ON THE GREEN'S FUNCTION OF THE BIHARMONIC OPERATOR

by

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SUMMARY — An estimation of Green's function of the biharmonic equation $\Delta\Delta u - \lambda u = 0$ in the complex λ plane is given.

1. *INTRODUCTION.* Let S denote a bounded domain in the xy-plane with a regular boundary S', so that the existence of the Green's function of the following boundary value problem

$$(1)\qquad \begin{gathered} \Delta\Delta u - \lambda u = 0 \\ u = 0, \quad \frac{\partial u}{\partial n} = 0, \quad \text{on } S \end{gathered}$$

is guaranteed.

Further, let Π, P and Q be points of S, r the distance between P and Q, and l_p the minimum distance between P and points on S'. m_1, m_2, $m_3, \ldots$; $M_1, M_2, M_3, \ldots$ are numbers not depending on P and Q.

If $G(P, Q; \lambda)$ denotes the Green's function of (1) and

$$(2)\qquad R(P, Q; -\lambda) = \frac{1}{\sqrt{\lambda}} \int_1^\infty e^{-\sqrt[4]{\lambda}\, r t/\sqrt{2}} \sin(\sqrt[4]{\lambda}\, r\, t/\sqrt{2}) \frac{dt}{\sqrt{t^2-1}}$$

the fundamental solution of $\Delta\Delta u + \lambda u = 0$, then we have [4]

$$(3)\qquad G(P, Q; -\lambda) = \frac{1}{2\pi} R(P, Q; -\lambda) - \gamma(P, Q; -\lambda),$$

$\gamma(P, Q; -\lambda)$ being a solution of $\Delta\Delta u + \lambda u = 0$ satisfying the following conditions on S':

$$\gamma(P, Q; -\lambda) = R(P, Q; -\lambda), \quad \frac{\partial \gamma(P, Q; -\lambda)}{\partial n} = \frac{\partial R(P, Q; -\lambda)}{\partial n}.$$

Pleijel [4] showed in 1940, by the use of the calculus of variation, that

$$|\gamma(P, P; -\lambda)| \leqslant \frac{m_1}{l_p} \lambda^{-3/4},$$

and Bojanić and Vučković [2] in 1953, by a similar method, that

$$|\gamma(P, Q; -\lambda)| \leqslant C(l_p)\, e^{-l_p \sqrt[4]{\lambda}/2\sqrt{2}},$$

both results holding for the real values of λ only.

The aim of this note is to give an inequality satisfied by $G(P, Q; -\lambda)$ and valid in the whole complex λ-plane except along the negative part of the real axis.

This idea to estimate the Green's function of an elliptic partial differential boundary value problem as a function of complex variable has been for the first time stated by V. Avakumović, in a not yet published paper [1]. Here it will be shown how a modification of his method can be applied on a more complicated case.

2.1. *RESULTS.* With the above notations we have the following

THEOREM I. *Let P be fixed, $\theta = \arg \lambda$ and*

$$\Omega(\lambda; l_p) = e^{-l_p/2 \Re(\sqrt[4]{\lambda} e^{-\pi i/4})} + e^{-l_p/2 \Re(\sqrt[4]{\lambda} e^{\pi i/4})}$$

then

$$\left| G(P, Q; -\lambda) - \frac{1}{2\pi} R(P, Q; -\lambda) \right| \leqslant \frac{C(l_p)}{\sqrt[8]{|\lambda|^5} \left| \sin \frac{\theta - \pi}{2} \right|} \Omega(\lambda; l_p)$$

for all $-\pi < \theta < \pi$, and $Q \in S$, where $C(l_p) = m_2/l_p^{15/2}$.

The proof of this theorem depends on the following three lemmas:

LEMMA 1. *Let λ_ν denote the eigenvalues of* (1), *and Φ_ν the associated eigenfunctions; then we have*

$$E(Q; x) = \sum_{\lambda_\nu \leqslant x} \Phi_\nu^2(Q) \leqslant M_1 \sqrt{x}$$

for all $x > 0$.

