SOME PRINCIPLES OF INDUCTION*

by

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SUMMARY — In the article we formulate the manner how to exhaust: 1) any set (Th. 1.1), 2) any ordered set (Th. 2.1), 3) any ranged set (Th. 3.1), 4) any ramified set (Th. 4.1). We get so 4 induction principles.

The main idea of induction consists in the problem how to exhaust a set. The most important induction principle is the principle of complete induction dealing with the set N of natural numbers. The principle can be expressed as follows (cf. Kurepa [4]).

If a set M satisfies the following two conditions:

A_1 $$1 \in M;$$

B_1 $$\text{if} \quad n \in N, \quad n \in M \quad \text{then} \quad n+1 \in M;$$

then the set M contains every natural number:

$$M \supseteq N.$$

In the paper we shall indicate some principles of induction each of which is dealing with a special kind of sets exhausted by special means, respectively. Especially, we shall indicate the *principle of induction for any set* (v. Th. 1.1), *for any ordered set* (Th. 2.1), *for any ranged set* (Th. 3.1), for any ordered set having no gaps and satisfying a certain condition called *ramification condition* (Th. 4.1); the last case presents a Khintchine's (Hinčin) induction principle dealing with the set of real numbers (cf. Khintchine [1] and [2]). The classical principle *(method of exhaustion)* can be interpreted as a certain exhaustion of all *„interior points“* of a given set.

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1. *PRINCIPLE OF INDUCTION FOR ANY SET S*

THEOREM 1.1. *In order that a set M contains S:*

$$M \supseteq S$$

the necessary and sufficient condition is that for every $X \subseteq S$ *satisfying* $X \neq S$, $X \subseteq M$, *there is a greater part* fX *of* S *so that* $fX \subseteq M$.

The condition is necessary. If $M \supseteq S$ and if $X \subset S$, it is sufficient to put $fX = S$ to be sure that

$$X \subset fX \subseteq M.$$

The condition is sufficient. Namely, if S_0 denotes the union of all $X \subseteq S \cap M$, obviously

(1.1) $$S_0 \subseteq S \cap M; \quad \text{but} \quad S_0 = S;$$

otherwise, by the condition of the theorem, there would be a

(1.2) $$f(S_0) \supset S_0 \quad \text{so that} \quad f(S_0) \subseteq M; \quad \text{thus} \quad f(S_0) \subseteq S \cap M$$

and consequently, by the definition of S_0:

$$f(S_0) \subseteq S,$$

contrarily to the hypothesis (1.2). Thus $S_0 = S$, and the relation (1.1) implies $S \subseteq M$.

2. *PRINCIPLE OF INDUCTION FOR ANY ORDERED SET S*

DEFINITION 2.1. For any point a of an ordered set $S = (S; \leqslant)$, let

(2.1) $$(-\infty, a]_S \quad \text{and} \quad (-\infty, a)_S$$

respectively denote the set of all $x \in S$ so that

(2.2) $$x \leqslant a \quad \text{and} \quad x < a$$

respectively.

DEFINITION 2.2. Every subset $P \subseteq S$ so that

$$\text{if} \quad x \in P \quad \text{then} \quad (-\infty, x]_S \subseteq P$$

is called *initial portion* of S. The empty set is considered also as an initial portion of S.

THEOREM 2.1. *In order that a set M contains an ordered set S, it is necessary and sufficient that for every initial portion P of S so that* $P \subseteq M$ *there is a greater initial portion fP of S contained in M, unless already* $P = S$.

In the case that S be completely ordered, the theorem 2.1. is identical with the theorem 1 p. 23 in Kurepa [3]. The proof in general case is the same as in this special case.

