$

$$\rho u_{,t} = \rho_\infty u_{1,t}; \quad \mu u_{,yy} = \mu_\infty u_{1,yy}.$$

In this approximation only terms $0(t^{-1})$ are retained. Having made the substitutions noted in (3.1.1) the following equation is obtained:

(3.1.2) $$\rho_\infty u_{1,t} - \mu_\infty u_{1,yy} = 0.$$

The solution of this equation is given by Blasius, as

(3.1.3) $$u_1 = U_0 \operatorname{erf} N = U_0 F_1(N), \; i.\,e., \; \operatorname{erf} N = F_1(N).$$

The error function erf N is given its usual meaning:

(3.1.4) $$\operatorname{erf} N = 2\pi^{-1/2} \int_0^N \exp(-t^2)\, dt.$$

N is defined as a dimensionless quantity and is written as:

(3.1.5) $$N = [2 (\nu_\infty t)^{1/2}]^{-1} y.$$

Thus N is dimensionless.

„U_0" is the function of x only which gives the velocity distribution outside the boundary layer. It may be assumed to be the function that igves the velocity distribution in an irrotational motion. In applications of this method $U_0 = 2\,V \sin(x R^{-1})$ might be used (V is the free stream velocity – R is the radius of the cylinder).

The boundary conditions require that u equals 0 when N equals zero or y equals zero. Thus:

(3.1.6) $$u|_0 = u_1|_0 = U_0 \operatorname{erf} N|_{N=0} = 0\,.$$

The boundary condition at ∞ requires that u is equal to U_0 at $N \to \infty$. Since $\operatorname{erf} N|_{N\to\infty} = 1$, then the boundary condition is fulfilled and u is equal to U_∞ at $N \to \infty$.

3.2. *First Approximation to v*

In the initial approximation, an incompressible flow is assumed. Thus for the initial approximation to v it is sufficient to satisfy the incompressible equation of continuity:

(3.2.1) $$u_{,x} + v_{,y} = 0\,.$$

Thus on the left hand side of this continuity equation the substitution $u = u_1$, $v = v_1$, is made. Therefore:

(3.2.2) $$u_{1,x} + v_{1,y} = 0\,.$$

The solution of this equation is:

(3.2.3) $$v_1 = -2\,(\nu_\infty t)^{1/2}\,U_0'\,\{N \operatorname{erf} N - \pi^{-1/2}[1 - \exp(-N^2)]\} = \\ = -t^{1/2}\nu_\infty^{1/2}\,U_0'\,F_2(N)\,.$$

The boundary condition at y is equal to zero is satisfied since v_1 is equal to zero at N equals zero. However, for $N \to \infty$, the boundary condition is not satisfied since v_1 is not equal to zero at $N \to \infty$. However, at moderate values of N for which u is approximately equal to U_0, v is of the order $2(\nu_\infty t)^{1/2} U_0' N$ [B l a s i u s, p. 20—37].

3.3. *First Approximation to Density*

The continuity equation for compressible flow is written:

(3.3.1) $$\rho_{,t} + \operatorname{div}(\rho \vec{V}) = 0\,,$$

and this equation is transformed to the new system of coordinates. Thus by equation (1.6):

(3.3.2) $$\rho_{,t} + e_1^{-1}\,[(\rho\, u)_{,x} + (e_1\, \rho\, v)_{,y}] = 0\,,$$

or

(3.3.3) $$\rho_{,t} + e_1^{-1}\,\rho_{,x}\,u + e_1^{-1}\,\rho\,u_{,x} + \rho_{,y}\,v + \rho\,v_{,y} = 0\,.$$

Finally the equation of continuity is written and is used in the form:

(3.3.4) $$\rho_{,t} = -\,\rho\,(u_{,x} + v_{,y}) - u\,\rho_{,x} - v\,\rho_{,y}.$$

For the first approximation to density ρ equal ρ_∞ is substituted on the right hand side and $\rho = \rho_1$, on the left hand side of (3.3.4).