LEMMA 2. *The Green's function* $G(P, Q; -\lambda)$ *satisfies the following relation*

$$\int_S |G(\Pi, Q; -\lambda)|^2 dF_\Pi \leqslant \frac{m_3}{|\lambda|^{3/2} \sin^2 \frac{\theta - \pi}{2}}.$$

LEMMA 3. *Let* $\xi(r)$ *be the function with first four continuous derivatives, having the following properties:*

(4) $$\xi(r) = \begin{cases} 1 & \text{for} \quad 0 \leqslant r \leqslant l_p/2, \\ 0 & \text{for} \quad r \geqslant l_p, \end{cases}$$

$$|\xi(r)| \leqslant 1, \quad |\xi'(r)| \leqslant \frac{m_4}{r}, \quad |\xi''(r)| \leqslant \frac{m_4}{r^2}, \quad |\xi'''(r)| \leqslant \frac{m_4}{r^3}, \quad |\xi^{IV}(r)| \leqslant \frac{m_4}{r^4}.$$

If $\varkappa_l$ *denotes the circular ring between two circles of radii* $l_p/2$, l_p *respectively, and*

(5) $$\Gamma(P, \Pi; -\lambda) = \frac{1}{2\pi} (\Delta \Delta + \lambda)_\Pi \xi(r) R(P, \Pi; -\lambda),$$

then

$$\int_{\varkappa_l} |\Gamma(P, \Pi; -\lambda)|^2 dF_\Pi \leqslant \frac{m_5 \sqrt[4]{|\lambda|}}{l_p^{15}} \Omega(\lambda; l_p),$$

2.2. *PROOF OF THE THEOREM I.* Instead of the usual expression of the Green's function (3), we shall write

(6) $$G(P, Q; -\lambda) = \frac{1}{2\pi} \xi(r) R(P, Q; -\lambda) - \gamma^*(P, Q; -\lambda).$$

Hence, according to the fundamental property of the Green's function, we have

$$(\Delta \Delta + \lambda)_\Pi \gamma^*(P, \Pi; -\lambda) = \Gamma(P, \Pi; -\lambda).$$

This can easily be transformed, using the corresponding Green's formula [4], into a linear homogenous integral equation, i. e. into

$$\gamma^*(P, Q; -\lambda) = -\int_S G(\Pi, Q; -\lambda) \Gamma(P, \Pi; -\lambda) dF_\Pi.$$

Hence, by the Cauchy-Schwarz's inequality and taking care of (4)

(7) $$|\gamma^*(P, Q; -\lambda)| \leqslant \left(\int_S |G(\Pi, Q; -\lambda)|^2 dF_\Pi \int_{\varkappa_l} |\Gamma(P, \Pi; -\lambda)|^2 dF_\Pi \right)^{1/2}.$$

Now, according to Lemmas 2 and 3 it follows from (7) that

$$(8)\qquad |\Upsilon^*(P,Q;-\lambda)| \leqslant \frac{m_6}{l_p^{15/2}\sqrt[8]{|\lambda|^5}\left|\sin\frac{\theta-\pi}{2}\right|}\,\Omega(\lambda;l_p)$$

and from (3), (6) and (4) that

$$(9)\quad \Upsilon(P,Q;-\lambda) = \begin{cases} \Upsilon^*(P,Q;-\lambda) & 0 \leqslant r \leqslant \frac{l_p}{2}, \\ \frac{1}{2\pi}(1-\xi(r)\,R(P,Q;-\lambda)+\Upsilon^*(P,Q;-\lambda) & \frac{l_p}{2} < r \leqslant l_p, \\ \frac{1}{2\pi}R(P,Q;-\lambda)+\Upsilon^*(P,Q;-\lambda) & r > l_p, \end{cases}$$

so that, according to (8), (4) and (9), we have

$$|\Upsilon(P,Q;-\lambda)| \leqslant \frac{C(l_p)}{\sqrt[8]{|\lambda|^5}\left|\sin\frac{\theta-\pi}{2}\right|}\,\Omega(\lambda;l_p),$$

where $C(l_p)=\frac{m_6}{l_p^{15/2}}$, and so the Theorem I is established.

We still have to prove the Lemmas 1—3.

