ON THE GENERALIZED FUNDAMENTAL EQUATIONS FOR THE INTERACTION BETWEEN DISSIPATIVE FLOWS AND EXTERNAL STREAMS

by

M. Z. KRZYWOBLOCKI (Urbana, Ill.)

SUMMARY. — The problem of interaction between a viscous or dissipative flow near the surface of a solid body, or in its wake, and an „outer" isentropic or nearly isentropic stream became very important. The author remodifies the Crocco-Lees approach [3] in order to include the rarefaction of the gas, the vertical velocity component and the pressure gradient in the vertical direction. The author uses Grad's equations based upon the kinetic theory of monatomic gases and preserves all the equations of momentum and energy in their full forms. A method, suitable for high speed computing machines, is developed, enabling one to calculate the velocity, density, pressure and temperature distributions in the boundary layer, as well as the components of the shearing stress tensor and heat flux vector. The convergence of the successive approximations process is proved; some remarks on the possible extension of the Crocco-Lees results in wakes to hypersonic flow regime close the paper.

INTRODUCTION

The problem of interaction between a viscous or dissipative flow near the surface of a solid body, or in its wake, and an „outer" isentropic or nearly isentropic stream became very important. A simplified approach to this problem was proposed by Crocco and Lees [3]. By means of a simplified theoretical model, their paper treats the general class of flow problems characterized by this kind of interaction. The external flow is taken to be a plane, steady supersonic flow, which makes a small angle with a plane surface or plane of symmetry. The internal dissipative flow is regarded as quasi-one-dimensional and parallel to the surface on the average, with a properly defined mean velocity and mean temperature. The nonuniformity of the actual velocity distribution is taken into account only approximately by means of a relation between mean temperature and mean velocity. Mixing, or the transport of momentum from outer stream to dissipative flow, is considered to be the fundamental physical process determining the pressure rise that can be supported by the flow. With the aid of this concept, a large number of flow problems is shown to be basically similar, such as boundary-layer-shock-wave interaction, wake flow behind blunt-based bodies (base pressure problem), flow separation in overexpanded supersonic nozzles, separation on wings and bodies, etc.

In the Crocco and Lees approach the equations of motion are reduced to a single nonlinear ordinary differential equation that can be integrated numerically. An important property of this equation is the existence of a „critical point" for supersonic wake flows, and, also, under certain conditions, for supersonic flows directed toward a solid surface. This critical point acts much like the „throat" of a nozzle in determining the base pressure, for example, or in some cases the surface pressure distribution in a boundary-layer-shock-wave interaction. One important reason for the marked difference between laminar and turbulent flows is the fact that the turbulent mixing rates are from five to ten times larger than the laminar mixing rates.

By introducing several reasonable physical assumptions, a „simplified" form of the mixing theory is developed particularly for separated and reattaching flows and wake flows. Separating flows, as well as reattaching flows, are found to be capable of supporting considerable pressure increases at high velocities. When the Crocco-Lees mixing theory is applied to the problem of determining the base pressure for a supersonic airfoil with a blunt trailing edge, it gives the correct fluid-mechanical explanation of the observed phenomena. Qualitative agreement is found between the theoretical calculations of the curve of base pressure versus Reynolds number and the data of Chapman and Bogdonoff on bodies of revolution and Chapman's data on blunt trailing-edge airfoils.

The results obtained in the base pressure problem for a supersonic airfoil with a blunt trailing edge open the way for application of the mixing theory to boundary-layer-shock-wave interactions, boundary layer separation, and many other phenomena. However, the dependence of the mixing rate and the mean velocity-mean temperature relation for the dissipative flow region on the flow parameters must be carefully investigated experimentally and theoretically, and the results incorporated into the analysis.

In their approach, Crocco and Lees use only one momentum equation and the energy equation in a simplified form. The second momentum equation and the vertical velocity component are neglected and the frictional stress at the surface (if a surface is present) appears in a form of a symbol. The correct form of the relation of the friction coefficient upon the derivatives of the velocity components is not used. A certain approximate relation is found between the friction coefficient along a flat plate in a compressible fluid flow and the corresponding friction coefficient along a flat plate in an incompressible fluid flow, using Stewartson's proposition [8a]. Thus, the whole approach is an approximate one and can be used only in the low or at most the high supersonic region. But some remarks are necessary when one approaches to the hypersonic regime.

Let the symbol „l" denote the mean free path and „a" a characteristic dimension of an object moving with a certain speed in a fluid. Then, the region where $l/a \ll 1$ is characterized as the one of ordinary gas dynamics. In that region the system of Navier-Stokes equations of motion (momentum), continuity and energy, derived fundamentally on the basis of mechanics of continuum is assumed to be adequate enough to describe the status of the motion. The associated boundary conditions are well known. A particular solution of the system in question must satisfy the following conditions:

(i) if the viscous, incompressible fluid extends to infinity, the value of the velocity, density and temperature must be specified at infinity;

(ii) all of the fluid particles which are adjacent to any solid surface have the same vector velocity and temperature as the corresponding elements of the solid boundary.

The second condition requires that there be no flow normal to any solid surface (unless there is suction or injection), and that there be no slip between the fluid and the wall. Various experimental physicists (Knudsen, Kundt, Warburg and others) have established that this „non-slip" condition is valid only if the mean free path of the gas is completely negligible relative to the characteristic macroscopic dimension. Hence, for a rarefied gas, the „non-slip" condition must be replaced by some relation which specifies the slip velocity of the gas relative to the solid wall. The region $l/a < 1$ up to $\cong 1$ or > 1, say, can be characterized as that of slip flow and of hypersonics in rarefied gases. The phenomenological assumption of A. Basset specifies that the slip velocity at the surface of a solid body be proportional to the shearing stress at the same surface. Similarly, the experiments of von Smoluchowski have shown that in a rarefied gas there exists a „temperature jump" between the wall temperature and the temperature of the gas layer immediately adjacent to the wall, and, that, to a first approximation, this temperature jump is proportional to the temperature gradient (normal to the wall) which exists in the gas at the vicinity of the wall.

It is today generally assumed that in this region the Navier-Stokes equations seem inadequate to describe the status of a motion of a body in a gas. The fundamental concepts of the kinetic theory of gases must be used. As it is wellknown, for monatomic gases this takes form of the Boltzmann equation. Due to the lack of an analogous adequate equation for the polyatomic gases, the remarks below are restricted to monatomic gases. Works on the Boltzmann equation by Hilbert, Enskog, Chapman, Burnett and others are confined to the region in which $l/a \ll 1$, or at most $l/a < 1$. Their procedure consists essentially of expansion in powers of (l/a). Hence these solutions are the thermodynamic approximations starting from the Boltzmann equation. It is expected that this procedure leads to an asymptotic solution. The first approximation leads to the Euler equation for a compressible, inviscid, non-heat conducting fluid flow; the second approximation yields the Navier-Stokes equations and the third approximation yields the so-called equations of slip flow or Burnett's equations. The polynomials used in that approach are numerical multiplies of Sonine's polynomials which arise in the study of Bessel's functions.

The most promising solution of the Boltzman equation was recently proposed by Grad [6] using Hermite polynomials. This solution may be considered to be between the thermodynamic and the full use of the Boltzman equation. It seems probable that the type of analysis proposed by Grad is preferable to the Hilbert-Enskog method when considering rapidly changing flow, for example, the internal structure of a shock wave.

Tests show that in the range in which the kinetic theory should be accepted as the fundamental concept of the phenomenon, the boundary layer is thick and the pressure gradient in the normal direction is no any longer small. Hence, the normal velocity component is of a non-negligible magnitude and consequently the second equation of momentum must be preserved. The thickness of the boundary layer in this regime may be of the order of the representative length dimension of the solid body.

In general, the structure of the boundary layer may be affected by the following characteristic features of the phenomenon in question: (i) rarefaction of the gas; (ii) interaction between boundary layer and the adjacent external stream; (iii) interaction between the boundary layer and front shock; (iv) boundary layer and/or body curvature; (v) gas imperfections. In the latter group one may distinguish three subgroups;

(*a*) conditions outside the molecule which can be taken into account by adopting a more complicated equation of state in place of the perfect gas equation; (*b*) changes inside the molecule, i. e., in addition to the energies of translation and rotation, energy can also be absorbed in vibration of the atoms in the molecules; this causes the variation of specific heats; the process is still more complicated by the phenomenon known as heat capacity lag or relaxation time; as the temperature increases with the increasing speed, the phenomenon of dissociation becomes apparent; (*c*) changes inside the atom indicator: there exists the possibility of ionization of the atoms of the gas at the highest velocities.

In the present paper the author re-modifies the Crocco-Lees approach in order to include the rarefaction of the gas, the vertical velocity component and the pressure gradient in the vertical direction. Briefly, the first two points from the five ones mentioned above are taken into account. This is accomplished by using Grad's proposition and preserving all the equations of motion (momentum and energy) in their full forms without any simplifyng assumptions. A method, suitable for high speed computing machines, is developed, enabling one to calculate the velocity, density, pressure and temperature distributions in the boundary layer, as well as the components of the shearing stress tensor and heat flux vector. The obtained system of ordinary differential equations may be solved by some iteration method or by successive approximations. Analogous functions to those derived by Crocco and Lees are proposed; the system of four differential equations is reduced to three by means of some consideration of the algebraic nature. As the initial values for each of those systems one can use the values from the compressible flow regime obtainable by means of the Crocco-Lees procedure. Some remarks on the possible extension of the Crocco-Lees results in wakes to hypersonic flow regime close the paper.

The author presumes that the reader is well acquainted with the Crocco-Lees paper; consequently, in the text below only a few items from that paper will be cited.

