A REMARK ON THE PRECEDING PAPER BY A. RÉNYI

by

M. KAC (Cornell University, Ithaca, N. Y., USA)

The curious and interesting identity

$$(1) \qquad \sum_{k=0}^{\infty} \rho_k z^k = \prod_p \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p-z}\right),$$

discovered and proved by A. Rényi in the preceding paper can be approached by an entirely different method based on the probabilistic notion of independence.

Let

$$n = \prod_p p^{\alpha_p(n)},$$

then it is easily seen that the functions $\alpha_p(n)$ are independent i. e.

$$(2) \qquad D\{\alpha_{p_1}(n) = k_1, \alpha_{p_2}(n) = k_2, \ldots, \alpha_{p_r}(n) = k_r\} = \prod_{s=1}^{r} D\{\alpha_{p_s}(n) = k_s\},$$

where $D\{\ldots\ldots\}$ denotes the density of the set of integers defined inside the braces.

Now,

$$(3) \qquad U(n) - V(n) = \sum_p \beta_p(n),$$

where

$$(4) \qquad \beta_p(n) = \begin{cases} d_p(n) - 1, & \alpha_p(n) \geqslant 1, \\ 0, & \alpha_p(n) = 0. \end{cases}$$

Clearly the functions $\beta_p(n)$ are also independent.

Now,

$$\rho_r = D\{U(n) - V(n) = k\} = \tag{5}$$
$$= \lim_{N\to\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{-i\xi k} \frac{1}{N} \sum_{n=1}^{N} e^{i\xi(U(n) - V(n))}\, d\xi$$

and if for every (or almost every ξ)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{i\xi(U(n) - V(n))} \tag{6}$$

exists, it follows (by dominated convergence) that

$$\rho_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\xi k} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} e^{i\xi(U(n) - V(n))}\, d\xi. \tag{7}$$

Formally, rising the independence of $\beta_p(n)$, we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{i\xi(U(n) - V(n))} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} exp\ \{i\xi \sum_p \beta_p(n)\} = \tag{8}$$
$$= \prod_p \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{i\xi\beta_p(n)},$$

and cleanly

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{i\xi\beta_p(n)} = 1 - \frac{1}{p} + \left(\frac{1}{p} - \frac{1}{p^2}\right) + \left(\frac{1}{p^2} - \frac{1}{p^3}\right) e^{i\xi} + \cdots = \tag{9}$$
$$= \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p - e^{i\xi}}\right)$$

which together with (8) and (7) gives

$$\rho_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\xi k} \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p - e^{i\xi}}\right) d\xi \tag{10}$$

which is equivalent to (1).

The problem is thus reduced to the justification of second equality in (8).

It follows trivially from the independence of the functions $\beta_p(n)$ that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} exp\{i\xi \sum_{p\leq p_r} \beta_p(n)\} = \prod_{p\leqslant p_r}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{i\xi\beta_p(n)} \tag{11}$$

which unfortunately does not immediately imply (8) because density is only a finitely additive measure.

However, because of a special property of functions one can deduce (8) from (11).

In fact, let

$$R_r(n) = \sum_{p>p_r} \beta_p(n) \tag{12}$$

and note that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} R_r(n) = \sum_{p>p_r}\sum_{k=2}(k-1)\left(\frac{1}{p^k}-\frac{1}{p^{k+1}}\right) < \tag{13}$$

$$< \sum_{p>p_r}\frac{1}{(p-1)^2} = o(r).$$

Thus the upper density of the set of integers for which $R_r(n) \geqslant 1$ is $o(r)$ and consequently

$$\overline{\lim_{N\to\infty}}\frac{1}{N}\left|\sum_{n=1}^{N} exp\{i\xi\Sigma\beta_p(n)\} - exp\{i\xi\sum_{p\leq p_r}\beta_p(n)\}\right| \leqslant$$

$$\leqslant \overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}|e^{i\xi R_r(n)}-1| \leqslant 2\overline{D}\{R_r(n)>0\} = 2o(r), \tag{14}$$

where $\overline{D}\{R_r(n)>0\}$ is the upper density of integers for which $R_r(n)>0$ {or equivalently $R_r(n)\geqslant 1$}. Formulas (14) and (11) clearly imply (8).

(Received 21 december 1955)