3.1. *PROOF OF THE LEMMA 1.* a) From the Pleijel's result

$$\left|\sum_{\nu=1}^{\infty}\frac{\Phi_\nu^2(Q)}{\lambda_\nu+\lambda}-\frac{1}{8\sqrt{\lambda}}\right| \leqslant \frac{m_1}{l_Q}\lambda^{-3/4}$$

[4] it follows

$$\sum_{\nu=1}^{\infty}\frac{\Phi_\nu^2(Q)}{\lambda_\nu+\lambda} \leqslant \frac{m_7}{8\sqrt{\lambda}}$$

for all $\lambda > m_8/l_Q^4$, where $m_7 = 1 + m_1$; but

$$E(Q;x) \equiv \sum_{\lambda_\nu \leqslant x}\Phi_\nu^2(Q) \leqslant (x+\lambda)\sum_{\lambda_\nu\leqslant x}\frac{\Phi_\nu^2(Q)}{\lambda_\nu+\lambda} \leqslant \frac{m_7(x+\lambda)}{8\sqrt{\lambda}}$$

and so, putting $\lambda = x$,

$$(10)\qquad E(Q;x) \leqslant M_1\sqrt{x}$$

for all $x > \frac{m_8}{l_Q^4} = \tau_Q$.

b) Let $G(P, Q)$ denote the Green's function of the following boundary value problem

$$\Delta \Delta u = 0 \quad , P \in S,$$

$$u = 0, \frac{\partial u}{\partial n} = 0, \; P \in S'.$$

Here $G(P, Q)$ is the kernel of the corresponding integral equation, satisfying the conditions of Mercer's theorem, so that it can be expanded in a bilinear series

$$G(P, Q) = \sum_{\nu=1}^{\infty} \frac{\Phi_\nu(P)\, \Phi_\nu(Q)}{\lambda_\nu},$$

which converges absolutely and uniformly. Hence

$$\sum_{\nu=1}^{\infty} \frac{\Phi_\nu^2(Q)}{\lambda_\nu} < M_2,$$

or

$$\text{(11)} \qquad M_2 > \int_{\lambda_1}^{\infty} \frac{dE(Q; x)}{x} = \int_{\lambda_1}^{\tau_Q} + \int_{\tau_Q}^{\infty} = J_1 + J_2 \geqslant J_1,$$

since J_2 is evidently positive. We have to prove that (10) still holds when $x \leqslant \tau_Q$. Suppose, on the contrary, that

$$\text{(12)} \qquad E(Q; x) \geqslant M(l_Q)\sqrt{x}$$

for all $x \leqslant \tau_Q$.

Integrating in J_1 by part we get

$$J_1 = \frac{E(Q; \tau_Q)}{\tau_Q} - \frac{E(Q; \lambda_1)}{\lambda_1} + \int_{\lambda_1}^{\tau_Q} x^{-2} E(Q; x)\, dx,$$

hence, according to (12),

$$J_1 > \frac{E(Q; \tau_Q)}{\tau_Q} - \frac{E(Q; \lambda_1)}{\lambda_1} + M(l_Q) \int_{\lambda_1}^{\tau_p} x^{-3/2}\, dx$$

$$> 2M(l_Q)\left(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\tau_Q}}\right) - \frac{E(Q; \lambda_1)}{\lambda_1}.$$

Since we can choose l_Q arbitrary small, we shall take it so that $\frac{1}{\sqrt{\tau_Q}} \leqslant \leqslant \frac{1}{2\sqrt{\lambda_1}}$. Hence it follows

$$J_1 > \frac{M(l_Q)}{\sqrt{\lambda_1}} - \frac{E(Q;\lambda_1)}{\lambda_1} \tag{13}$$

for $\frac{1}{\tau_Q} \leqslant \frac{1}{4\lambda_1}$.

From (11) and (13) we have at once

$$M_2 \geqslant \frac{M(l_Q)}{\sqrt{\lambda_1}} - \frac{E(Q;\lambda_1)}{\lambda_1},$$

or

$$M(l_Q) \leqslant M_2 \sqrt{\lambda_1} + \frac{E(Q;\lambda_1)}{\sqrt{\lambda_1}} \leqslant M_3,$$

and so

$$E(Q;x) \leqslant M_3 \sqrt{x}$$

for all $x \leqslant m_3/l_Q^4$, which completes the proof.