1. *FUNDAMENTAL HYDRODYNAMIC EQUATIONS*

1.1. *Hydrodynamic Equations*

If a body moves in a gas, it exercises an important influence on the whole character of the motion of the gas molecules. The hydrodynamic expressions for the fact that the conditions of motion of the gas are changed by the presence of the body consist of the appearance of a system of hydrodynamic stresses and heat flow. The fundamental equations of hydrodynamics are the following: equation of motion:

$$(1.1.1) \qquad u^*_{i,t} + u^*_r \, u^*_{i,r^*} + \rho^{*-1} (p^*_{,i^*} + p^*_{ij,j^*}) = 0;$$

equations of continuity and state of a perfect gas:

$$(1.1.2) \qquad \rho^*_{,t^*} + (\rho^* u^*_i)_{,i^*} = 0; \quad p^* = R^* \rho^* T^*;$$

equation of energy

$$(1.1.3) \qquad \rho^* \frac{D}{Dt^*}(c^*_v T^*) + p^* u^*_{i,i^*} + p^*_{ij} u^*_{i,j^*} + \frac{1}{2} s^*_{i,i^*} = 0,$$

where the symbol D/Dt^* donotes the operator:

$$(1.1.3\,a) \qquad D/Dt^* = \partial/\partial t^* + u^*_i \, \partial/\partial x^*_i;$$

various forms of the equations for the stress tensor, p^*_{ij}, and heat flux vector, s^*_i, were proposed by Navier-Stokes (continuum), Burnett and Grad (kinetic theory, see [2], [6]). They are given in the Appendix (1.1.4; 1.1.5; 1.1.6). In the systems, presented above, a standard tensor notation is used. A subscript with comma denotes a partial differentiation. The asterisk means that the physical quantity in question has proper physical dimensions. Notice, that in the case of a steady motion, the first two terms in the energy equation can be presented in the form (with $c^*_v = c^*_p - R^*$):

$$(1.1.7) \qquad Z = \rho^* u^*_i (c^*_v T^*)_{,i^*} + p^* u^*_{i,i^*} =$$

$$\rho^* u^*_i (c^*_p T^*)_{,i^*} - \rho^* u^*_i (R^* T^*)_{,i^*} + p^* u^*_{i,i^*};$$

adding the expression $(\rho^* u^*_i)_{,i^*} R^* T^* = 0$, gives the result $-(p^* u^*_i)_{,i^*}$ and:

$$(1.1.8) \qquad Z = \rho^* u^*_i (c^*_p T^*)_{,i^*} - p^*_{,i^*} u^*_i.$$

Not cited here is the system proposed by Truesdell [9]. This is due to the fact that the coefficients (analogous to μ^* and λ^*) in Truesdell's expansion are unknown, as yet, and possibly must be determined by some experiments.

1.2. *Dimensionless Magnitudes*

Introduce the magnitudes in a steady, uniform motion of the fluid in question: U^*_0; p^*_0; ρ^*_0; T^*_0; μ^*_0; λ^*_0, and a representative length L^*. This enables one to introduce the dimensionless quantities:

$$x_i = x^*_i L^{*-1} (i = 1, 2, 3); \quad u_i = u^*_i U^{*-1}_0; \quad \rho = \rho^* \rho^{*-1}_0;$$

$$T = T^* T_0^{*-1};\ R = R^* R_0^{*-1} = 1;\ \mu = \mu^* \mu_0^{*-1};\ p = p^* p_0^{*-1};$$
$$t = t^* U_0^* L^{*-1};\ \lambda = \lambda^* \lambda_0^{*-1};\quad \lambda_0^* = R_0^* \mu_0^*;$$
(1.2.1)
$$p_{ir} = p_{ir}^* p_0^{*-1};\ s_i = s_i^* (p_0^* U_0^*)^{-1};\ \rho_0^* U_0^{*2} p_0^{*-1} = \alpha = \gamma_0 M_0^2;$$
$$M_0^2 = U_0^{*2} a_0^{*-2};\ a_0^{*2} = \gamma_0 R_0^* T_0^*;$$
$$\gamma = c_p^* c_v^{*-1} = \gamma_0 = c_{p0}^* c_{v0}^{*-1};\ L^* \rho_0^* U_0^* \mu_0^{*-1} = R_e = \beta;$$
$$\chi = c_{p0}^* R^{*-1}.$$

Notice, that the symbol a_0^* denotes the speed of sound in isentropic gas conditions, M_0 the Mach number, R_e the Reynolds number. To obtain the dimensionless forms of equations multiply the original system of equations by the following expressions:

conservation of momentum by	$L^* (\rho_0^* U_0^{*2})^{-1}$;
continuity by	$(\rho_0^* U_0^*)^{-1}$;
state by	$p_0^{*-1} = (R_0^* \rho_0^* T_0)^{-1}$;
energy by	$L^* (p_0^* U_0^*)^{-1}$;
stress tensor by	$L^* (p_0^* U_0^*)^{-1}$;
heat flux vector by	$L^* (p_0^* U_0^{*2})^{-1}$.

1.3. *Two Dimensional Steady Motion*

Grad's system of equations in this case has the following dimensionless form:

(1.3.1) $$\rho (u u_{,x} + v u_{,y}) + \alpha^{-1} (p_{,x} + p_{xx,x} + p_{xy,y}) = 0;$$

(1.3.2) $$\rho (u v_{,x} + v v_{,y}) + \alpha^{-1} (p_{,y} + p_{yx,x} + p_{yy,y}) = 0;$$

(1.3.3) $$(\rho u)_{,x} + (\rho v)_{,y} = 0;\quad p = R \rho T = \rho T;$$

(1.3.4) $$\chi \rho [(c_p T)_{,x} u + (c_p T)_{,y} v] - p_{,x} u - p_{,y} v + p_{xx} u_{,x} + p_{xy} (u_{,y} + v_{,x}) + p_{yy} v_{,y} + \frac{1}{2} (s_{x,x} + s_{y,y}) = 0.$$

Equations for p_{xx}, p_{xy}, p_{yy}, s_x, s_y, are given in the Appendix (1.3.5) to (1.3.9). Adding the expressions $u (\rho u_i)_{,i} = 0$ $(i = 1, 2)$ and $v (\rho u_i)_{,i} = 0$ to equations (1.3.1, 2), respectively, gives the following forms of these equations:

(1.3.1 *a*) $$(\rho u^2)_{,x} + (\rho u v)_{,y} + \alpha^{-1} (p_{,x} + p_{xx,x} + p_{xy,y}) = 0;$$

(1.3.2 a) $$(\rho v^2)_{,y} + (\rho u v)_{,x} + \alpha^{-1} (p_{,y} + p_{yx,x} + p_{yy,y}) = 0 .$$

Adding the expression $\chi c_p T (\rho u_i)_{,i} = 0$ $(i = 1, 2)$ to the energy equation, gives the following form of that equation:

(1.3.4 a) $$(\chi c_p T \rho u)_{,x} + (\chi c_p T \rho v)_{,y} - u p_{,x} - v p_{,y} + \frac{1}{2}(s_{x,x} + s_{y,y}) + \Phi = 0 ;$$

(1.3.4 b) $$\Phi = p_{xx} u_{,x} + p_{xy} (u_{,y} + v_{,x}) + p_{yy} v_{,y} .$$

The above system is applied to a flow along a flat plate located along the x-axis.

1.4. *Transformation of Fundamental Equations*

Let $\delta = \delta(x)$ denote the variable finite thickness of the boundary layer along a flat plate. The following boundary conditions are assumed with

$$\chi c_p T \rho u = h_x , \qquad \chi c_p T \rho v = h_y :$$

(1.4.1) $y = 0$: continuum: $u = 0$; $v = 0$ (no suction); $v \neq 0$ (suction);
slip flow: $u = u_w$; $v = 0$ (no suction); $v \neq 0$ (suction);
$\rho = \rho_w$; $T = T_w$; $p = p_w$;

(1.4.2) $y = \delta$: $u = u_e$; $v = v_e$; $\rho = \rho_e$; $p = p_e$; $T = T_e$;
$p_{xx} = p_{xxe} = p_{xy} = p_{xye} = p_{yy} = p_{yye} = 0$;
$s_x = s_{xe} = s_y = s_{ye} = 0$; $h_x = h_{xe}$; $h_y = h_{ye}$.

In continuum with no suction the boundary conditions are very simple: $u_w = v_w = 0$. In the slip flow regime and/or with suction, when $u_w \neq 0$, $v_w \neq 0$, these values must be given as linear or nonlinear boundary conditions thus furnishing linear or nonlinear boundary value problems.

The rate at which the mass $\overline{m}_e$ is transported from the external stream to the internal flow, i.e., to the boundary layer region, is equal to:

(1.4.3) $$\frac{d}{dx}\overline{m}_e = \rho_e \left(u_e \frac{d\delta}{dx} - v_e \right) = \rho_e u_e \left(\frac{d\delta}{dx} - \tan\theta \right),$$

with the positive value of u_e pointing in the direction of increasing δ and positive value of v_e directed upward, and θ denoting the local angle between the external streamline at $y = \delta$ and the x-axis. Introduce the notion of the mass flux $\overline{m}_u$ in the internal flow in the horizontal direction and of the momentum flux in both horizontal and vertical directions, I_{xx} and I_{xy}, respectively:

$$\overline{m}_u = \int_0^\delta \rho\, u\, dy; \quad I_{xx} = \int_0^\delta \rho\, u^2\, dy; \quad I_{xy} = \int_0^\delta \rho\, u\, v\, dy. \tag{1.4.4}$$

Similarly, the flux H_x of enthalpy per unit volume $\chi c_p \rho T$ is:

$$\chi c_p \rho u T = h_x; \quad \chi c_p \rho v T = h_y; \quad \int_0^\delta h_x\, dy = H_x. \tag{1.4.5}$$

Integrate all the equations (1.3.1. to 4) with respect to y between the limits 0 and δ with the use of the formula:

$$\begin{aligned} \frac{d}{dc}\int_a^b f(x;c)\,dx &= \int_a^b \frac{\partial}{\partial c} f(x;c)\,dx + \\ &+ f(b;c)\frac{db}{dc} - f(a;c)\frac{da}{dc}. \end{aligned} \tag{1.4.6}$$