The necessity of the condition being obvious, let us prove that the condition of the theorem 2.1 is sufficient to conclude that $S \subseteq M$. In fact, let S_0 be the union of all initial portions P of S so that $P \subseteq M$. Obviously, S_0 is an initial portion of S and $S_0 \subseteq M$. If $S_0 \subset S$ and not $S_0 = S$, fS_0 would be also an initial portion of S contained in M. By the definition of S_0 the portion fS_0 would occur as a term in the union for S_0, thus $S_0 \supseteq fS_0$, contrarily to the hypothesis that $f(S_0) \supset S_0$.

3. *RANGED SETS*

DEFINITION 3.1. For any ordered set S we denote by

$$R_0 S \tag{3.1}$$

the *initial row* (rangée) of S consisting of all initial points of S (i. e. of points $x \in S$ satisfying $(-\infty, x)_S = \text{void}$).

DEFINITION 3 2. The initial row $R_0 S$ of S is *complete*, if every point of S is comparable with at least one point of $R_0 S$.

DEFINITION 3.3. An ordered set S is *ranged*, if every non void subset X of S has an *initial* point i. e. if $o \neq X \subseteq S$ implies $R_0 X \neq o$.

THEOREM 3.1. (Principle of induction for ranged sets: conclusion from $(-\infty, x)_S$ to x). *Provided that the initial row, $R_0 S$, of ordered set S is complete, the following two properties are equivalent:*

Property $A_3(S)$: The set S is ranged;

Property $B_3(S)$: The following induction property holds: If a set M satisfies the folowing two conditions:

3.I. *M contains the set $R_0 S$;*

3.II. *M constains x provided that $x \in S$ and $(-\infty, x)_S \subseteq M$;*
then M contains S.

What regards the proof of the theorem 3.1 for the case of well ordered sets, see Sierpinski [5], p. 164.

§ 3.1. The property $A_3(S)$ implies the property $B_3(S)$: for any ranged set S, and for any set M satisfying 3.I and 3.II one has $M \supseteq S$. To prove it, let

$$P = \bigcup_x (-\infty, x]_S, \tag{3.1.2}$$

x running over all points x of S so that

(3.1.3) $$(-\infty, x]_S \subseteq M.$$

Obviously,

(3.1.4) $$P \subseteq M.$$

We say that $P = S$ and thus $S \subseteq M$. In the opposite case one would have $P \subset S$ and the non void set

(3.1.5) $$S \setminus P$$

as a part of the ranged set S would have at least one initial point, say i. Thus, P being an initial portion of S:

$$(-\infty, i)_S \subseteq P: \quad \text{therefore} \quad (-\infty, i)_S \subseteq M$$

and by the condition 3.II one would have $i \in M$. Consequently

$$(-\infty, i]_S \subseteq M \text{ and } i \in P,$$

contrarily to the fact that i be a first element of (3.1.5). Concludingly, the set (3.1.5) is empty, $S = P$ and, according to (3.1.4), $S \subseteq M$.

§ 3.2. The property $B_3(S)$ implies the property $A_3(S)$: if $R_0 S$ is complete and if every M satisfying 3.I and 3.II contains S, S is ranged. In opposite case, there would be a non-void subset $X \subseteq S$ having no initial point:

(3.2.1) $$R_0 X = \text{void}, \quad X \neq o.$$

Let

(3.2.2) $$M = \bigcup_x (-\infty, x]_S,$$

x running over all points of S satisfying

(3.2.3) $$(-\infty, x]_S \cap X = o.$$

Now, we should conclude that $M \supseteq S$, particularly $M \supseteq X \supset o$, contrarily to the immediate consequence

$$M \cap X = o$$

of the definition of M.

To prove $M \supseteq S$, it is sufficient, by hypothesis, to see that M would satisfy the conditions 3.I and 3.II.

At first, X having none initial point, X contains no point of $R_0 S$; thus, every $x \in R_0 S$ satisfies (3.2.2), hence $R_0 S \subseteq M$; i. e. M satisfies 3.I. Moreover, M satisfies 3.II too: if $a \in S$ and $(-\infty, a)_S \subseteq M$, thus $(-\infty, a)_S \cap X = \text{void}$, then $a \in M$ i. e. a satisfies (3.2.3).