The velocity functions u and v assume their forms given by u_1 and v_1. With these substitutions equation (3.3.4) reduces to:

(3.3.5) $$\rho_{1,t} = \rho_\infty\,(u_{1,x} + v_{1,y}) - u_1\,\rho_{\infty,x} - v_1\,\rho_{\infty,y} = 0\,;$$
$$\rho_1 = \text{const.} = 0\,.$$

The final results will be presented in the form of a series. Thus $\rho = \rho_\infty + \sum_{i=1}^{i=n} \rho_i$, where each ρ_i is a successive approximation. The first approximation to density is represented by $\rho = \rho_\infty + \rho_1$, therefore $\rho = \rho_\infty$ everywhere. This result, that the density is constant, was expected since the incompressible continuity equation was used to determine u_1 and v_1.

3.4. *First Approximation to Temperature Distribution*

The temperature terms are determined from the energy equation. The temperature is expanded in the same form as density, i. e.,

(3.4.1) $$T = T_s + \sum_{i=1}^{n} T_i\,,$$

where T_s is the temperature on the surface of the body.

The energy equation is written:

(3.4.2) $$J c_v\,\rho\,T_{,t} - J K\,T_{,yy} = -\,J c_v\,\rho\,(u\,T_{,x} + v\,T_{,y})$$
$$-\,R\,\rho\,T\,(u_{,x} + v_{,y}) + J\,(K_{,x}\,T_{,x} + K_{,y}\,T_{,y} + K\,T_{,xx}) + \Phi\,,$$

where Φ is given in the simpler form:

(3.4.3) $$\Phi = \mu\left[\frac{4}{3}\,(v_{,y})^2 + 2\,v_{,x}\,u_{,y} + (u_{,y})^2\right].$$

On the right hand side of equation the quantities

$$u = u_1\,;\quad v = v_1\,;\quad \rho = \rho_\infty + \rho_1 = \rho_\infty\,;\quad T = T_\infty\,;\quad K = K_\infty\,;\quad \mu = \mu_\infty\,,$$

are substituted. On the left hand side the substitutions:

(3.4.4) $$T = T_s + T_1\,;\quad K = K_\infty\,;\quad \rho = \rho_\infty\,,$$

were made so that the equation could be reduced to ordinary differential equation.

In this first approximation only terms of the order (t^{-1}) are retained*. Equation (3.4.3) reduces to:

$$J \mu_{\infty}^{-1} (c_v \rho_{\infty} T_{1,t} - K_{\infty} T_{1,yy}) = \frac{4}{3} (v_{1,y})^2 + 2 v_{1,x} u_{1,y} + (u_{1,y})^2. \tag{3.4.5}$$

The constant coeffficients on the left hand side of (3.4.5) linearize the partial differential equation. Retaining only the terms of order (t^{-1}) equation (3.4.5) becomes:

$$J \mu_{\infty}^{-1} (c_v \rho_{\infty} T_{1,t} - K_{\infty} T_{1,yy}) = (u_{1,y})^2 = U_0^2 (\nu_{\infty} t)^{-1} G_1 (N). \tag{3.4.6}$$

$T_1 = \overline{T}_1$ is substituted for T_1 and $\overline{T}_1$ is made a function of a new variable $\overline{N}$. The following substitution is made:

$$\overline{T}_1 = T_{\infty} A_1 U_0^2 \overline{F}_3 (\overline{N}). \tag{3.4.7}$$

„A_1" has units $t^2 l^{-2}$ and $\overline{N}$ is defined as:

$$\overline{N} = A_2 N. \tag{3.4.8}$$

„A_2" is a dimensionless number. The constants A_1 and A_2 have the following expression:

$$A_1 = (J c_v T_{\infty})^{-1}; \quad A_2 = (\nu_{\infty} c_v \rho_{\infty} K_{\infty}^{-1})^{1/2}. \tag{3.4.9}$$

The expression for $\overline{T}_1$ equation (3.4.7), is substituted into equation (3.4.6). In what follows is to be understood that the functions F, G, H, I, etc., are functions of N and $\overline{F}$, $\overline{G}$, $\overline{H}$, $\overline{I}$, etc., are functions of $\overline{N}$.

