PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 119 (133) (2026), 99–104 DOI: https://doi.org/10.2298/PIM2633099C

SLOWLY OSCILLATING DOUBLE SEQUENCES IN METRIC SPACES

Hüseyin Çakallı and Şebnem Yıldız Yar

Abstract. A double sequence $\mathbf{x} = \{x_{k,l}\}$ of points in a metric space $\mathbf{X}$ is slowly oscillating if for any given $\varepsilon > 0$, there exist $\alpha(\varepsilon) > 0$, $\delta = \delta(\varepsilon) > 0$, and $N = N(\varepsilon)$ such that $d(x_{k,l}, x_{s,t}) < \varepsilon$ whenever $k, l \geqslant N(\varepsilon)$ and $k \leqslant s \leqslant (1+\alpha)k$, $l \leqslant t \leqslant (1+\delta)l$. We study continuity type properties of factorable double functions defined on a double subset $E \times E$ of $\mathbf{X}^2$ into $\mathbf{X}$, and obtain interesting results related to uniform continuity, sequential continuity, and a newly introduced type of continuity of factorable double functions defined on a double subset $E \times E$ of $\mathbf{X}^2$ into $\mathbf{X}$.

1. Introduction

Pringsheim [**18**] introduced the concept of convergence of real double sequences and then Hardy [**10**] introduced the notion of regular convergence for double sequences in the sense that double sequence has a limit in Pringsheim's sense and has one sided limits (see also [**9**, **19**]). Most of papers which appeared in recent years study double sequences from various points of view (see [**1**, **7**, **8**, **12**–**15**, **17**]). Some results in the investigation are generalizations of known results concerning simple sequences to certain classes of double sequences, while other results reflect a specific nature of the Pringsheim convergence (e.g., the fact that a double sequence may converge without being bounded). A single sequence (x_n) of points in $\mathbf{R}$ is slowly oscillating if

$$\lim_{\lambda\to 1^+} \overline{\lim_{n}} \max_{n+1\leqslant k\leqslant[\lambda_n]} |x_k - x_n| = 0$$

where $[\lambda_n]$ denotes the integer part of λ_n [**2**]. This is equivalent to, for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geqslant N(\varepsilon)$ and $n \leqslant m \leqslant (1+\delta)n$.

We investigate slowly oscillating double sequences and newly defined types of continuities for factorable double functions in metric spaces. Throughout this paper, $\mathbf{X}$ will denote a metric space with a metric d. We begin with giving some definitions which deal with double sequences in metric spaces.

2020 *Mathematics Subject Classification:* Primary 40B05, Secondary 40C05.
Key words and phrases: Double sequences, slowly oscillating, P-convergent, continuity.
Communicated by Stevan Pilipović.

2. Preliminaries

In this section, we recall main definitions.

Definition 2.1. A double sequence $\mathbf{x} = \{x_{k,l}\}$ in a metric space $\mathbf{X}$ is Cauchy provided that, given an $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $d(x_{k,l}, x_{s,t}) < \epsilon$ whenever $k, l, s, t > N$.

Definition 2.2. A double sequence $\mathbf{x} = \{x_{k,l}\}$ in a metric space $\mathbf{X}$ has a *Pringsheim limit* L (denoted by P-$\lim x = L$) provided that, given an $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $d(x_{k,l}, L) < \epsilon$ whenever $k, l > N$. Such an $\mathbf{x}$ is described more briefly as "P-convergent".

A double sequence $\mathbf{x} = \{x_{m,n}\}$ is bounded if there exist an element x_0 of X, and an $M > 0$ such that $d(x_{m,n}, x_0) < M$ for all $m, n \in \mathbf{N}$. We note that a P-convergent double sequence need not be bounded.

