REMARK ON FATOU-RIESZ'S THEOREM

by

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1. *INTRODUCTION.* Let $A(u)$ be for $u \geqslant 0$ of bounded variation over every finite segment and

$$L(s) = \int_0^\infty e^{-su}\, d\{A(u)\}$$

convergent for $\Re(s) > 0$.

Let $L(s)$ satisfy the following assumptions:

I) Within $|\Im(s)| \leqslant 2T$, $\Re(s) > 0$ with a fixed T we have

$$|L(s)| \leqslant M$$

so that

$$\lim_{\sigma = +0} L(\sigma + it) = Q(t) \quad \text{for nearly all} \quad |t| \leqslant 2T$$

exists.

II)

$$H(y) = \frac{1}{\pi} \int_{-2T}^{+2T} \frac{\sin yt}{t}\, Q(t)\, dt \to Q, \qquad y \to \infty .$$

M. Riesz has proved the following fundamental theorem:

THEOREM 1. *From* I), II) *and*

(1.1) $$A(u') - A(u) \to 0 \quad \textit{for all} \quad u \leqslant u' \leqslant u + h,\ u \to \infty$$

follows

$$A(u) \to Q, \quad u \to \infty. \tag{1.2}$$

On the other side, A. E. Ingham [3] and J. Karamata [4] have proved, in connection with the method introduced by N. Wiener [5] and S. Ikehara [2] that the Riesz's Theorem may be proved on the basis of the following two theorems:

THEOREM A. *From* I), II) *and*

$$A(u) = O(u) \tag{1.3}$$

follows that

$$\int_0^{\infty} A(u)\, du \, \frac{T}{\pi} \int_{u-y}^{u+y} \left(\frac{\sin Tt}{Tt}\right)^4 dt = Q + o(1), \quad y \to \infty. \tag{1.4}$$

THEOREM B. *From* (1.1) *and* (1.4) *it follows* (1.2).

From Theorems A and B follows the Riesz's Theorem 1 whilst again it will be shown that (1.3) is an elementary consequence of (1.1) and $L(s) \to Q,\ s \to 0$.

In what follows I will replace 1) the in (1.4) appearing $\left(\frac{\sin x}{x}\right)^4$ by an as general as possible $k(x)$ and 2) the in (1.3) behind the sign O figuring u by an as great as possible $A^*(u)$. In order to state briefly the Theorem which is to be proved, let us define three classes of functions $\varkappa$, $\varkappa_1$ and α as follows:

DEFINITION. 1. $k(x) \in \varkappa$ if

a) $k(x) \geqslant 0$ for $-\infty < x < \infty$,

b) $k(x) \in L^1$, [1])

c) $\displaystyle\int_{-\infty}^{+\infty} \frac{|\lg K(x)|}{1+x^2}\, dx < \infty$ with $\displaystyle K(x) = \int_x^{\infty} k(\tau)\, d\tau$

and

d) $\sqrt{K(x)} \in L^2$.

[1]) $g(x) \in L^p\ (p > 0)$ means: $|g(x)|^p$ is over $(-\infty, +\infty)$ integrable.

DEFINITION 2. $k(x) \in \varkappa_1$ if

$$\text{a) } k(x) \in \varkappa,$$

$$\text{b) } \sqrt{K(x)} \in L^1,$$

and

$$\text{c) } K(x) \lg(1+|x|) \in L^1.$$

DEFINITION 3. $A(x) \in \alpha$ if

a) a such $A^*(x) \geqslant 0$ exists that $A(x) + A^*(x) \geqslant 0$,

and

$$\text{b) } A^*(u) \int_{u-y}^{u+y} k(\tau)\, d\tau \quad \in L^1 \text{ for every } y.$$

Besides $A^*(u) \equiv 0$ for $u < 0$.

Then we have the following generalisation of Karamata's [4] Theorem A.

THEOREM I. *From* I), $A(x) \in \alpha$ *and* $k(x) \in \varkappa$ *it follows: There exists a* $h_1^*(t) \in L^1$ *with* $h_1^*(t) \equiv 0$ *for* $|t| \geqslant 2T$ *such that*

$$H_1(y) = \frac{1}{\pi} \int_{-2T}^{+2T} \frac{\sin yt}{t} h_1^*(t)\, Q(t)\, dt = \int_0^\infty A(u)\, du \int_{u-y}^{u+y} k(\tau)\, d\tau.$$

B) *Moreover, provided that* II) *holds and* $k(x) \in \varkappa_1$ *then we have*

$$H_1(y) = Q + o(1), \quad y \to \infty.$$

It is easy to verify that the Theorem I implies the Theorem A as a special case.

The proof of Theorem I is based on the following

LEMMA (Wiener-Paley [5]). *Let, for* $-\infty < x < \infty$, $f(x) \geqslant 0$ *and* $f(x) \in L^2$.