The following system is obtained:

$$\begin{aligned} \frac{d}{dx} I_{xx} = u_e \frac{d}{dx}\overline{m}_e - \alpha^{-1}\Bigg[\frac{d}{dx}\int_0^\delta (p + p_{xx})\,dy \\ - p_e\frac{d\delta}{dx} - p_{xyw}\Bigg] + \rho_w u_w v_w; \end{aligned} \tag{1.4.7 a}$$

$$\frac{d}{dx} I_{xy} = v_e \frac{d}{dx}\overline{m}_e - \alpha^{-1}\left[\frac{d}{dx}\int_0^\delta p_{yx}\,dy + p_e - (p_w + p_{yyw})\right] + \rho_w v_w^2; \tag{1.4.7 b}$$

$$\frac{d}{dx}\overline{m}_u = \frac{d}{dx}\overline{m}_e + \rho_w v_w: \tag{1.4.7 c}$$

$$\begin{aligned} \frac{d}{dx} H_x = \chi c_p T_e \frac{d}{dx}\overline{m}_e - \frac{1}{2}\frac{d}{dx}\int_0^\delta s_x\,dy + h_{yw} + \frac{1}{2} s_{yw} \\ - \int_0^\delta \left(\Phi - \frac{Dp}{Dt}\right) dy; \quad \frac{D}{Dt} = u_i \frac{\partial}{\partial x_i} \quad (i = 1, 2). \end{aligned} \tag{1.4.7 d}$$

Introduce the following new variables:

$$u_1 = I_{xx}\,\overline{m}_u^{-1}; \quad v_1 = I_{xy}\,\overline{m}_u^{-1}; \quad \overline{m}_u = \rho_1 u_1 \delta = p_1 u_1 \delta T_1^{-1}; \tag{1.4.8 a}$$

(1.4.8 b) $$H_x = \chi c_p \delta \rho_1 u_1 T_1 = \chi c_p T_1 \bar{m}_u .$$

Here formally $p_1 = \rho_1 T_1$, but basically, only the magnitude ρ_1 is defined in terms of $\bar{m}_u$, u_1 and δ.

Some useful expressions are introduced under the assumption that all the characteristic quantities on the edge of the boundary layer, (i. e., for $y = \delta$) of the external flow, ρ_e, u_e, etc., are independent of y and depend only on x:

(1.4.9) $$\delta_u^* = \int_0^\delta [1 - \rho u (\rho_e u_e)^{-1}] \, dy = \delta - \bar{m}_u (\rho_e u_e)^{-1};$$

(1.4.10) $$\delta_u^{**} = \int_0^\delta (\rho u)(\rho_e u_e)^{-1} (1 - u u_e^{-1}) \, dy = \delta - \delta_u^* - I_{xx} (\rho_e u_e^2)^{-1};$$

(1.4.11) $$\delta_v^* = \int_0^\delta [1 - \rho u (\rho_e v_e)^{-1}] \, dy = \delta - \bar{m}_u (\rho_e v_e)^{-1};$$

(1.4.12) $$\delta_v^{**} = \int_0^\delta (\rho u)(\rho_e v_e)^{-1} (1 - v u_e^{-1}) \, dy = \delta - \delta_v^* - I_{xy} (\rho_e u_e v_e)^{-1}.$$

From these equations one can derive the expressions:

(1.4.13) $$\bar{m}_u = \rho_e u_e (\delta - \delta_u^*) = \rho_e v_e (\delta - \delta_v^*);$$

(1.4.14) $$I_{xx} = \rho_e u_e^2 (\delta - \delta_u^* - \delta_u^{**}); \quad I_{xy} = \rho_e u_e v_e (\delta - \delta_v^* - \delta_v^{**}),$$

and with the use of (1.4.8):

(1.4.15 a) $$u_1 u_e^{-1} = k_1 = (\delta - \delta_u^* - \delta_u^{**})(\delta - \delta_u^*)^{-1} = I_{xx} (\bar{m}_u u_e)^{-1};$$

(1.4.15 b) $$v_1 u_e^{-1} = k_{12} = (\delta - \delta_v^* - \delta_v^{**})(\delta - \delta_v^*)^{-1} = I_{xy} (\bar{m}_u u_e)^{-1};$$

(1.4.15 c) $$\rho_1 \rho_e^{-1} = p_1 T_e (p_e T_1)^{-1} = k_1^{-2} k_2^{-1}; \quad k_2 = \delta (\delta - \delta_u^* - \delta_u^{**})^{-1}.$$

1.5. *Isentropic Flow Relations*

Introduce the dimensionless functions:

(1.5.1) $$w = u a_s^{-1}; \quad z = v a_s^{-1}; \quad W_e^2 = V_e^2 a_s^{-2} = w_e^2 + z_e^2; \quad a_s^2 = \gamma T_s,$$

with the subscript „s" denoting the stagnation conditions and „a" the local velocity of sound. The following equations are valid for the flow of the isentropic gas outside the boundary layer:

$$\frac{1}{2} V_e^2 + (\gamma - 1)^{-1} \gamma T_e = \gamma (\gamma - 1)^{-1} T_s; \tag{1.5.2}$$

or with the relation $p_e = \rho_e T_e$:

$$p_e = \rho_e T_s \left[1 - \frac{1}{2} (\gamma - 1) W_e^2\right]. \tag{1.5.3}$$

From the fundamental equation $\rho_e V_e d V_e + d p_e = 0$, with the use of eq. (1.5.3) one gets:

$$\varphi_e^{-1} d W_e = - p_e^{-1} d p_e; \quad \varphi_e = \left[1 - \frac{1}{2}(\gamma - 1) W_e^2\right](\gamma W_e)^{-1} = T_e (\gamma W_e T_s)^{-1}; \tag{1.5.4}$$

or

$$\varphi_e^{-1} p_e = \rho_e V_e a_s; \quad \rho_e V_e^2 = \varphi_e^{-1} p_e W_e. \tag{1.5.5}$$

1.6. *Second Transformation of Equations*

Introduce the symbols:

$$m_e = \overline{m}_e a_s; \tag{1.6.1a}$$

$$m_u = \overline{m}_u a_s = p_1 \delta \varphi_1^{-1} = I_{xx} w_1^{-1}; \quad w = u_1 a_s^{-1}; \tag{1.6.1b}$$

$$\varphi_1 = T_1 (\gamma T_s w_1)^{-1}; \tag{1.6.1c}$$

$$H_x = \chi c_p \int_0^\delta \rho u T d y = \chi c_p \delta \rho_1 u_1 T_1 = \chi c_p T_1 \overline{m}_u. \tag{1.6.1d}$$

The last relation determines the value of T_1 and consequently, using ρ_1, the value of p_1. The system (1.4.7) jointly with equations (1.4.4) and (1.6.1*b*) is associated with the system (1.6.2):

$$\frac{d}{dx}(m_u w_1) = w_e \frac{d m_e}{dx} - \alpha^{-1} \left[\frac{d}{dx} \int_0^\delta (p + p_{xx}) dy - p_e \frac{d \delta}{d x} - p_{xyw}\right] + a_s^2 \rho_w w_w z_w; \tag{1.6.2 a}$$

$$\frac{d}{dx}(m_u z_1) = z_e \frac{d m_e}{dx} - \alpha^{-1} \left[\frac{d}{dx} \int_0^\delta p_{yx} dy + p_e - (p_w + p_{yyw})\right] + a_s^2 \rho_w z_w^2; \tag{1.6.2 b}$$

(1.6.2 c) $$\frac{d\,m_u}{dx} = \frac{d\,m_e}{dx} + a_s^2\,\rho_w\,z_w;$$

(1.6.2 d) $$\frac{d\,m_e}{dx} = p_e\,\varphi_e^{-1}\cos\theta\left(\frac{d\delta}{dx} - \tan\theta\right);$$

(1.6.2 e) $$m_u = p_1\,\delta\,\varphi_1^{-1};$$

(1.6.2 f) $$\frac{d}{dx}(\chi\,c_p\,T_1\,m_u) = \chi\,c_p\,T_e\frac{d\,m_e}{dx} + a_s\left[\left(h_{yw} + \frac{1}{2}s_{yw}\right) + \int_0^\delta\left(\frac{Dp}{Dt} - \Phi\right)dy - \frac{1}{2}\frac{d}{dx}\int_0^\delta s_x\,dy\right].$$

The following magnitudes are given:

(a) at the surface of the body:
(i) continuum regime: $w_w = 0$; $z_w = 0$ (no suction), $= a$ (given when there is suction); ρ_w, T_w, p_w given;
(ii) slip flow regime: w_w, z_w, ρ_w, T_w, p_w are given in form of numbers or specified by means of some equations involving the unknown functions w, z, ρ, T, p; this latter case leads to a nonlinear boundary value problem.

(b) at the outer edge of the boundary layer:
(i) and (ii): p_e, ρ_e, T_e and θ are given..

The unknowns are: m_e, m_u, δ, W_e, w_1, z_1, T_1. Since $p_e = p_e(W_e)$, $T_e = T_e(W_e)$, and $\theta = \theta(W_e)$ are known from the Prandtl-Meyer relations, the above system furnishes 6 relations for 7 unknowns. As in [3], an additional relation between these variables is introduced by means of an assumption regarding the mixing rate $d\,m_e/dx$. This will be discussed below. Assume that

$$\int_0^\delta (p + p_{xx})\,dy;\quad p_{xyw};\quad \int_0^\delta p_{yx}\,dy;\quad (p_w + p_{yyw});$$

$$\int_0^\delta (Dp/Dt - \Phi)\,dy \quad \text{and} \quad \int_0^\delta s_x\,dy$$

can be expressed in terms of the seven unknowns; then, in general, the above system can be solved by means of some iteration process.