Now,

$$(-\infty, a]_S = (-\infty, a]_S \cup (a),$$

thus

$$(-\infty, a]_S \cap X = (-\infty, a)_S \cap X \cup (a) \cap X = (a) \cap X = 0,$$

because if $(a) \cap X \neq o$, i. e. $a \in X$, the point a would be a first one in X, contrarily to the hypothesis (3.2.1).

Remark 3.2.1. In the special case that S is any well ordered set, particularly any set of ordinal numbers (e. g. the set of all ordinals $< \omega_1$), the second half of the theorem 3.1 is called the principle of transfinite induction.

4. *NON-LACUNARITY OF A CLASS OF ORDERED SETS AND AN INDUCTION PRINCIPLE*

We shall consider ordered sets S satisfying the following

DEFINITION 4.1. *(Ramification condition):* If $x \in S$, the set $(-\infty, x)_S$ of all points of S each of which precedes x is a *chain* of S (i. e. is a totally ordered set)[1].

DEFINITION 4.2. Let A, B be subsets of an ordered set S; we shall say that A is *coinitial* with B, if

$$B = \bigcup_a [a, \infty)_B, \ (a \in A), \tag{4.1}$$

$[a, \infty)_B$ denoting the set of all $x \in B$ so that $a \leqslant x$.

LEMMA 4.1. *S satisfying the ramifacation condition, let $x \in S$; every maximal chain of S so that $x \in C$ contains the whole set $(-\infty, x)_S$ as a part and is contained in the set $(-\infty, x, \infty)_S$ of all points of S, each of which is comparable with x.*

Khintchine [1] has proved that Dedekind's axiom of continuity is equivalent with a certain kind of exhaustion of the continuum of all real numbers. In a previous paper we have proved that the Khintchine's principle is a characteristic property of totally ordered sets having no gaps (v. Kurepa, [3], p. 23 Th. 3). Now, we shall prove the

THEOREM 4.1. *Let S be any ordered set satisfying the ramification condition; the following two properties of S are equivalent:*

Property $A_4(S)$: No maximal chain C of S presents a gap[2] *(non lacunarity property of S);*

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[1] The ramification condition occurs in the study of Suslin problem.

[2] That means, if A is any proper initial portion of C, it is impossible that neither A has a last element nor $S \setminus A$ has the first element.

Property $B_4(S)$: The following implication (induction principle) holds: Let M be any set satisfying the following two conditions:

4.I. *M contains an initial portion of S coinitial with S;*

4.II. *If $x \in S$ and $(-\infty, x)_S \subseteq M$, then $x \in M$; moreover, for every $x' \in S$ so that $x < x'$ there is a $x'' \in (x, x']_S$[1] satisfying $(-\infty, x'']_S \subseteq M$; Conclusion: $M \supseteq S$.*

As every totally ordered set satisfies the ramification condition the theorem 4.1 implies the

COROLLARY 4.1. *For every totally ordered set S the following two properties are equivalent:*

Property $A(S)$: none Dedekind's cutting of S presents a gap;

Property $B(S)$: The following induction principle in S holds: Let M be any set so that

1) *there is a non-void initial portion of S contained in M;*

2) *if $x \in S$ and $(-\infty, x)_S \subseteq M$ then $x \in M$; moreover if x is not the last element of S, there is a $x' \in (x, \infty)_S$ so that $(-\infty, x')_S \subseteq M$;*

then $M \supseteq S$ (cf. Khintchine [1], [2] in the case that S is the ordered set of all real numbers and Kurepa [3] p. 23 for any totally ordered set S).