Thus,

$$\overline{T}_{1,t} = A_1 U_0^2 \overline{F}_3' \overline{N}_{,t}; \tag{3.4.10}$$

$$\overline{T}_{1,yy} = A_1 U_0^2 \overline{F}_3'' (\overline{N}_{,y})^2, \tag{3.4.11}$$

are substituted into (3.4.6) and the result is:

$$U_0^2 (\overline{F}_3'' + 2 \overline{N} \overline{F}_3') = U_0^2 \overline{\overline{G}}_1; \quad \overline{\overline{G}}_1 = - 4 \overline{G}_1. \tag{3.4.12}$$

The term $\overline{G}_1$ is G_1 with the substitution $\overline{N} = A_2 N$. The differential equation:

$$\overline{F}_3'' + 2 \overline{N} \overline{F}_3' = \overline{\overline{G}}_1, \tag{3.4.13}$$

determines the function $\overline{F}_3$.

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*) The order of t preserved is arbitrary, but one must be consistent.

Any constant is a solution of the complementary equation of (3.4.13). The second complementary solution is $C_1 = \int_0^{\overline{N}} \exp(-z^2)\, dz = C_1 \operatorname{erf} \overline{N}$. Reducing the order of (3.4.13) the equation becomes:

(3.4.14) $$\overline{f}' + 2\overline{N}\overline{f} = \overline{\overline{G}}_1,$$

where

(3.4.14a) $$\overline{f} = \overline{F}_3' \quad \text{and} \quad \overline{f}' = \overline{F}_3''.$$

Equation (3.4.14) has the solution:

(3.4.15) $$f = [\int \overline{\overline{G}}_1 (\exp \int 2\overline{N}\, d\overline{N})\, d\overline{N}] \exp(-\int 2\overline{N}\, d\overline{N}) = \\ = [\int \overline{\overline{G}}_1 (\exp \overline{N}^2)\, d\overline{N}] (\exp - \overline{N}^2).$$

Therefore the function $\overline{F}_{3p}$ is given by:

(3.4.15a) $$\overline{F}_{3p} = \int_0^{\overline{N}} [\int \overline{\overline{G}}_1 (\exp \overline{N}^2)\, d\overline{N}] (\exp - \overline{N}^2)\, d\overline{N}.$$

$\overline{F}_{3p}$ is the particular integral in this case. The solution of equation (3.4.13) is given by the particular integral and the two complementary solutions. Thus the general solution is:

(3.4.16) $$\overline{F}_3 = \overline{F}_{3p} + C_1 \operatorname{erf} \overline{N} + C_2,$$

where C_1 and C_2 are determined by the boundary conditions on T_1. The boundary conditions on T are as follows:

(3.4.17) $$y = 0: \quad N = \overline{N} = 0; \quad T = T_s;$$
$$y \to \infty: \quad N = \overline{N} \to \infty; \quad T = T_\infty.$$

Temperature is represented by the binomial $T = T_s + T_1$. The function equal to $\overline{T}_1$ is substituted and the result is:

(3.4.18) $$T = T_s + T_\infty A_1 U_0^2 \overline{F}_3 = T_s + T_\infty A_1 U_0^2 (\overline{F}_{3p} + C_1 \operatorname{erf} \overline{N} + C_2).$$

When $y = 0$, $N = \overline{N} = 0$, and F_{3p} is equal to zero. The value of T becomes:

(3.4.19) $$T = T_s + T_\infty A_1 U_0^2 C_2,$$

and it is evident that C_2 is equal to zero and T is equal to T_s. When y approaches infinity, N and $\overline{N}$ approach infinity, equation (3.4.18) becomes:

(3.4.20) $$T = T_s + T_\infty A_1 U_0^2 (\overline{F}_{3p} + C_1) \,|_{\overline{N} \to \infty}.$$

It is assumed that the integral representing $\bar{F}_{sp}$ has a finite value for the limit ∞. Briefly, we restrict the calculation to large values of y, excluding $y = \infty$, if this leads to indefinite integrals. The second boundary condition is satisfied by determining C_1 so that $T = T_\infty = T_s + T_\infty =$ constant.