In order that, for every $T > 0$ *could be found such a* $h(t)$ *with* $h(t) \equiv 0$ *for* $|t| \geqslant T$ *that*

$$H(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(t)\, e^{-ixt}\, dt,$$

should satisfy the condition

$$|H(x)| = f(x)$$

it is necessary and sufficient that

$$\int_{-\infty}^{+\infty} \frac{|\lg f(x)|}{1+x^2}\, dx < \infty .$$

For the present purpose, we shall not use but a half of Wiener-Paley's Lemma (sufficiently).

2. *LEMMAS.* Let $C_1, C_2, \ldots$ be fixed numbers resp. functions of variables appearing within the brackets.

Moreover, let us for brevity sake put

$$h_1(t) = \int_{-T}^{+T} \bar{h}(-u)\, h(t-u)\, du \tag{2.1}$$

so that

$$h_1(t) \equiv 0 \quad \text{for} \quad |t) \geqslant 2T .$$

LEMMA 1. *Provided that the significance of $f(x)$ and $h(x)$ be that, as explained in Wiener-Paley's Lemma and $h_1(t)$ be defined by* (2.1), *then we have*

$$\frac{i}{\pi} \int_{-2T}^{+2T} e^{-iut} h_1(t) \sin yt\, dt = - \int_{u-y}^{u+y} \frac{d}{d\xi} \{f^2(\xi)\}\, d\xi.$$

Proof: As

$$\bar{H}(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{+T} \bar{h}(-t)\, e^{-ixt}\, dt \tag{2.2}$$

so then, while $h(t) \in L^1$,

$$|H(x)|^2 = H(x)\bar{H}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixt}\, dt \int_{-T}^{+T} \bar{h}(-u)\, h(t-u)\, du =$$

$$= \frac{1}{2\pi} \int_{-2T}^{+2T} e^{-i(u+y)t} h_1(t)\, dt =$$

$$= \pi\, |H(u-y)|^2 - \pi\, |H(u+y)|^2 .$$

But as $|H(x)| = f(x)$, the assertion is so proved.

LEMMA 2. *Let $f(x)$ have the significance assigned to in Wiener-Paley's Lemma and $h_1(t)$ be defined by* (2.1). *Moreover, let $f(x) \in L^1$ and $f^2(x) \lg(1+|x|) \in L^1$. Then $h_1(t)$ satisfies the condition*

$$\int_{-2T}^{+2T} \left| \frac{h_1(t+\varepsilon) - h_1(t)}{t} \right| d\varepsilon < C_3(T).$$

Proof. Since

$$|H(x+\varepsilon) - H(x)| \leqslant \frac{1}{\sqrt{2\pi}} \int_{-T}^{+T} |h(t)|\,|1 - e^{-i\varepsilon t}|\,dt \to 0, \quad \varepsilon \to 0,$$

$H(x)$ is continuous in x. Besides $H(x) \in L^1$ because $f(x) \in L^1$. Therefore ([1], p. 51)

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H(t)\, e^{ixt}\, dt$$

for every x. Consequently

$$h_1(t+\varepsilon) - h_1(t) = \int_{-T}^{+T} \bar{h}(-u)\{h(t+\varepsilon-u) - h(t-u)\}\,du =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-T}^{+T} \bar{h}(-u)\,du \int_{-\infty}^{+\infty} H(\tau)\{e^{i(t+\varepsilon-u)\tau} - e^{i(t-u)\tau}\}\,d\tau,$$

Further, since $|H(x)| \in L^1$ and (2.2), we have

$$h_1(t+\varepsilon) - h_1(t) = \int_{-\infty}^{+\infty} H(\tau)\{e^{i(t+\varepsilon)\tau} - e^{it\tau}\} \frac{d\tau}{\sqrt{2\pi}} \int_{-T}^{+T} \bar{h}(-u)\, e^{-iu\tau}\,du =$$

$$= \int_{-\infty}^{+\infty} |H(\tau)|^2 \{e^{i(t+\varepsilon)\tau} - e^{it\tau}\}\,d\tau =$$

$$= 2 \int_{-\infty}^{+\infty} |H(\tau)|^2 \left\{\sin\left(t + \frac{\varepsilon}{2}\right)\tau + i\cos\left(t + \frac{\varepsilon}{2}\right)\right\} \sin\frac{\varepsilon\tau}{2}\,d\tau$$

and also

$$|h_1(t+\varepsilon)-h_1(t)| \leqslant 4 \int_{-\infty}^{+\infty} |H(\tau)|^2 \left|\sin\frac{\varepsilon\tau}{2}\right| d\tau .$$

Hence

$$\int_{-2T}^{+2T} \left|\frac{h_1(t+\varepsilon)-h_1(t)}{\varepsilon}\right| d\varepsilon \leqslant 4 \int_{-\infty}^{+\infty} f^2(\tau)\, d\tau \int_{-T\tau}^{+T\tau} \left|\frac{\sin u}{u}\right| du$$

$$\leqslant C_2(T) \int_{-\infty}^{+\infty} f^2(\tau)\, d\tau + \int_{-\infty}^{+\infty} f^2(\tau) \lg(1+|\tau|)\, d\tau = C_3(T).$$