1.7. *Iteration Process*

We shall show below, that, in general, one can prove the existence of a solution of the system (1.6.2) obtained by means of a method of successive approximations. We consider this system subject to the initial conditions. To prove the convergence, let us generalize the proof used by Hamel [7, p. 11] for the solution of Volterra's kind of equation. Assume a system of ordinary differential equations

$$\frac{dy_i}{dx} = f_i(y_j, x) \quad (i, j = 1, 2, \ldots, n), \tag{1.7.1}$$

or

$$y_i = \int f_i(y_j, x)\, dx + C_i \quad (i, j = 1, 2, \ldots, n), \tag{1.7.2}$$

with $C_i\,(i = 1, \ldots, n)$ denoting the constants of integration chosen with reference to the initial conditions. Let the symbols $y_{j(o)}, y_{j(1)}, \ldots, y_{j(n)}$, denote the values obtained in successive approximations. As the given approximation $y_{j(0)}$ we may assume the values from the domain of the compressible fluid flow, i. e., those obtained by the method of Crocco and Lees [3]. This assures a greater rate of convergence of the process of successive approximations. Then one has:

$$\begin{aligned} y_{i(1)} &= \int_X f_{i(0)}(y_{j(0)}, x)\, dx + C_i; \\ y_{i(2)} &= \int_X f_{i(1)}(y_{j(1)}, x)\, dx + C_i; \\ &\cdots\cdots\cdots\cdots\cdots \\ y_{i(n)} &= \int_X f_{i(n-1)}(y_{j(n-1)}, x)\, dx + C_i, \end{aligned} \tag{1.7.3}$$

where the symbol X denotes the interval of integration. Construct the differences:

$$y_{i(n+1)} - y_{i(n)} = \int_X (f_{i(n)} - f_{i(n-1)})\, dx. \tag{1.7.4}$$

The ratio:

$$\frac{f_{i(n)} - f_{i(n-1)}}{y_{i(n)} - y_{i(n-1)}} = \frac{\partial f_{iF}}{\partial y_i}, \tag{1.7.5}$$

where f_{iF} denotes a value of f_i located between $y_{i(n-1)}$ and $y_{i(n)}$. Let

$$max \left| \frac{\partial f_{iF}}{\partial y_i} \right| = N_i; \quad max\,|N_i| = N \quad (i = 1, \ldots, n); \tag{1.7.6}$$

then:

(1.7.7) $$|f_{i(n)} - f_{i(n-1)}| \leqq N |y_{i(n)} - y_{i(n-1)}|,$$

and from (1.7.4):

(1.7.8) $$|y_{i(n+1)} - y_{i(n)}| \leqq N \int_X |y_{i(n)} - y_{i(n-1)}| \, dx.$$

Let the functions f_i's be bounded and let:

(1.7.9) $$max |f_i| = M,$$

in the region considered. Then, with the assumptions:

(1.7.10 *a*) $$y_{i(0)} = const. = C_i;$$

(1.7.10 *b*) $$|y_{i(1)} - y_{i(0)}| = \int_X f_{i(0)} \, dx \leqq M X;$$

for $n = 1$:

(1.7.10 *c*) $$|y_{i(2)} - y_{i(1)}| \leqq N \int_X |y_{i(1)} - y_{i(0)}| \, dx \leqq M N \int_X X \, dx = M N \frac{1}{2!} X^2;$$

for $n = 2$:

(1.7.10 *d*) $$|y_{i(3)} - y_{i(2)}| \leqq N \int_X |y_{i(2)} - y_{i(1)}| \, dx \leqq M N^2 \frac{1}{3!} X^3, \text{ etc.};$$

(1.7.10 *e*) $$|y_{i(n)} - y_{i(n-1)}| \leqq M N^{n-1} \frac{1}{n!} X^n.$$

Consider the infinite series:

(1.7.11) $$y_{i(0)} + (y_{i(1)} - y_{i(0)}) + \ldots + (y_{i(n)} - y_{i(n-1)}) + \ldots,$$

whose sums to n terms are $y_{i(n-1)}$. By force of the formulas (1.7.10) these series are dominated by the series:

(1.7.12) $$C_i + M X + \frac{1}{2!} M N X^2 + \ldots + \frac{1}{n!} M N^{n-1} X^n + \ldots,$$

which can be presented in the forms:

(1.7.13) $$C_i + M N^{-1} [\exp (N X) - 1].$$

These are exponential series which converge for all finite values of X. Hence, put:

(1.7.14) $$y_i = C_i + (y_{i(1)} - y_{i(0)}) + \ldots = \lim_{n \to \infty} [C_i + (y_{i(n)} - y_{i(n-1)})] = \lim_{n \to \infty} y_{i(n)};$$

the functions $y_i's$ are continuous since they are the limits of uniformly convergent sequences of continuous functions. Due to the continuity and uniform convergence, it is possible to take limits under the integral signs in equations (1.7.3), thus obtaining:

$$(1.7.15)\qquad \lim_{n\to\infty} y_{i(n)} = y_i = \int_X \lim_{n\to\infty} f_{i(n-1)}(y_{j(n-1)}, x)\,dx + C_i.$$
$$= \int_X f_i(y_j, x)\,dx + C_i;$$

this differentiated, furnishes the formula:

$$(1.7.16)\qquad dy_i/dx = f_i(y_j, x),$$

which proves that $y_j's$ fulfill the original system of equations.

To prove the uniqueness of the solution suppose that there are two solutions: $y_i^{(1)} \neq y_i^{(2)}$. Setting $w_i = y_i^{(1)} - y_i^{(2)}$ yields the inequality:

$$(1.7.17)\qquad |w_i| = |\int_X (f_i^{(1)} - f_i^{(2)})\,dx| \leqq N \int_X |y_i^{(1)} - y_i^{(2)}|\,dx,$$

which furnishes the inequality:

$$(1.7.18)\qquad max\,|w_i| \leqq NX\, max\,|w_i|.$$

Since it is assumed that $max|w_i| \neq 0$, this implies that $1 \leqq NX$. One obtains immediately a contradiction if $NX < 1$, which means that the proof refers to a sufficiently small interval X. By repetition and continuation of the proof it is not too difficult to prove the uniqueness in a larger interval [7]. The above discussion gives the length of the interval X in which the process of successive approximations converges. Knowing the values of M (1.7.9) and of N (1.7.6) and the values of the n-th and $(n-1)$-th approximation one must find such a magnitude of the interval of integration X (1.7.3) with the use of whose the inequality (1.7.10 e) is satisfied. One may also use the results of Whyburn [11]. These are: Assume a system of ordinary, first order differential equations:

$$(1.7.18\,a)\qquad D[u]: dy_i(x)/dx = f_i(x; y_1(x), \ldots, y_n) + \sum_{j=1}^{j=n} A_{ij}(x; y_1, \ldots, y_n)\,y_j \qquad (i = 1, \ldots, n),$$

where between others the following conditions are imposed:

(i) There exists a function $M(x)$ such that for all x and for all $(y_1, y_2, \ldots, y_n)$:

$$|A_{ij}|,\ |f_i| \leqq M(x), \qquad (i, j = 1, \ldots, n).$$

(ii) There exists a function $L(x)$ such that for each x and for every $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$:

(1.7.18 b)
$$\left\{ \begin{array}{c} |A_{ij}(x;\, y_1, \ldots, y_n) - A_{ij}(x;\, z_1, \ldots, z_n)| \\ |f_i(x;\, y_1, \ldots, y_n) - f_i(x;\, z_1, \ldots, z_n)| \end{array} \right\} \leqq L(x) \sum_{r=1}^{r=n} |y_r - z_r|, \quad (i, j = 1, \ldots, n).$$

Let $N(x)$ be the function that is equal to the greater of $M(x)$ and $L(x)$ for each x. If $x = c$ is any point and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrarily assigned constants, then Whyburn had shown that there exists a unique solution of the above system satisfying the conditions:

(1.7.18 c)
$$y_i(c) = \alpha_i .$$

Moreover, if $[a, b]$, $a \leqq x \leqq b$, denotes a certain interval in which the functions f_i and A_{ij} are continous in $(y_1, \ldots, y_n)$, then Whyburn had shown that the following process of successive approximations $y_i^{(k)}$, $(k = 0, 1, \ldots)$, converges to the proper limit:

(1.7.18 d)
$$y_i^{(0)}(x) = \alpha_i \quad (i = 1, 2, \ldots, n);$$

(1.7.18 e)
$$y_i^{(k)}(x) = \alpha_i + \int_a^b [f_i(t, y^{(k-1)}) + \sum_{j=1}^{j=n} A_{ij}(t;\, y_1^{(k-1)}, y_2^{(k-1)}, \ldots, y_n^{(k-1)})\, y_j^{(k-1)}(t)]\, dt$$
$$(i = 1, 2, \ldots, n), \quad (k = 1, 2, 3, \ldots),$$

provided that the following conditions are fulfilled:

(i) Let a constant K be chosen greater than unity and so that

(1.7.18 f)
$$K > \sum_{i=1}^{i=n} |\alpha_i| + \int_a^b M(t)\, dt;$$

(ii) Let the constant G be calculated:

(1.7.18 g)
$$G = K \exp [n \int_a^b M(t)\, dt];$$

(iii) the following inequality must be fulfilled:

(1.7.18 h)
$$|y_i^{(k)}(x)| < K\, [(n^2 G + 2n)]\, |\int_a^b N(t)\, dt\, |\,]^k / k\,!.$$

The length of the interval $[a, b]$ must be chosen so as to satisfy all the above conditions.

The next problem is the determination of the functions describing the velocity —, temperature distributions, etc. As an example, assume, that the value of the function $\overline{m}_u$ is known and given by means of the process of successive approximations, discussed above. Two schemes will be proposed: (*a*) expansion in Taylor series (valid near the origin; and (*b*) Fourier series (valid in a certain interval).

(*a*) Taylor series.