§ 4.1. *The property $A_4(S)$ implies the property $B_4(S)$.* S satisfying the condition of ramification and having the property $A_4(S)$, let M by any set satisfying 4.I and 4.II; we say that M contains S. To prove it, let

(4.1.1) S_0 be the set of all points $x \in S$ so that

(4.1.2) $$(-\infty, x]_S \subseteq M.$$

Of course

(4.1.3) $$S_0 \subseteq M, \quad S_0 \subseteq S.$$

We say that

(4.14) $$S_0 = S.$$

Otherwise there would be a point

(4.1.5) $$x' \in S \setminus S_0;$$

let

(4.1.6) C be a *maximal chain* containing the point x';

the decomposition

(4.1.6) $$C = (S_0 \cap C) \cup (S \setminus S_0) \cap C$$

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1) Of course, $(x, x']_S$ denotes the set of all $y \in S$ satisfying either $x < y \leqslant x'$ or $x > y \geqslant x'$.

is a cutting of the chain C. None of the two terms in (4.1.6) would be void: the second containing x' and the first containing a predecessor, say a, of x'. Namely, the coinitiality of M with S implies the existence of a point $a \in M$ so that $a \leqslant x'$; the set S satisfying the ramification condition, the relation $x' \in C$ implies (v. Lemma 4.1) $a \in C$.

Now, the set $(S \setminus S_0) \cap C$ has no initial element i, because otherwise one would have $(-\infty, i)_S \subseteq M$ and, by the condition 4.II, also $i \in M$, thus $(-\infty, i]_S \subseteq M$ and consequently $i \in S_0$.

The second component of (4.1.6) having no initial element, the first one would have, according to the non-lacunarity of C, the *last* element, say l. Thus $(-\infty, l]_S \subseteq M$. Now $l \leqslant x' \in S$ and in virtue of the condition 4.II, there is a $x'' \in S$ so that $l < x'' \leqslant x'$ and $(-\infty, x'')_S \subseteq M$, and hence, according to the same condition, $x'' \in M$.

Now, by ramification condition the set $(-\infty, x'']_S$ is totally ordered; as it is a part of the chain C (v. lemma 4.1), one has

$$(-\infty, x'']_S = (-\infty, x'']_C,$$

and the point l would be no last point of the first component, the point $x'' > l$ belonging to M and thus to S_0.

§ 4.2. *The property* $B_4(S)$ *implies the property* $A_4(S)$; S satisfying both the ramification condition and the condition $B_4(S)$, every maximal chain of S is non-lacunar. In opposite case, there would be a maximal chain $C \subseteq S$ so that at least one proper initial portion I of C would not have the terminal point nor the remainder $C \setminus I$ the initial point. The proof will be achieved, when we shall prove that the set

$$M = S \setminus \bigcup_x [x, \infty)_S, \quad (x \in C \setminus I) \tag{4.2.1}$$

would satisfy

$$M \supseteq S, \tag{4.2.2}$$

what is a non-sense, because $S \supseteq C \setminus I$ and $M \cap (C \setminus I) = \text{void}$. Now, to prove (4.2.2) it is sufficient, in virtue of the property $B_4(S)$, to show that M satisfies the conditions 4.I and 4.II. At first, 4.I is satisfied. M is coinitial with S: if $t \in S$, there is an $a \in M$ so that $a \leqslant t$; this is obvious if t does not belong to the set

$$\bigcup_x [x, \infty)_S, \quad (x \in C \setminus I); \tag{4.2.3}$$

and, if t belongs to the set (4.2.3), we can choose for a any point of the set I – the above initial portion of C. The set M being also an initial portion of S, the condition 4.I is satisfied.