3.5. *First Approximation to Viscosity and K*

Viscosity is represented by a power series in T_1 where the coefficients are chosen so that the series is convergent. Thus,

$$\mu = \bar{\mu} \sum_{i=0}^{i=n} A_i T^i, \quad A_i = a, b, c, \ldots, i = 1, 2, \ldots, \tag{3.5.1}$$

where the A_i' s are constants having dimensions 1/degree, 1/(degree)^2, etc.,. Then μ_1 becomes:

$$\mu_1 = \bar{\mu} [1 + a (T_s + T_1) + b (T_s + T_1)^2 + \ldots], \tag{3.5.2}$$

and

$$\mu_{,T} = \bar{\mu} [a + 2b (T_s + T_1) + 3c (T_s + T_1)^2 + \ldots]. \tag{3.5.3}$$

Assuming K to be expressed by $K = K_0 c_p \mu$, then it is possible to write:

$$K_1 = K_0 c_p \mu_1, \tag{3.5.4}$$

as the first approximation to the coefficient of heat conductivity.

4. *SECOND APPROXIMATIONS*

4.1. *Second Approximation to u*

For the second approximation the values determined in the first approximation are substituted into the right hand side of equation (2.12). Thus the substitutions $\rho = \rho_\infty$, $u = u_1$, $\mu = \mu_1$, $v = v_1$, are made on the right hand side. On the left hand side of equation (2.12) the substitutions $\mu = \mu_1$, $u = u_1 + u_2$, $\rho = \rho_\infty$, are made. The coefficients on the left hand side are made constant so that the resulting equation is linear. Thus equation (2.12) becomes:

$$\begin{aligned} \rho_\infty u_{1,t} - \mu_1 u_{1,yy} + \rho_\infty u_{2,t} - \mu_\infty u_{2,yy} = &- R [\rho_\infty (T_s + T_1)]_{,x} \\ &+ \frac{4}{3} \mu_{1,x} u_{1,x} + \frac{2}{3} \mu_1 u_{1,xx} + \mu_{1,y} v_{1,x} - \rho_\infty u_1 u_{1,x} \\ &- \rho_\infty v_1 u_{1,y} - \frac{1}{3} \mu_1 v_{1,xy} + \mu_{1,y} u_{1,y} - \frac{2}{3} \mu_{1,x} v_{1,y}. \end{aligned} \tag{4.1.1}$$

The final form is:

(4.1.2) $$u_{2,t} - \nu_\infty u_{2,yy} = - u_{1,t} + \rho_\infty^{-1} (\mu_1 u_{1,y})_{,y} - R T_{1,x} +$$
$$+ 2 \rho_\infty^{-1} \mu_{1,x} u_{1,x} + \rho_\infty^{-1} (\mu_1 u_{1,xx} + \mu_{1,y} v_{1,x}) - u_1 u_{1,x} - v_1 u_{1,y}.$$

In this approximation the terms of order (t^0) are preserved. The right hand side of equation (4.1.2) contains terms composed of the product of three quantities. The typical terms is composed of a first factor $A^l_{jkl,}$ which is composed of physical constants or properties of the fluid. The second factor in the product is one composed of the velocity U_0, powers of this velocity, and powers of the derivatives of this velocity. The extent to which these factors (U_0, U_0', U_0'', etc.) enter would be determined by the problem at hand. The last factor in the product is some function of N. Thus using a summation convention on repeated scripts, the most general form of such equation as is described above is:

(4.1.3) $$u_{2,t} - \nu_\infty u_{2,yy} = - \frac{1}{4} A^j_{rsq..} U_0^r U_0'^s U_0''^q \cdots G_j.$$

The scripts act as exponents and indicate powers of the velocity terms. The function

(4.1.4) $$u_2 = A_4^p \, t \, U_0 U_0' F_{21\,p}(N),$$

is introduced. The factor A_4^p is dimensionless. F_{21} indicates the second approximation to the velocity component in the $\vec{i}_1$ or $\vec{i}$ direction, i. e. the tangent direction.