3. *PROOF OF THEOREM* I. According to the definition 1 $\sqrt{K(x)}$ fulfils the same assumption as $f(x)$ of Wiener-Paley's Lemma. We put

$$K(x) = |H(x)|^2 = f(x)$$

and have to prove the Theorem I with

$$h_1^*(x) = h_1(x) \quad \text{and} \quad k(x) = -\frac{d}{dx}(|H(x)|^2).$$

A. a) Without restraining the assumptions let $A(0)=0$. Then we have (integration by parts)

$$L(s) = s\int_0^{\infty} e^{-su} A(u)\, du \quad \text{for} \quad \Re(s) > 0.$$

b) For $|t| \leqslant 2T$ and $\sigma > 0$ we have

$$\left|\frac{\sin yt}{\sigma+it} h_1(t) L(\sigma+it)\right| = \left|\frac{\sin yt}{t} \frac{t}{\sigma+it} h_1(t) L(\sigma+it)\right|$$

$$\leqslant C_4(y) M = C_5(y).$$

So we have

$$\left|\frac{i}{\pi}\int_{-2T}^{+2T} \frac{\sin yt}{\sigma+it} h_1(t) L(\sigma+it)\, dt\right| \leqslant C_6(y, T) \tag{3.1}$$

and (Lebesgue's Theorem)

$$H_1(y) = \lim_{\sigma=+0} \frac{i}{\pi} \int_{-2T}^{2T} \frac{\sin yt}{\sigma + it} h_1(t) L(\sigma + it)\, dt = \frac{1}{\pi} \int_{-2T}^{2T} \frac{\sin yt}{t} h_1(t) Q(t)\, dt.$$

c) From the convergence of $L(s)$ for $\Re(s) > 0$ we have

$A(u) \exp[-\sigma u] \in L^1$ (with $A(u) \equiv 0$ for $u < 0$)

and therefore

$$\frac{i}{\pi} \int_{-2T}^{+2T} \frac{\sin yt}{\sigma + it} h_1(t) L(\sigma + it)\, dt = \int_0^\infty A(u) e^{-\sigma u}\, du \, \frac{i}{\pi} \int_{-2T}^{2T} e^{-iut} h_1(t) \sin yt\, dt$$

so that

$$= \int_0^\infty A(u) e^{-\sigma u}\, du \int_{u-y}^{u+y} k(\tau)\, d\tau$$

according to Lemma 1.

d) On account of the definition 3

$$\left| \frac{1}{\pi} \int_0^\infty A^*(u) e^{-\sigma u}\, du \int_{u-y}^{u+y} k(\tau)\, d\tau \right| \leqslant C_7$$

and besides uniformly in $\sigma \geqslant 0$, so that on account of (3.1)

$$\psi_\sigma(y) = \int_0^\infty e^{-\sigma u} (A(u) + A^*(u))\, du \int_{u-y}^{u+y} k(\tau)\, d\tau$$

fulfils the inequality

$$|\psi_\sigma(y)| \leqslant C_6(y, T) + C_7.$$

But, since $A(u) + A^*(u) \geqslant 0$, $\psi_\sigma(y) \uparrow$ if $\sigma \downarrow 0$ there exists

$$\psi(y) = \lim_{\sigma=+0} \psi_\sigma(y).$$

Hence

$$\int_0^\infty (A(u) + A^*(u))\, du \int_{u-y}^{u+y} k(\tau)\, d\tau$$

is converging and yields the value

$$H_1(y) + \int_0^\infty A^*(u)\, du \int_{n-y}^{u+y} k(\tau)\, d\tau.$$

It follows then

(3.2) $$H_1(y) = \int_0^{\infty} A(u)\, du \int_{u-y}^{u+y} k(\tau)\, d\tau$$

and so the first part of Theorem I is proved.

B. I have still to prove that: *When* $f(x) \in L^1$, $f^2(x) \lg(1 + |x| \in L^1$ *and* $H(y) \to Q$, $y \to \infty$, *then we have*

$$H_1(y) = \frac{1}{\pi} \int_{-2T}^{+2T} \frac{\sin yt}{t} h_1(t)\, Q(t)\, dt \to Q \int_{-\infty}^{+\infty} f^2(x)\, dx.$$

On account of Lemma 2 and $|Q(t)| \leqslant M$

$$\frac{h_1(t) - h_1(0)}{t} Q(t)$$

is absolutely integrable over $(-2T, +2T)$. Therefore (Dini's Theorem)

$$H_1(y) - h_1(0)\, H(y) = \frac{1}{\pi} \int_{-2T}^{+2T} \frac{h_1(t) - h_1(0)}{t} Q(t) \sin yt\, dt \to 0, \quad y \to \infty.$$

Hence, the second part of Theorem I is also proved, since (Plancherel's Theorem)

$$h_1(0) = \int_{-T}^{+T} h(-u)\, \overline{h}(-u)\, du = \int_{-\infty}^{+\infty} |H(x)|^2\, dx$$

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