Let us prove that 4.II is satisfied too. Well,

(4.2.4) $$\text{if} \quad x \in S \quad \text{and} \quad (-\infty, x)_S \subseteq M, \quad \text{then} \quad x \in M$$

i. e. the point x does not belong to (4.2.3). Namely from $x \in$ (4.2.3), one would conclude that there is $a \in C \setminus I$ so that $a \leqslant x$. But the inequality $a < x$ is impossible, because, in virtue of (4.2.4), $a \in M$, what is in contradiction with $a \in C \setminus I$, the sets M and $C \setminus I$ having no common point. The equality $a = x$ is impossible too, because from $a = x$ we should conclude that a is the first point in $C \setminus I$, contrarily to the hypothesis that $C \setminus I$ has no such point. Thus the implication (4.2.4) is true. It remains to consider the case that there is a $x' \in S$ satisfying $x < x'$; we shall prove the existence of a $x'' \in (x, x']_S$ so that $(-\infty, x'')_S \subseteq M$. Obviously, it suffices to suppose $x' \in$ (4.2.3). Let then $a \in C \setminus I$ be so that $a \leqslant x'$. In virtue of the ramification condition, the points x, a as predecessors of x' are comparable: either $a \leqslant x$ or $a > x$. The case $a \leqslant x$ is not possible, because from $a \in C \setminus I$ we should conclude that $x \in$ (4.2.3), contrarily to $x \in M$. Thus $x < a$. This means that $x \in I$, because

$$I = M \cap (-\infty, a)_S, \quad (a \in C \setminus I).$$

The set I having no terminal point, it is sufficient to take for x'' any point of I coming after x, to be assured that $(-\infty, x'')_S \subseteq M$.

Thus condition 4.I and 4.II would hold, hence the contradiction (4.2.2).

The theorem 4.1 is completely proved.

5. *THE ROLE OF THE RAMIFICATION CONDITION IN THEOREM 4.1*

It is very interesting to observe that if an ordered set does not satisfy the ramification condition, the theorem 4.1 must not be true.

THEOREM 5.1. *For general ordered sets the conditions $A_4(S)$ and $B_4(S)$ are independent: there is an ordered set satisfying $B_4(S)$ and not $A_4(S)$* (v. example 5.1); *there is an ordered set satisfying $A_4(S)$ and not $B_4(S)$* (v. example 5.2).

Example 5.1. Let $C(o)$ be the set C of real numbers in which the ordering is changed inasmuch the point o is considered to be incomparable to every point of the set $(-\infty, o)_S$ of all points $< o$ in C.

Of course, $C(o) \setminus (o)$ is a maximal chain of $C(o)$ presenting a gap in the decomposition

$$C(o) \setminus (o) = (-\infty, o)_{C(o)} \cup (o, \infty)_{C(o)}.$$

What regards the set $C(o)$, the induction principle B_4 holds: If M satisfies 4.I and 4.II, then $M \supseteq C(o)$.

At first, M containing an initial portion of $C(o)$ coinitial with $C(o)$, M contains the point $o \in C(o)$, o being a first point of $C(o)$. In virtue of the condition 4.II the relation $o \in M$ and void set $= (-\infty, o)_{C(o)} \subseteq M$ implies for every $x > o$ the existence of a $x'' > o$ and $\leqslant x$ so that $(-\infty, x'\cdot)_{C(o)} \subseteq M$. Consequently, we conclude that what regards the set M, and the non-lacunar ordered sets $A = (-\infty, o)_{C(o)}$ and $B = (o, \infty)_{C(o)}$ respectively, we can apply the corollary 4.1 and conclude that

$$M \supseteq A, \quad M \supseteq B.$$

Thus

$$M \supset A \cup B.$$

That inclusion with $o \in M$ implies:

$$M \supseteq A \cup (o) \cup B = C(o).$$

Shortly, the induction principle B_4 is applicable for $S = C(o)$, although $C(o)$ is not non-lacunar. The reason is that $C(o)$ does not satisfy the ramification condition: for instance every $x > o$ is preceded by o and every $a < o$, on the one hand, and on the other hand, o and a are incomparable in $C(o)$.