Substituting $u_{2,t}$ and $u_{2,yy}$ into equation (4.1.3) it is found that:

(4.1.5) $$A_4^p [U_0 U_0' (F_{21\,p} - \frac{1}{2} N F'_{21p}) - \frac{1}{4} U_0 U_0' F''_{21p}] =$$
$$= - \frac{1}{4} A^j_{rsq\ldots} U_0^r U_0'^s U_0''^q \cdots G_j,$$

or

(4.1.5a) $$U_0 U_0' A_4^p (F''_{21p} + 2 N F'_{21p} - 4 F_{21\,p}) =$$
$$= A^j_{rsq\ldots} U_0^r U_0'^s U_0''^q \cdots G_j.$$

The method of finding the general solution of (4.1.5*a*) is to find the particular integrals of each term on the right hand side. Terms of like coefficients are grouped and particular integrals found for each term with a distinct coefficient. Two complementary solutions of (4.1.5*a*) are known and consequently the general solution can be obtained. Since equation (4.1.5a) is linear, solutions can be added. As an illustration consider:

$$(4.1.6) \qquad U_0 U_0' A_4^{p1} (F_{21p1}'' + 2 N F_{21p1}' - 4 F_{21p1}) = A_{(r1)(s1)(q1)}^{j1} U_0^{(r1)} U_0'^{(s1)} U_0''^{(q1)} \ldots\ldots G_{j1},$$

considered as a typical term. The factor A_4^{p1} is determined uniquely as:

$$(4.1.7) \qquad A_4^{p1} = A_{(r1)(s1)(q1)}^{j1} U_0^{(r1)-1} U_0'^{(s1)-1} U_0''^{q1)} \ldots;$$

thus the equation becomes:

$$(4.1.8) \qquad F_{21p1}'' + 2 N F_{21p1}' - 4 F_{21p1} = G_{j1}.$$

The particular integral of this equation is written as:

$$F_{21p1} = \left\{ \int_0^N [\int G_{j1} (2N^2+1)(\exp N^2) dN] (\exp - N^2)(2N^2+1)^{-1} dN \right\} (2N^2+1).$$

The complementary solutions are:

$$F_{21p1} = 2N^2 + 1;$$

$$F_{21p1} = \frac{1}{2} \pi^{-1/2} (2N^2+1) \text{ erf } N + \pi^{-1} N (\exp - N^2).$$

For the particular term G_{j_1} then the solution is:

$$(4.1.9) \quad F_{21p1} = \left\{ \int_0^N [\int G_{j1} (2N^2+1)(\exp N^2) dN] (\exp - N^2)(2N^2+1) dN \right\} \times$$

$$\times (2N^2+1) + C_1 (2N^2+1) + C_2 \left\{ \frac{1}{2} \pi^{-1/2} (2N^2+1) \text{ erf N} + \pi^{-1} N (\exp - N^2) \right\}.$$

The second approximation to velocity is written as follows:

$$(4.1.10) \qquad u_2 = (t U_0 U_0') (A_4^{p1} F_{21p1} + A_4^{p2} F_{21p2} + \ldots + A_4^{pn} F_{21pn}).$$

The functions F_{21pi} are solutions of the set of differential equations:

$$(4.1.11) \qquad F_{21p1}'' + 2 N F_{21p1}' - 4 F_{21p1} = G_{j1};$$

. . . .

. . . .

. . . .

$$F_{21pn}'' + 2 N F_{21pn}' - 4 F_{21pn} = G_{jn}.$$

The following factors A_4^{p1}, $A_4^{p2}, \ldots, A_4^{pn}$, are uniquely determined by:

$$A_4^{p1} = A_{rsq}^{j1} \, U_0^r \, U_0'^s \, U_0''^q \ldots ,$$
$$\cdot \quad \cdot \quad \cdot \quad \ldots ,$$
$$\cdot \quad \cdot \quad \cdot \quad \ldots ,$$
$$A_4^{pn} = A_{rsq}^{jn} \, U_0^r \, U_0'^s \, U_0''^q \ldots .$$

All A_4^{pl} are dimensionless and they are composed of physical constants and derivatives of the velocities etc.