Let:

$$\overline{m}_u = \int_0^{\delta} \rho\, u\, dy = f^{(2)}(x) = f^{(2)}(0) + \frac{1}{1!} x f^{(2)\prime}(0) + \frac{1}{2!} x^2 f^{(2)\prime\prime}(0) + \ldots, \tag{1.7.19}$$

and

$$\begin{aligned} \rho\, u = f^{(1)}(x, y) = f^{(1)}(0, 0) + \frac{1}{1!} x f^{(1)}_{,x}(0, 0) + \frac{1}{1!} y f^{(1)}_{,y}(0, 0) \\ + \frac{1}{2!} x^2 f^{(1)}_{,xx}(0, 0) + \frac{1}{2!} 2 x y f^{(1)}_{,xy}(0, 0) + \frac{1}{2!} y^2 f^{(1)}_{,yy}(0, 0) + \ldots, \end{aligned} \tag{1.7.20}$$

which implies that

$$\begin{aligned} \int_0^{\delta} \rho\, u\, dy = f^{(1)}(0,0)\,\delta + \frac{1}{1!} x f^{(1)}_{,x}(0, 0)\,\delta + \frac{1}{2!} x^2 f^{(1)}_{,xx}(0, 0)\,\delta \\ + \frac{1}{2!} f^{(1)}_{,y}(0, 0)\,\delta^2 + \frac{1}{2!} x f^{(1)}_{,xy}(0, 0)\,\delta^2 + \frac{1}{3!} f^{(1)}_{,yy}(0, 0)\,\delta^3 + \ldots \end{aligned} \tag{1.7.21}$$

But:

$$\delta = \delta(x) = \delta(0) + \frac{1}{1!} x\, \delta'(0) + \frac{1}{2!} x^2 \delta''(0) + \ldots , \tag{1.7.22}$$

Inserting the expansion (1.7.22) into the expansion (1.7.21), one gets:

$$\begin{aligned} \int_0^{\delta} \rho\, u\, dy = f^{(1)}(0,0)\left[\delta(0) + \frac{1}{1!} x\, \delta'(0) + \frac{1}{2!} x^2 \delta''(0) + \ldots\right] \\ + \frac{1}{1!} f^{(1)}_{,x}(0, 0)\, x \left[\delta(0) + \frac{1}{1!} x\, \delta'(0) + \frac{1}{2!} x^2 \delta''(0) \ldots\right] \end{aligned} \tag{1.7.23}$$

$$+\frac{1}{2!}f^{(1)}_{,xx}(0,0)\,x^2\,[\delta(0)+\ldots]+\ldots$$

$$+\frac{1}{2!}f^{(1)}_{,y}(0,0)\,[\delta(0)+\ldots]^2+\ldots.$$

Comparison of equal powers in x in the expansions (1.7.19) and (1.7.23) furnishes a system of simultaneous algebraic equations:

$$f^{(2)}(0)=f^{(1)}(0,0)\,\delta(0)+\frac{1}{2!}f^{(1)}_{,y}(0,0)\,\delta^2(0)+\frac{1}{3!}f^{(1)}_{,yy}(0,0)\,\delta^3(0)+\ldots; \tag{1.7.24 a}$$

$$f^{(2)\prime}(0)=f^{(1)}(0,0)\,\delta'(0)+f^{(1)}_{,x}(0,0)\,\delta(0)+\frac{1}{2!}f^{(1)}_{,xy}(0,0)\,\delta^2(0)+\ldots; \tag{1.7.24 b}$$

$$\frac{1}{2!}f^{(2)\prime\prime}(0)=\frac{1}{2!}f^{(1)}(0,0)\,\delta''(0)+\frac{1}{1!}f^{(1)}_{,x}(0,0)\,\delta'(0)+\frac{1}{2!}f^{(1)}_{,xx}(0,0)\,\delta(0)+\frac{1}{2!}f^{(1)}_{,y}(0,0)\,\delta'^2(0)+\ldots. \tag{1.7.24 c}$$

Preserving the sufficient number of terms in the system (1.7.24) enables one to calculate approximately the values of $f^{(1)}(0,0)$, $f^{(1)}_{,x}(0,0)$, $f^{(1)}_{,y}(0,0)$, etc. This gives the approximate value of the function ρu. Application of the procedure to the calculation of the velocity components, density, pressure and temperature from the system (1.6.2) furnishes a set of systems of the kind of system (1.7.24), which set of systems should be solved simultaneously. This task may be achieved by means of high speed computing devices.

(*b*) Fourier series.

The functions $f^{(2)}(x)$ and $\delta(x)$ being known functions can be expressed in terms of a Fourier series:

$$f^{(2)}(x)=\frac{1}{2}a_0^{(2)}+a_1^{(2)}\cos x+b_1^{(2)}\sin x+\ldots+a_m^{(2)}\cos mx+\ldots, \tag{1.7.25a}$$

(1.7.25 *b*) $$\delta(x) = \frac{1}{2}\delta_0 + \delta_1 \cos x + \theta_1 \sin x + \ldots + \delta_m \cos m x + \theta_m \sin m x + \ldots;$$

here, the coefficients $a_n^{(2)}$, $b_n^{(2)}$, δ_n, θ_n, should be found by means of the known formulas.

Let the function $f^{(1)}(x, y)$ be given in form of a double trigonometric series:

(1.7.26) $$f^{(1)}(x, y) = \frac{1}{2} a_{00} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [a_{mn}^{(1)} \cos m x \cos (n \pi y \delta^{-1}) + b_{mn}^{(1)} \sin m x \sin (n \pi y \delta^{-1}) + c_{mn}^{(1)} \cos m x \sin (n \pi y \delta^{-1}) + d_{mn}^{(1)} \sin m x \cos (n \pi y \delta^{-1})],$$

with:

(1.7.26 *a*) $$\int_0^{\delta} \cos (n \pi y \delta^{-1})\, dy = 0;$$

$$\int_0^{\delta} \sin (n \pi y \delta^{-1})\, dy = -(\pm 1 - 1)\, \delta\, (\pi n)^{-1}.$$

Representing in all the series the trigonometric terms by means of power series in x, which series, as it is well known, are convergent for all finite values of x, and comparing the equal powers of x enables one to find the approximate values of the coefficients $a_{mn}^{(1)}$, $b_{mn}^{(1)}$, $c_{mn}^{(1)}$ and $d_{mn}^{(1)}$.

The following procedure may be adopted:

from $\overline{m}_u$ we find ρu;
from $I_{xx} = u_1 \overline{m}_u$ we find ρu^2;
from $I_{xy} = v_1 \overline{m}_u$ we find $\rho u v$;
from H_x we find h_x and T.

This procedure enables one to find the approximate expressions for the following functions: u, v, ρ, T; using these expressions one can find the values of p, p_{xx}, p_{xy}, p_{yy} etc., from equations (1.3.5 to 1.3.9) satisfying the boundary conditions at $y = 0$ and $y = \delta$ by the method described in [8].

As mentioned above, to begin the successive approximations, one can use the values of the compressible fluid flow, derived by Crocco and

Lees [3, p. 653]:

$$u, \rho u\, dy \div [3, \text{eq. } (2.\ 10a)]; \quad \delta \div [3, \text{eq. } (2.14)];$$

$$T_1\, T_s^{-1} \div [3, \text{eq. } (2.\ 17)]; \quad k\, (\equiv k_1) \div [3, \text{eq. } (2.16)].$$

1.8. *The f-and F-functions.*

In the approach of Crocco and Lees, the function f [3, p. 654, eq. (2.18)] plays an important role:

$$f = (\delta_i - \delta_i^* - \delta_i^{**})\, \delta_i\, (\delta_i - \delta_i^*)^{-2}, \tag{1.8.1}$$

and the relation $f = f(k)$ (the function k in Crocco-Lees paper corresponds to the function k_1 in the present work). The subscript i denotes the incompressible flow to which the corresponding compressible flow may be reduced by means of Stewartson's method, [3, p. 653]. Below, we shall derive the expression for the generalized form of the function f, corresponding to eq. (1.8.1).

From eq. (1.4.15c) with the use of eq. (1.5.3) one easily obtains:

$$T_1\, T_s^{-1} = f - \frac{1}{2}(\gamma - 1)\, p_e^{-1}\, p_1\, k_1^2\, k_2\, W_e^2, \tag{1.8.2}$$

or

$$T_1\, T_s^{-1} = f - \frac{1}{2}(\gamma - 1)\, p_e^{-1}\, p_1\, k_1^2\, k_2\, w_1^2 \cos^{-2}\theta, \tag{1.8.2 a}$$

with

$$f = p_e^{-1}\, p_1\, k_1^2\, k_2. \tag{1.8.2 b}$$

This is an analogy to eq. (2.17) in [3]. In each of successive approximations one can calculate the functions f, k_1, k_2, p_1, p_e and thus have the relation $f = f(k_1;\ k_2)$ in hypersonic region corresponding to the Crocco-Lees function $f = f(k)$ in the supersonic regime. It seems possible, that many characteristic features of the boundary layer flow, discussed by Crocco and Lees, such as: the behaviour of and the conditions in the boundary layer in the vicinity of the separation point, the reattachment point, wakes, ect., can be generalized to the regime of the hypersonic flow by means of the formula (1.8.2) and the process of successive approximations.