Example 5.2. *(Construction of a non-lacunar set without property B_4).*

Let

(5.1) A be any ordered set satisfying the following conditions:

a) A is non lacunar,

b) D denoting the set of all integers, A contains a sequence s_n, $(n \in D)$ of pairwise incomparable points so that the sets

(5.2) $$(-\infty, s_n)_A, \quad (n \in D)$$

are $\neq o$, pairwise disjoint and

(5.3) $$A = \bigcup_n (-\infty, s_n, \infty)_A, \quad n \in D; \quad \text{here} \quad (-\infty, s_n, \infty)_A$$

denotes the set of all points of A comparable with s_n.

c) No $(s_n, \infty)_A$ has a last point.

The existence of such a set A is obvious; e. g. we can suppose that

$$A = \bigcup_{n \in D} (n, n+1)_C,$$

the sets $(n, n+1)_C$, $(n \in D)$ being considered pairwise incomparable: one can put $s_n = n + {}^1/_2$.

Now, let

(5.4) $$B = \{\dots b_{-n} < b_{-n+1} < \dots b_{-1} < b_0 < b_1 < b_1 \dots\}$$

be a totally ordered set similar with the ordered set D of all integers and so that

(5.5) $$A \cap B = \text{void}.$$

We shall embed the ordered sets A, B into

(5.6) $$S = A \cup B$$

ordered as follows:

(5.7) Each b_r for $r \geqslant n$ is preceded by $(-\infty, s_n]_A$ and is incomparable with $A \setminus (\infty, s_n]_A$.

LEMMA 5.1. *No point of B is succeeded by a point of A:*

$$A \cap (b, \infty)_S = \textit{void}, \quad (b \in B).$$

LEMMA 5.2. *B is a maximal chain of S: there is no point of A comparable with every point of B.*

Let $x \in A$; in virtue of (5.3) one has

$$\text{either} \quad x \in \bigcup_n [s_n, \infty)_A, \quad (n \in D)$$

$$\text{or} \quad x \in \bigcup_n (-\infty, s_n)_A, \quad (n \in D).$$

In the first case x is incomparable with every $b \in B$. In the second case, the sets being pairwise disjoint there is a single $n \in D$ so that $x \in (-\infty, s_n)$; by the definition (5.7) of ordering, the point b_{n-1} of B is incomparable with x.

LEMMA 5.3. *The set S has no gap.*

Let m be any maximal chain of S; in virtue of (5.6) one has

$$m = m \cap A \cup m \cap B. \tag{5.8}$$

If $m \cap A =$ void, then $m = B$; if $m \cap B =$ void, then m is a maximal chain in A; in both cases m has no gap. It remains the case that both terms in (5.8) are non-void. In this case $m \cap B$ is a terminal portion of B, because there is no point of A succeding to a point of B; in virtue of the lemma 5.2, $m \cap B \neq B$. Thus $m \cap B$ is a proper terminal portion of B and consequently has a first point (the set B being isomorphic to the set D of integers). Briefly, m has no gap.

LEMMA 5.4. *The set A is an initial portion of S and is coinitial with S.*

The lemma 5.4 is obvious.

LEMMA 5.5. *The set A satisfies the conditions* 4.II *in $B_4(S)$: if*

$$x \in S, \quad (-\infty, x)_S \subseteq A, \tag{5.9}$$

then $x \in A$; moreover, because no $x \in S$ is a terminal point of S, for every $x' > x$ there is

$$x'' \in (x, x']_S \quad \text{so that} \quad (-\infty, x'']_S \subseteq A. \tag{5.10}$$

At first, the relations (5.9) imply $x \in A$, because, for any $b_n \in B$ one has $b_{n-1} < b_n$ and $b_{n-1} \in S \setminus A$.

Secondly, because of the condition c), we can assume $x' \in B$, say $x' = b_n$. Consequently, $s_n \leqslant b_n$. Now $x \leqslant s_n$; otherwise, the points x, b_n would be incomparable. Now, it suffices to take $x'' \in (x, s_n)_A$ to be assured that the relations (5.10) are fulfilled.

Briefly, the set S satisfies the condition $A_4(S)$ (cf. lemma 5.3) and does not satisfy the condition $B_4(S)$ cf. lemmas (5.4—5.5).

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