The boundary conditions are the same, that is, when N is equal to zero, u must be equal to zero. When N approaches infinity, u must be equal to U_0.

Since u is equal to $u_1 + u_2$, then when y is equal to zero N is equal to zero and it is necessary that u equals zero. From the first approximation at y equal to zero u_1 is equal to zero; therefore, for y equal to zero, u_2 must equal zero, and each $F_{21\,pi}$ must be equal to zero. For equation (4.1.9) when N is equal to zero:

(4.1.12) $$F_{21\,pi} = C_1 \, (2\, N^2 + 1),$$

therefore the constant $C_1 = 0$.

When N approaches infinity it is necessary to assume that the integral in $F_{21\,pi}$ remained finite and in such a case the expression for $F_{21\,pi}$ becomes:

(4.1.13) $$F_{21\,pi} \,|_{\sim\infty} \cong \left\{ \int_0^N [\ldots] \right\} (2\, N^2 + 1) + C_2 \, \frac{1}{2} \, \pi^{1/2} (2\, N^2 + 1) \,|_{\sim\infty} \cong 0 .$$

C_2 is determined from this expression since it is necessary that (4.1.13) equals zero in order that u is equal U_0 when N approaches infinity.

The second approximation to velocity becomes:

(4.1.14) $$u = U_0 \text{ erf } N - t \, A_4^p \, U_0 \, U_0' \, F_{21\,p} .$$

4.2. *Second approximation to v*

For the second approximation to v the procedure is the same as with second approximation to u. The following quantities are substituted into the equation of motion normal to the surface (equation (2.13)): on the right hand side the quantities:

(4.2.1) $$\rho = \rho_\infty ; \quad \mu = \mu_1 ;$$

(4.2.2) $$u = u_1 + u_2 ; \quad v = v_1 ; \quad T = T_s + T_1 ; \quad K = K_1 ,$$

are substituted. On the left hand side the quantities:

(4.2.3) $$\nu = \nu_1 + \nu_2; \quad \rho = \rho_\infty; \quad \mu = \mu_1,$$

are substituted. Equation (2.13) becomes:

(4.2.4) $$\rho_\infty \nu_{2,t} - \mu_\infty \nu_{2,yy} = - \rho_\infty \nu_{1,t} + \mu_1 \nu_{1,yy} - R \rho_\infty T_{1,y} - \rho_\infty (\nu_1 \nu_{1,y} + u_1 \nu_{1,x} + u_2 \nu_{1,x}) + \mu_1 \nu_{1,xx} - \frac{1}{3} \mu_1 u_{2,xy} - \frac{2}{3} \mu_{1,y} u_{2,x} + \mu_{1,x} u_{2,y} + \mu_{1,y} \nu_{1,y}.$$

This equation is of the same form as equation (4.1.3) and is represented by the expression:

(4.2.5) $$\nu_{2,t} - \nu_\infty \nu_{2,yy} = B^j_{rsq} U_0^r U_0'^s U_0''^q \cdots H_j.$$

In equation (4.2.5) terms of the order (t^0) are preserved.

The procedure is identical as with u_2 and so the function:

(4.2.6) $$\nu_2 = t U_0 U_0' A_5^p F_{22p},$$

is defined. This function is substituted into equation (4.2.5) and the differential equation:

(4.2.7) $$A_5^p (F''_{22p} + 2 N F'_{22p} - 4 F_{22p}) = B^j_{rsq} U_0^{r-1} U_0'^{s-1} U_0''^q \ldots H_j,$$

is obtained. Again the solution of a set of differential equations:

(4.2.8) $$\begin{array}{l} F''_{22p1} + 2 N F'_{22p1} - 4 F_{22p1} = H_{j1}; \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ F''_{22pn} + 2 N F'_{22pn} - 4 F_{22pn} = H_{jn}, \end{array}$$

determines the functions F_{22pi}. The factors A_5^p are determined uniquely by the system:

(4.2.9) $$\begin{array}{l} A_5^{p1} = B^{j1}_{rsq} U_0^{r-1} U_0'^{s-1} U_0''^q \ldots, \\ \quad \vdots \qquad \vdots \qquad \qquad \ldots, \\ \quad \vdots \qquad \vdots \qquad \qquad \ldots; \\ A_5^{pn} = B^{j2}_{rsq} U_0^{r-1} U_0'^{s-1} U_0''^q \ldots. \end{array}$$

The expression for ν_2 is identical with (4.1.10) and is written as:

(4.2.10) $$\nu_2 = t U_0 U_0' (A_5^{p1} F_{22p1} + A_5^{p2} F_{22p2} + \ldots A_5^{pn} F_{22pn}).$$

By the identical line of reasoning used for u_2 the boundary conditions for ν_2 are satisfied at N is equal to zero, but are not satisfied as N approaches infinity.

The entire solution for v is written as:

(4.2.11) $$v = -2(\nu_\infty t)^{1/2} U_0' \{N \operatorname{erf} N - \pi^{-1/2}[1 - (\exp - N^2)]\} + + t U_0 U_0' A_5^p F_{22p} .$$

4.3. *Second Approximation to Density*

The continuity equation is written in the form given by equation (3.3.4). On the right hand side of this equation the following substitutions are made:

(4.3.1) $$\rho = \rho_\infty + \rho_1; \quad u = u_1 + u_2; \quad v = v_1 + v_2 .$$

And on the left hand side $\rho_{,t} = \rho_{2,t}$. Preserving terms of the order (t^0) the result is:

(4.3.2) $$\rho_{2,t} = 0, \quad \text{or} \quad \rho_2 = 0.$$

Since it is not necessary to assume integer powers of t, fractional powers such as $t^{1/2}$, $t^{3/2}$, etc., could have been used. As an illustration assume that $t^{1/2}$ terms are preserved.

Equation (3.3.4) with the substitutions (4.3.1)) becomes:

(4.3.3) $$\rho_{2,t} = -\rho_\infty v_{2,y} = -\rho_\infty (t U_0 U_0' A_5^p F_{22p}') (2 \nu_\infty^{1/2} t^{1/2})^{-1} .$$

The function ρ_2 is defined as:

(4.3.4) $$\rho_2 = t^{3/2} U_0'^{3/2} \rho_\infty A_6^p F_{23p},$$

(4.3.5) $$\rho_{2,t} = \frac{3}{2} t^{1/2} U_0'^{3/2} \rho_\infty A_6^p F_{23p} - t^{3/2} U_0'^{3/2} \rho_\infty A_6^p \cdot \frac{1}{2} t^{-1} N F_{23p}' .$$

Substitute (4.3.5) into (4.3.3) and the differential equation becomes:

(4.3.6) $$A_6^p(-N F_{23p}' + 3 F_{23p}) = (U_0 U_0')^{-1/2} A_5^p F_{22p} .$$

This equation leads to a set of solutions of the form:

(4.3.7) $$F_{23pi} = \left(\int_0^N N^{-3} F_{22pi}\, dN + C_1\right) N^3,$$

and a set of constants determined uniquely by:

(4.3.8) $$A_6^{pi} = (U_0 U_0')^{-1/2} \nu_\infty^{-1/2} A_5^{pi}.$$

When N is equal to zero the boundary conditions require that C_1 equals zero. It is not likely that the boundary condition as N approaches infinity can be fulfilled since there is only one arbitrary constant.

If one preserves terms of order ($t^{1/2}$, $t^{3/2}$, etc.) one must be consistent and preserve the same order terms in all equations. In this approximation (t^0) is retained therefore:

(4.3.9) $$\rho = \rho_\infty + \rho_1 + \rho_2 = \rho_\infty .$$

4.4, *Second Approximation to Temperature*

The energy equation (2.23) and the general expression for Φ is used to determine T. On the left hand side of (2.23) $\rho = \rho_\infty$, $T = T_s + T_1 + T_2$, $K = K_1$, are substituted. And on the right hand side the substitutions $\rho = \rho_\infty$, $u = u_1 + u_2$, $v = v_1 + v_2$, $T = T_s + T_1$, $\mu = \mu_1$, $K = K_1$, are made. The terms of order (t^0) are preserved and the resulting expression is of the form:

(4.4.1) $$J\, c_v\, \rho_\infty\, T_{2,t} - J\, K_\infty\, T_{2,yy} = -\frac{1}{4} D^j_{rsq\ldots}\, U^r_0\, U'^s_0\, U''^q_0 \ldots \overline{I}_j .$$

The D's are composed of physical constants, U^r_0, U'^s_0, U''^q_0, . . . , are velocity terms, and I's are functions of N only. The following function is defined:

(4.4.2.) $$\overline{T}_2 = T_\infty\, A_8\, A^p_7\, t\, U'^2_0\, \overline{F}_{24\,p} ,$$

(4.4.3) $$A_8 = (J\, c_v\, \rho_\infty\, T_\infty)^{-1} .$$

Equation (4.4.2) is substituted into (4.4.1) and the result is:

(4.4.4) $$A^p_7\, (U'_0)^2\, (\overline{F}''_{24\,p} + 2\, \overline{N}\, \overline{F}'_{24\,p} - 4\, \overline{F}_{24\,p}) = D^j_{rsq\ldots}\, U^r_0\, U'^s_0\, U''^q_0 \ldots \overline{I}_j .$$

The functions $\overline{F}_{24\,p_i}$ are determined from the equations:

(4.4.5) $$\overline{F}''_{24\,pi} + 2\, \overline{N}\, \overline{F}_{24\,pi} - 4\, \overline{F}_{24\,pi} = \overline{I}_{ji} \qquad (i = 1, \ldots, n),$$

and the constants are determined uniquely from the equations:

(4.4.6) $$A^{pi}_7 = D^{jl}_{rsq\ldots}\, U^r_0\, U'^{s-2}_0\, U''^q_0 \cdots \qquad (i = 1, \ldots, n).$$

Equation (4.4.5) is a linear differential equation and its solution is of the form:

(4.4.7) $$\overline{F}_{24\,pi} = \left\{ \int_0^{\overline{N}} \left[\int \overline{I}_{ji}\, (2\, \overline{N}^2 + 1)\, (\exp - \overline{N}^2)\, (2\, \overline{N}^2 + 1)^{-1}\, d\overline{N} \right\} (2\, \overline{N}^2 + 1) +$$
$$+ C_1\, (2\, \overline{N}^2 + 1) + C_2\, [(2\, \pi^{1/2})^{-1}\, (2\, \overline{N}^2 + 1)\, \mathrm{erf}\, \overline{N} + \pi^{-1}\, \overline{N}\, (\exp - \overline{N}^2)].$$

The boundary conditions for this type of function were discussed previously for u_2. For $\overline{N}$ equal to zero it is necessary that C_1 equal zero. When $\overline{N}$ approaches infinity the boundary condition is satisfied if the integral is finite. Assuming that the integral is finite, then as $\overline{N}$ approaches infinity the constant C_2 is determined from the condition that:

$$\left\{\int_0^{\overline{N}} \dots\right\}(2\overline{N}^2+1)+C_2(2\pi^{1/2})^{-1}(2\overline{N}^2+1)\Big|_{\overline{N}=\sim\infty}\cong 0. \tag{4.4.8}$$

The entire expression for T becomes:

$$T = T_s + T_1 + T_2. \tag{4.4.9}$$

When y is equal to zero N equal $\overline{N}$ equal zero and T is equal to T_s. When y approaches infinity, $T = T_s + T_1 + T_2$ and the constants for T_1 are determined so that $T = T_\infty$ at $\overline{N}_{\rightarrow\infty}$ (see Section 3.4).

4.5. *Second Approximation to* μ *and* K

The second approximation to μ is written:

$$\mu_2 = \overline{\mu}[1 + a(T_s + T_1 + T_2) + b(T_s + T_1 + T_2)^2 + \dots]. \tag{4.5.1}$$

As in the first approximation to K the following expression is given for K_2:

$$K_2 = K\, c_v\, \mu_2. \tag{4.5.2}$$

The extension of this method to higher derivatives is only a matter of repetition.

The method presented above is adaptable to computing devices.

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