It proves to be more convenient later to work with the quantity F, which is related to f by:

$$F = f\, k_1^{-2} - p_e^{-1}\, p_1, \tag{1.8.3}$$

or

(1.8.3 *a*) $$F = p_e^{-1} p_1 (\delta_u^* + \delta_u^{**}) (\delta - \delta_u^* - \delta_u^{**})^{-1},$$

or

(1.8.3 *b*) $$F = p_e^{-1} p_1 (k_1^2 k_2 - 1) = T_1 p_e^{(1-\gamma)/\gamma} (1 - k_1^{-2} k_2^{-1}).$$

1.9. *Mixing Rate*

As it is known, Blasius' consideration [1] leads to the result that in an incompressible fluid:

(1.9.1) $$\delta_i \sim C_1 x (Re_x)^{-1/2}; \quad \delta_i^* \sim C_2 x (Re_x)^{-1/2};$$

which implies that:

(1.9.2) $$\delta_i - \delta_i^* \sim (C \mu_0 x \rho_0^{-1} u_{e\,incom}^{-1})^{1/2}.$$

On the base of this consideration, Crocco and Lees have shown that in the boundary layer in a compressible fluid:

(1.9.3) $$d\overline{m}/dx = k \rho_e u_e; \quad k = C \mu_e \overline{m}^{-1},$$

where μ_e denotes the value of the coefficient of viscosity at the outer edge of the boundary layer. In the general case, one should consider not a constant c but a function $c = c(k)$. In the present case we preserve this form of relation and we put:

(1.9.4) $$d\overline{m}_e/dx = k \rho_e V_e; \quad k = \mu_e \overline{m}_e^{-1} C(k_1),$$

where the parametric function $c(k_1)$ supposed to be given. A comparison of equations (1.4.3) and (1.9.4) furnishes the expression:

(1.9.5) $$k = (d\delta/dx - \tan\theta) \cos\theta,$$

or

(1.9.5 *a*) $$k \cos^{-1}\theta + \tan\theta = d\delta/dx.$$

2. *REDUCTION OF THE SYSTEM OF EQUATIONS*

2.1. *Transformation of the System*

With the aid of eqs. (1.4.15*a*), (1.8.2) and (1.8.3) equation (1.6.2*e*) is rewritten in the form:

(2.1.1) $$m_u k_1 \left\{F + p_e^{-1} p_1 \left[1 - \frac{1}{2} k_2 (\gamma - 1) W_e^2\right]\right\} = $$
$$= p_1 \delta \gamma w_e = p_1 \delta \gamma W_e \cos\theta.$$

This corresponds to eq. (2.27) in [3, p. 658]. Perform the following operations:

(*a*) differentiate eq. (2.1.1) with respect to x, divide the so obtained expression by $d\,m_u/dx$ in form of eqs. (1.6.2*c* and *d*), using eqs. (1.6.2*e*) and (1.9.5);

(*b*) divide eq. (1.6.2*a*) by $d\,m_u/dx$, using eqs. (1.6.2*e*) and (1.9.5);

(*c*) apply similar procedure to eqs. (1.6.2*b* and *f*).

In all the equations so obtained, eliminate the pressure p_1 by means of eq. (1.4.15*c*) with $T_e = p_e \rho_e^{-1}$, $\rho_e = p_e^{1/\Upsilon}$, etc., thus obtaining:

(2.1.2)
$$p_1 = p_e^{1/\Upsilon}\, T_1\, k_1^{-2}\, k_2^{-1},$$

with

(2.1.3)
$$p_e = p_s\left[1 - \frac{1}{2}\,(\Upsilon - 1)\, W_e^2\right]^{\Upsilon/(\Upsilon-1)}.$$

The so obtained system of four equations (2.1.4 – 10) is given in Appendix.

2.2. *Algebraic Systems*

Consider the system (2.1.4 – 10) to be an algebraic system of equations in four unknowns:

(2.2.1)
$$\frac{d\,k_1}{d\,ln\,m_u} = x_1;\quad \frac{d\,k_{12}}{d\,ln\,m_u} = x_2;\quad \frac{d\,W_e}{d\,ln\,m_u} = x_3;\quad \frac{d\,T_1}{d\,ln\,m_u} = x_4;$$

(2.2.2)
$$\sum_{n=1}^{n=4} a_{mn}\, x_n = b_m \quad (m = 1, 2, 3, 4).$$

The values of the coefficients a_{mn} and b_m are given in Appendix.

Using matrix representation the system (2.2.2) is given by:

(2.2.3)
$$A\,x = B;$$

(2.2.4)
$$A = \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix};$$

(2.2.5)
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix};\qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix},$$

x and B being column vectors. One can easily verify the value of the fundamental determinant:

(2.2.6) $$D[A] \equiv D = a_{32}\, a_{44}\, (a_{13}\, a_{21} - a_{11}\, a_{23}),$$

or

(2.2.6*a*) $$D = \chi\, c_p\, k_1\, W_e \cos^2\theta\, \{\gamma A + F)\, [1 - \frac{1}{2}(\gamma - 1)\, W_e^2]^{-1}$$
$$- A(\gamma - 1)\, [k_2^{-1} - \frac{1}{2}(\gamma - 1)\, W_e^2]^{-1}\}\, W_e^2 - (4F + 2A).$$

Since, in general, $D \neq 0$, except possibly at a finite set of points, the system (2.1.4 to 10) has a solution. The remaining four determinants are:

(2.2.7) $$D_1 = - a_{32}\, [a_{23}\, (a_{44}\, b_1 - a_{14}\, b_4) - a_{13}\, a_{44}\, b_2];$$

(2.2.8) $$D_2 = a_{11}\, a_{44}\, (a_{33}\, b_2 - a_{23}\, b_3) - a_{21}\, a_{33}\, (a_{44}\, b_1 - a_{14}\, b_4);$$

(2.2.9) $$D_3 = - a_{32}\, [a_{11}\, a_{44}\, b_2 + a_{21}\, (a_{14}\, b_4 - a_{44}\, b_1)];$$

(2.2.10) $$D_4 = - a_{32}\, b_4\, (a_{11}\, a_{23} - a_{13}\, a_{21}),$$

and

(2.2.11) $$x_i = D^{-1} D_i \quad (i = 1, 2, 3, 4).$$

Since, in general, the x_i's ($i = 1, 2, 3, 4$) are determined, except possibly at a finite set of points, one can write:

(2.2.12) $$d/d\, ln\, m_u = (d\, W_e / d\, ln\, m_u)\, d/d\, W_e = x_3\, d/d\, W_e\,;$$

(2.2.13) $$d/d\, W_e = (d\, F/d\, W_e)\, d/d\, F.$$

This implies:

(2.2.14) $$d\, F/d\, W_e = D_1\, D_3^{-1}\, d\, F/d\, k_1\,;$$

(2.2.15) $$d\, k_{12}/d\, W_e = D_2\, D_3^{-1};$$

(2.2.16) $$d\, T_1/d\, W_e = D_4\, D_3^{-1}.$$

This corresponds to eq. (2.30) in [3]. Other values for these derivatives are given in the Appendix (eqs. 2.2.16 to 18). In all the expressions for the coefficients a_i's and b_i's, the differentiation with respect to $ln\, m_u$ is achieved by means of eq. (2.2.12); the value of k_2 may be calculated after other functions are known; the value of δ may be calculated by means of a modified eq. (1.9.5*a*):

(2.2.17) $$d\,\delta / d\, W_e = [k\, (\cos\theta)^{-1} + \tan\theta]\, (d\, W_e/dx)^{-1}.$$

To calculate the derivative $d\, W_e/dx$, the following procedure is applied:

(2.2.18) $$d\, W_e/dx = (d\, W_e / d\, ln\, m_u)\, (d\, ln\, m_u/dx) = x_3\, m_u^{-1}\, d\, m_u/dx\,;$$

with the use of formulas (1.6.2*c*, *d*), (1.9.5), (2.1.1), (2.1.2) and (2.1.3) one gets:

$$\frac{d\,m_u}{dx} = k_1^3 k_2 (F + A)\, M^{-1} \left\{ k\, p_s \left[1 - \frac{1}{2}(\gamma - 1)\, W_e^2 \right]^{1/(\gamma-1)} \gamma\, W_e + \rho_w z_w a_s^2 \right\}, \tag{2.2.19}$$

with A given by (2.1.8) (Appendix), F by (1.8.3b) as $F = F(W_e)$ and M by (2.2.20):

$$M = T_1 p_s^{1/\gamma} \delta \gamma\, W_e \cos\theta \left[1 - \frac{1}{2}(\gamma - 1)\, W_e^2 \right]^{1/(\gamma-1)}. \tag{2.2.20}$$

Since k, $F = F(k_1)$ are presumed to be known and $\theta = \theta(W_e)$ is given by the Prandtl-Meyer relation or by any other relation, the system (2.2.14, 15, 16) is integrable by some numerical methods.

2.3. *Summary*

Fundamentally, the above system furnishes the following functions:

$m_u = m_u(W_e)$; $F = F(W_e)$; $k_{12} = k_{12}(W_e)$; $T_1 = T_1(W_e)$, $k_1 = k_1(W_e)$,

with $\theta = \theta(W_e)$ or $W_e = W_e(\theta)$ being known. Hence one may calculate:

u_1 from eq. (1.4.15*a*);
v_1 from eq. (1.4.15*b*);
ρ_1 from eq. (1.4.15*c*) or from (1.4.8a);
p_1, T_1 from eq. (1.8.3*b*).

Next, one can apply the procedure explained in Section 1.7. to find the values of u, v, ρ, T, p, which inserted into the set of initial equations (1.3.5 to 9) jointly with the boundary conditions furnish the values of the functions p_{xx}, p_{xy}, p_{yy}, s_x, s_y. As the initial values for u, v, ρ, etc., one can use the values from the compressible flow regime obtainable by means of the Crocco-Lees procedure. The process of successive approximations may be carried on to the required degree of accuracy.

3. *PARTICULAR CASES*

3.1. *Wakes. Critical Point*

In wakes the values of the components of the shearing stress tensor at the wall and the values of the components of the heat flux vector at the wall are equal to zero. Crocco and Lees [3] assume that in the wake the value of the coefficient $k(\equiv k_1)$ is constant but this may be subject to some objections. Without this assumption but with the knowledge of the value of the function $F = F(k_1)$, the above system furnishes the values of the functions in question in the wake.