3.2 *PROOF OF THE LEMMA 2.* From

$$\int_S |G(\Pi, Q; -\lambda)|^2 \, dF_\Pi = \frac{G(Q, Q; -\lambda) - G(Q, Q; -\bar{\lambda})}{\bar{\lambda} - \lambda}$$

([3], p. 141) and from

$$G(Q, Q; -\lambda) = \sum_{\nu=1}^{\infty} \frac{\Phi_\nu^2(Q)}{\lambda_\nu + \lambda}$$

[2], it follows

$$\int_S |G(\Pi, Q; -\lambda)|^2 \, dF_\Pi = \sum_{\nu=1}^{\infty} \frac{\Phi_\nu^2(Q)}{|\lambda_\nu + \lambda|^2},$$

Hence, by the evident inequality

$$|\lambda_\nu + \lambda| = |\lambda_\nu - \lambda e^{-\pi i}| \geqslant \sin\frac{\theta - \pi}{2} (\lambda_\nu + |\lambda|),$$

valid for all θ 's between $-\pi$ and π,

$$J_1 \equiv \int_S |G(P,Q;-\lambda)|^2 dF_\Pi \leqslant \frac{1}{\sin^2 \frac{\theta-\pi}{2}} \sum_{\nu=1}^{\infty} \frac{\Phi_\nu^2(Q)}{(\lambda_\nu + |\lambda|)^2} =$$

$$= \frac{1}{\sin^2 \frac{\theta-\pi}{2}} \int_0^\infty \frac{d\,E(Q;x)}{(x+|\lambda|)^2} = \frac{2}{\sin^2 \frac{\theta-\pi}{2}} \int_0^\infty \frac{E(Q;x)}{(x+|\lambda|)^3} dx;$$

but according to Lemma 1

$$\int_0^\infty \frac{E(Q;x)}{(x+|\lambda|)^3} \leqslant M_1 \int_0^\infty \frac{\sqrt{x}}{(x+|\lambda|)^3} dx.$$

Putting $|\lambda| t = x$ we finally have

$$J_1 \leqslant \frac{m_3}{|\lambda|^{3/2} \sin^2 \frac{\theta-\pi}{2}}. \qquad \text{q. e. d.}$$

3.3 *PROOF OF THE LEMMA 3.* We can split the fundamental solution $R(P,Q;-\lambda)$ as follows ([4], p. 92)

$$R(P,Q;-\lambda) = \frac{1}{\sqrt{\lambda}} \frac{K_0(r\sqrt[4]{\lambda}\, e^{-\pi i/4}) - K_0(r\sqrt[4]{\lambda}\, e^{\pi i/4})}{2i},$$

where

$$K_0(z) = \int_0^\infty e^{-zt} \frac{dt}{\sqrt{t^2-1}}.$$

Now, we can, according to (14), express $R(P,Q;-\lambda)$ and the first four derivatives of it in terms of the modified Bessel's functions of the first kind $K_\nu(z)$, so that the right side of (5) can be effectively calculated without difficulty.

Moreover, by the use of the well-known relation

$$K_\nu(z) \leqslant \frac{m_9}{\sqrt{|z|}} e^{-\Re(z)}, \quad |\arg z| < \frac{\pi}{2}$$

([5] p. 202), we can estimate the right side of (5), so that, after a considerable calculation, we get

$$2\pi \Gamma(P,\Pi;-\lambda) \leqslant$$

$$\leqslant \frac{m_9}{\sqrt{r}\,\sqrt[8]{|\lambda|}\,\sqrt{|\lambda|}}\left(\frac{m_{10}}{r^4}+\frac{m_{11}\sqrt[4]{|\lambda|}}{r^3}+\frac{m_{12}\sqrt{|\lambda|}}{r^2}+\frac{m_{13}\sqrt{|\lambda|}\,\sqrt[4]{|\lambda|}}{r}\right)\Omega(\lambda,2r) \leqslant$$

$$\leqslant \frac{m_{14}\sqrt[4]{|\lambda|}}{l_p^4\sqrt{r}}\,\Omega(\lambda;2r) \quad \text{for} \quad \frac{l_p}{2} < r \leqslant l_p,$$

but

$$\Gamma(P,\Pi;-\lambda) = \begin{cases} 0 & \text{for } 0 \leqslant r \leqslant \frac{l_p}{2}, \\ 0 & \text{for } r > l_p, \end{cases}$$

according to (4).

Hence it follows

$$J_2 \equiv \int_{\varkappa_l} |\Gamma(P,\Pi;-\lambda)|^2\,dF_\Pi \leqslant \frac{m\sqrt{|\lambda|}}{l_p^{16}}\int_{l_p/2}^{l_p}\Omega(\lambda;2r)\,dr,$$

or

$$J_2 \leqslant \frac{m_{15}\sqrt[4]{|\lambda|}}{l_p^{15}}\,\Omega(\lambda;l_p), \qquad \text{q. e. d.}$$

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