When $D = 0$, all D_i's $(i = 1, 2, 3, 4)$ must vanish. This furnishes a system of 5 equations in 5 unknown functions and as the additional information furnished one can assume θ_{cr}. The singular point obtained in this way, is a critical point, which probably will have the same meaning as that one in the original approach of Crocco and Lees, i. e., „throat" of the wake. It seems obvious that the conclusions derived by Crocco and Lees in the present approach certainly are valid to „the first approximations". Due to the complexity of the equations it is impossible to derive general conclusions from the systems, presented above; certain number of particular, numerical examples may help enormously in the extension of the Crocco-Lees results from the regime of the supersonic flow to the regime of the hypersonic flow.

3.2. *Final Remarks*

A similar procedure as the one explained above, can be applied to all the other phenomena, discussed in [3]: base pressure problem, separated and reattaching flows, recompression in wakes, etc. Using the results given in [3] all these problems can be reproduced and extended to the hypersonic flow regime. The slip-flow regime can also be treated; the corresponding boundary conditions will complicate the problem enormously; but the previous papers of the author on some problems of the analogous character [8] have shown that there is possible to obtain an approximate solution.

APPENDIX

Section 1.1.

Navier-Stokes:

(1.1.4 *a*)
$$\overset{*}{p}_{ij} = -\left[\mu^* (\overset{*}{u}_{i,j^*} + \overset{*}{u}_{j,i^*}) - \frac{2}{3}\mu^* \overset{*}{u}_{l,l^*}\delta_{ij}\right];$$

(1.1.4 *b*)
$$\frac{1}{2}\overset{*}{s}_i = \overset{*}{q}_i = -\lambda^* \, T^*_{,i^*};$$

Burnett (with the coefficients corrected according to [10]):

$$(1.1.5\,a)\qquad \overset{\star}{p}_{ij} = -\,2\,\mu^{\star}\,\overset{\star}{e}_{ij} + K_1\,\mu^{\star 2}p^{\star -1}\,\overset{\star}{u}_{k,k^\star}\,\overset{\star}{e}_{ij}$$

$$+ K_2\,\mu^{\star 2}p^{\star -1}\big[-\overline{(\rho^{\star -1}\overset{\star}{p}_{,j^\star})_{,i^\star}} - \overline{\overset{\star}{u}_{k,i^\star}\,\overset{\star}{u}_{j,k^\star}} - \overline{2\,\overset{\star}{e}_{ik}\,\overset{\star}{u}_{k,j^\star}}\big]$$

$$+ K_3\,\mu^{\star 2}\,(\rho^{\star}\,T^{\star})^{-1}\,\overline{\overset{\star}{T}_{,i^\star j^\star}} + K_4\,\mu^{\star 2}\,(p^{\star}\,\rho^{\star}\,T^{\star})^{-1}\,\overline{p_{,i^\star}\,\overset{\star}{T}_{,j^\star}}$$

$$+ K_5\,\mu^{\star 2}\,(\rho^{\star}\,T^{\star 2})^{-1}\,\overline{\overset{\star}{T}_{,i^\star}\,\overset{\star}{T}_{,j^\star}} + K_6\,\mu^{\star 2}\,p^{\star -1}\,\overline{\overset{\star}{e}_{ik}\,\overset{\star}{e}_{kj}}\,;$$

where the used symbols denote:

$$(1.1.5b)\qquad K_1 = \frac{4}{3}\left(\frac{7}{2} - T^{\star}\,\mu^{\star -1}\,\overset{\star}{\mu}_{,T^\star}\right);\quad K_2 = 2,\quad K_3 = 3;$$

$$K_4 = 0;\quad K_5 = 3\,T^{\star}\,\mu^{\star -1}\,\overset{\star}{\mu}_{,T^\star};\quad K_6 = 8\,;$$

$$(1.1.5c)\qquad \overset{\star}{e}_{ij} = \frac{1}{2}(\overset{\star}{u}_{i,j^\star} + \overset{\star}{u}_{j,i^\star}) - \frac{1}{3}\,\overset{\star}{u}_{k,k^\star}\,\delta_{ij}\,;$$

$$\overline{A}_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}\,A_{kk}\,\delta_{ij}\,;$$

$$(1.1.5d)\qquad \overset{\star}{q}_i = -\,\lambda^{\star}\,\overset{\star}{T}_{,i^\star} + \theta_1\,\mu^{\star 2}\,(\rho^{\star}\,T^{\star})^{-1}\,\overset{\star}{u}_{j,j^\star}\,\overset{\star}{T}_{,i^\star}$$

$$+ \theta_2\,\mu^{\star 2}\,(\rho^{\star}\,T^{\star})^{-1}\left[\frac{2}{3}(T^{\star}\,\overset{\star}{u}_{j,j^\star})_{,i^\star} + 2\,\overset{\star}{u}_{j,i^\star}\,\overset{\star}{T}_{,j^\star}\right]$$

$$+ [\theta_3\,\mu^{\star 2}\,(\rho^{\star}\,p^{\star})^{-1}\,\overset{\star}{p}_{,j^\star} + \theta_4\,\mu^{\star 2}\,\rho^{\star -1}\,\partial/\partial\,\overset{\star}{x}_j$$

$$+ \theta_5\,\mu^{2}\,(\rho^{\star}\,T^{\star})^{-1}\,\overset{\star}{T}_{,j^\star}]\,\overset{\star}{e}_{ji}\,;$$

$$(1.1.5e)\qquad \theta_1 = \frac{15}{4}\left(\frac{7}{2} - T^{\star}\,\mu^{\star -1}\,\overset{\star}{\mu}_{,T^\star}\right) = \frac{45}{16}K_1\,;\quad \theta_2 = -\,\frac{45}{8}:$$

$$\theta_3 = -\,3;\quad \theta_4 = 3;\quad \theta_5 = 3\left(\frac{35}{4} + T^{\star}\,\mu^{\star -1}\,\overset{\star}{\mu}_{,T^\star}\right);$$

Grad:

$$(1.1.6a)\qquad \overset{\star}{p}_{ij,t^\star} + (\overset{\star}{p}_{ij}\,\overset{\star}{u}_r)_{,r^\star} + \frac{1}{5}(\overset{\star}{s}_{i,j^\star} + \overset{\star}{s}_{j,i^\star} - \frac{2}{3}\,\overset{\star}{s}_{r,r^\star}\,\delta_{ij})$$

$$+ \overset{\star}{p}_{ir}\,\overset{\star}{u}_{j,r^\star} + \overset{\star}{p}_{jr}\,\overset{\star}{u}_{i,r^\star} - \frac{2}{3}\,\overset{\star}{p}_{rs}\,\overset{\star}{u}_{r,s^\star}\,\delta_{ij}$$

$$+ p^{*}\,(\overset{\star}{u}_{i,j^\star} + \overset{\star}{u}_{j,i^\star} - \frac{2}{3}\,\overset{\star}{u}_{r,r^\star}\,\delta_{ij}) + p^{*}\,\mu^{*-1}\,\overset{\star}{p}_{ij} = 0;$$

(1.1.6 *b*)
$$\overset{*}{s}_{i,t^*} + (\overset{*}{s}_i \overset{*}{u}_r)_{,r^*} + \frac{7}{5}\overset{*}{s}_r \overset{*}{u}_{i,r^*} + \frac{2}{5}\overset{*}{s}_r \overset{*}{u}_{r,i^*}$$
$$+ \frac{2}{5}\overset{*}{s}_i \overset{*}{u}_{r,r^*} + 2 R^* T^* \overset{*}{p}_{ir,r^*} + 7 \overset{*}{p}_{ir}(R^* T^*)_{,r^*}$$
$$- 2 \rho^{*-1} \overset{*}{p}_{ir}(\overset{*}{p}_{rs} - p^* \delta_{rs})_{,s^*} + 5 p^* (R^* T^*)_{,i^*} + \frac{5}{2} R^* p^* \lambda^{*-1} \overset{*}{s}_i = 0.$$

Section 1.3.

(1.3.5)
$$(p_{xx} u)_{,x} + (p_{xx} v)_{,y} + \frac{2}{15}(2 s_{x,x} - s_{y,y})$$
$$+ \frac{2}{3}(2 p_{xx} u_{,x} + 2 p_{xy} u_{,y} - p_{yx} v_{,x} - p_{yy} v_{,y})$$
$$+ \frac{2}{3} p (2 u_{,x} - v_{,y}) + \alpha^{-1} \beta \mu^{-1} p p_{xx} = 0;$$

(1.3.6)
$$(p_{xy} u)_{,x} + (p_{xy} v)_{,y} + \frac{1}{5}(s_{x,y} + s_{y,x}) + p_{xx} v_{,x}$$
$$+ p_{xy}(u_{,x} + v_{,y}) + p_{yy} u_{,y} + p (u_{,y} + v_{,x})$$
$$+ \alpha^{-1} \beta \mu^{-1} p p_{xy} = 0;$$

(1.3.7)
$$(p_{yy} u)_{,x} + (p_{yy} v)_{,y} + \frac{2}{15}(2 s_{y,y} - s_{x,x})$$
$$+ \frac{2}{3}(2 p_{yy} v_{,y} + 2 p_{yx} v_{,x} - p_{xy} u_{,y} - p_{xx} u_{,x})$$
$$+ \frac{2}{3} p (2 v_{,y} - u_{,x}) + \alpha^{-1} \beta u^{-1} p p_{yy} = 0;$$

(1.3.8)
$$(s_x u)_{,x} + (s_x v)_{,y} + \frac{11}{5} s_x u_{,x} + \frac{2}{5} s_x v_{,y} + \frac{7}{5} s_y u_{,y}$$
$$+ \frac{2}{5} s_y v_{,x} + 2 \alpha^{-1} R T (p_{xx,x} + p_{xy,y}) + 7 \alpha^{-1} [p_{xx} (R T)_{,x}$$
$$+ p_{xy} (R T)_{,y}] - 2 \alpha^{-1} \rho^{-1} \{p_{xx} [(p_{xx} + p)_{,x} + p_{xy,y}]$$
$$+ p_{xy} [p_{yx,x} + (p + p_{yy})_{,y}]\} + 5 \alpha^{-1} p (R T)_{,x}$$
$$+ \frac{5}{2} \alpha^{-1} \beta \lambda^{-1} R p s_x = 0;$$

(1.3.9) $$(s_y u)_{,x} + (s_y v)_{,y} + \frac{2}{5} s_x u_{,y} + \frac{7}{5} s_x v_{,x} + \frac{2}{5} s_y u_{,x}$$

$$+ \frac{11}{5} s_y v_{,y} + 2\alpha^{-1} R T (p_{yx,x} + p_{yy,y})$$

$$+ 7\alpha^{-1} [p_{yx} (RT)_{,x} + p_{yy} (RT)_{,y}]$$

$$- 2\alpha^{-1} \rho^{-1} \{p_{yx} [(p + p_{xx})_{,x} + p_{xy,y}] + p_{yy} [p_{yx,x}$$

$$+ (p + p_{yy})_{,y}]\} + 5\alpha^{-1} p (RT)_{,y} + \frac{5}{2} \alpha^{-1} \beta \lambda^{-1} R p s_y = 0.$$

Section 2.1

(2.1.4) $$\frac{d}{d\, ln\, m_u} [k_1 (F + A)] - k_1 (F + A) \frac{d}{d\, ln\, m_u} \times$$

$$\times \{ln [p_s^{1/\Upsilon} T_1 (1 - \frac{1}{2} (\Upsilon - 1) W_e^2)^{1/(\Upsilon-1)} k_1^{-2} k_2^{-1} W_e \cos\theta]\} =$$

$$= (1 + k^{-1} \sin\theta) p_s^{(1-\Upsilon)/\Upsilon} T_1 k_1^{-2} k_2^{-1} \{1 + \rho_w z_w a_s^2 \left[1 - \frac{1}{2} (\Upsilon - 1) W_e^2\right] \times$$

$$\times \left[\Upsilon p_s \left(1 - \frac{1}{2} (\Upsilon - 1) W_e^2\right]^{\Upsilon/(\Upsilon-1)} k W_e\right]^{-1} \Big\}^{-1} - k_1 (F + A);$$

(2.1.5) $$\frac{d}{d\, ln\, m_u} (k_1 W_e \cos\theta) + k_1 (F + A) (\alpha \Upsilon \varphi_e \cos\theta)^{-1} \times$$

$$\times \frac{d\, ln\, W_e}{d\, ln\, m_u} k_1^2 k_2 B^{-1} \left[d \int_0^\delta (p + p_{xx}) dy - p_s \left(1 - \frac{1}{2} (\Upsilon - 1) W_e^2\right)^{\Upsilon/(\Upsilon-1)} d\delta\right] / d W_e =$$

$$= W_e \cos\theta (1 - k_1 - \varphi_e \rho_w z_w a_s^2 C^{-1}) + \varphi_e (\alpha^{-1} p_{xyw} + \rho_w w_w z_w a_s^2) C^{-1};$$

(2.1.6) $$\frac{d}{d\, ln\, m_n} (k_{12} W_e \cos\theta) + k_1 (F + A) (\alpha \Upsilon \varphi_e \cos\theta)^{-1} \times$$

$$\times \frac{d\, ln\, W_e}{d\, ln\, m_u} k_1^2 k_2 B^{-1} d \int_0^\delta p_{xy}\, d y / d W_e =$$

$$= W_e \sin\theta (1 - k_1 - \varphi_e \rho_w z_w a_s^2 C^{-1}) + \varphi_e \{\alpha^{-1} [p_w + p_{yyw}$$

$$- p_s \left(1 - \frac{1}{2} (\Upsilon - 1) W_e^2\right)^{\Upsilon/(\Upsilon-1)}\Big] + \rho_w z_w^2 a_s^2\} C^{-1};$$

$$(2.1.7)\quad \frac{d(\chi c_p T_1)}{d \ln m_u} + \frac{1}{2} a_s k_1^3 k_2 (F + A) (\Upsilon \varphi_e B \cos \theta)^{-1} \cdot \frac{d \ln W_e}{d \ln m_u} \frac{d \int_0^\delta s_x \, d y}{d W_e}$$

$$= \chi c_p T_e (1 - \varphi_e \rho_w z_w a_s^2 C^{-1})$$

$$- \chi c_p T_1 + a_s \varphi_e \left[\frac{1}{2} s_{y_w} + h_{y_w} + \int_0^\delta \left(\frac{D p}{D t} - \Phi\right) d y\right] C^{-1};$$

$$(2.1.8)\quad A = p_s^{(1-\Upsilon)/\Upsilon} T_1 \left[1 - \frac{1}{2}(\Upsilon - 1) W_e^2\right]^{-1} k_1^{-2} \left[k_2^{-1} - \frac{1}{2}(\Upsilon - 1) W_e^2\right];$$

$$(2.1.9)\quad B = \delta \Upsilon p_s^{1/\Upsilon} T_1 W_e \left[1 - \frac{1}{2}(\Upsilon - 1) W_e^2\right]^{(2-\Upsilon)/(\Upsilon-1)};$$

$$(2.1.10)\quad C = k p_s \left[1 - \frac{1}{2}(\Upsilon - 1) W_e^2\right]^{\Upsilon/(\Upsilon-1)} + \varphi_e \rho_w z_w a_s^2.$$

Section 2.2.

$$a_{11} = (3F + A); \quad a_{12} = 0;$$

$$a_{13} = k_1 W_e (\Upsilon A + F) [1 - \frac{1}{2}(\Upsilon - 1) W_e^2]^{-1}$$

$$- A(\Upsilon - 1) k_1 W_e [k_2^{-1} - \frac{1}{2}(\Upsilon - 1) W_e^2]^{-1} - k_1 (F + A) W_e^{-1};$$

$$a_{44} = - k_1 F T_1^{-1};$$

$$b_1 = (1 + k^{-1} \sin \theta) p_s^{(1-\Upsilon)/\Upsilon} T_1 k_1^{-2} k_2^{-1} \{1 +$$

$$+ \rho_w z_w a_s^2 [1 - \frac{1}{2}(\Upsilon - 1) W_e^2]^{1/(1-\Upsilon)} \cdot \Upsilon p_s k W_e\}^{-1}$$

$$- k_1 (F + A) (1 + \tan \theta \, d\theta / d \ln m_u) - k_1 \, dF / d \ln m_u$$

$$+ k_1 k_2^{-1} \{A [1 - \frac{1}{2}(\Upsilon - 1) k_2 W_e^2]^{-1} - (F + A)\};$$

$$a_{21} = W_e \cos \theta; \quad a_{22} = 0; \quad a_{23} = k_1 \cos \theta; \quad a_{24} = 0;$$

$$b_2 = W_e \cos \theta (1 - k_1 - \varphi_e \rho_w z_w a_s^2 C^{-1})$$

$$+ \varphi_e (\alpha^{-1} p_{xy_w} + \rho_w w_w z_w a_s^2) C^{-1} + k_1 W_e \sin \theta \frac{d\theta}{d \ln m_u}$$

$$- k_1 (F + A)(\alpha \gamma \varphi_e \cos\theta)^{-1} k_1^2 k_2 (W_e B)^{-1} \times$$

$$\times \{d \int_0^\delta (p + p_{xx})\, d y - p_s [1 - \frac{1}{2}(\gamma - 1) W_e^2]^{\gamma/(\gamma-1)} d\delta\}/d \ln m_u ;$$

$$a_{31} = 0; \quad a_{32} = W_e \cos\theta; \quad a_{33} = k_{12} \cos\theta; \quad a_{34} = 0;$$

$$b_3 = W_e \sin\theta (1 - k_1 - \varphi_e \rho_w z_w a_s^2 C^{-1}) + \varphi_e \{\alpha^{-1} [p_w + p_{yyw}$$

$$- p_s \left(1 - \frac{1}{2}(\gamma - 1) W_e^2\right)^{\gamma/(\gamma-1)}] + \rho_w z_w^2 a_s^2\} C^{-1} - k_{12} W_e \frac{d\theta}{d \ln m_u} \sin\theta$$

$$- k_1 (F + A)(\alpha \varphi_e \gamma \cos\theta)^{-1} k_1^2 k_2 (W_e B)^{-1} d \int_0^\delta p_{xy}\, dy/d \ln m_u ;$$

$$a_{41} = 0; \quad a_{42} = 0; \quad a_{43} = 0; \quad a_{44} = \chi c_p ;$$

$$b_4 = \chi c_p T_e (1 - \varphi_e \rho_w z_w a_s^2 C^{-1}) - \chi c_p T_1$$

$$+ a_s \varphi_e \left[\frac{1}{2} s_{yw} + h_{yw} + \int_0^\delta \left(\frac{Dp}{Dt} - \Phi\right) dy\right] C^{-1}$$

$$- \frac{1}{2} a_s k_1^3 k_2 (F + A)(\gamma \varphi_e B W_e \cos\theta)^{-1} d \int_0^\delta s_x\, dy/d \ln m_u ;$$

(2.2.16)
$$d F/d W_e = [a_{23} (a_{44} b_1 - a_{14} b_4) - a_{13} a_{44} b_2] \times$$
$$\times [a_{21} (a_{14} b_2 - a_{44} b_1) + a_{11} a_{44} b_2]^{-1};$$

(2.2.17)
$$d k_{12}/d W_e = [a_{11} a_{44} (a_{33} b_2 - a_{23} b_3)$$
$$+ a_{21} a_{33} (a_{14} b_4 - a_{44} b_1)]\{- a_{32} [a_{21} (a_{14} b_2 - a_{44} b_1) + a_{11} a_{44} b_2]\}^{-1};$$

(2.2.18)
$$d T_1/d W_e = b_4 (a_{11} a_{23} - a_{13} a_{21}) [a_{21} (a_{14} b_4 - a_{44} b_1) + a_{11} a_{44} b_2]^{-1}.$$

The subscript w denotes the values at the surface of the solid body (wall of the flat plate).

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