Definition 2.3 (Patterson [**16**]). A double sequence $\mathbf{y}$ is a *double subsequence* of $\mathbf{x}$ provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $\{x_j\} = \{x_{n_j,k_j}\}$, then $\mathbf{y}$ is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & - \end{array}$$

3. Results

We introduce slowly oscillatingness and quasi-Cauchyness of a double sequence in a metric space X.

Definition 3.1. A double sequence $\mathbf{x} = \{x_{k,l}\}$ of points in a metric space $\mathbf{X}$ is called slowly oscillating for any given $\varepsilon > 0$, there exist $\alpha = \alpha(\varepsilon)$, $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $d(x_{k,l}, x_{s,t}) < \varepsilon$ if $k, l \geqslant N(\varepsilon)$ and $k \leqslant s \leqslant (1+\alpha)k$, $l \leqslant t \leqslant (1+\delta)l$.

Definition 3.2. A double sequence $\mathbf{x} = \{x_{k,l}\}$ of points in metric space X is called quasi-Cauchy if given an $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\Delta'' x_{k,l} < \epsilon$ for $k, l > N$, where $\Delta'' x_{k,l} = \max_{r,s=1 \text{ and/or } 0} d(x_{k+r}, x_{l+s})$.

We see that any slowly oscillating double sequence of points in X is quasi-Cauchy, any convergent double sequence of points in X is quasi-Cauchy, so any regularly convergent double sequence is quasi-Cauchy. Any Cauchy double sequence is quasi-Cauchy. Any subsequence of a convergent double sequence is convergent. Any subsequence of a Cauchy double sequence is Cauchy. But the situation is different for quasi-Cauchy double sequences. There are subsequences of a quasi-Cauchy double sequence which are not quasi-Cauchy.

Definition 3.3. A subset of a metric space is called double ward compact if any double sequence of points in X has a quasi-Cauchy double subsequence.

Definition 3.4. A subset of a metric space is called double slowly oscillating compact if any double sequence of points in X has a slowly oscillating double subsequence.

We see that any slowly oscillating double compact is ward double compact.

First, we note that any finite subset of X is double ward compact, the union of two double ward compact subsets of X is double ward compact and the intersection of any family of double ward compact subsets of X is double ward compact. Any G-sequentially compact subset of X is double ward compact for a regular subsequential method G (see [**3**–**5**]). Furthermore any subset of double ward compact set is double ward compact, any totally bounded subset of X is double ward compact, any slowly oscillating compact subset of X is double ward compact. These observations suggest to us the following. A subset of a metric space is double slowly oscillating compact if and only if it is totally bounded.

Theorem 3.1. *If a factorable double function f defined on a double subset $E \times E$ of $\mathbf{X}^2$ is uniformly continuous, then it preserves factorable slowly oscillating double sequences from $E \times E$.*

Proof. Suppose that f is uniformly continuous, and let

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be any slowly oscillating factorable double sequence. To prove that $\{f(x_{n,m})\}$ is slowly oscillating, take any $\varepsilon > 0$. Uniform continuity of f implies that there exists a $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$ for $x, y \in E \times E$. Since $\{x_{n,m}\}$ is slowly oscillating, for this $\delta = \delta(\varepsilon) > 0$, there exist $\alpha = \alpha(\delta)$, and $N = N(\delta)$ such that $d(x_{k,l}, x_{s,t}) < \delta$ if $k, l \geqslant N(\varepsilon)$ and $k \leqslant s \leqslant (1 + \alpha)k$, $l \leqslant t \leqslant (1 + \delta)l$. Hence $d(f(x_{k,l}), f(x_{s,t})) < \varepsilon$. □

Theorem 3.2. *If a factorable double function f defined on a double subset $E \times E$ of $\mathbf{X}^2$ preserves factorable slowly oscillating double sequences from $E \times E$, then it preserves factorable P-convergent double sequences from $E \times E$.*

Proof. Suppose that f preserves factorable slowly oscillating double sequences from $E \times E$. Let

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be any P-convergent factorable double sequence with P-limit L. Then the sequence

$$\begin{array}{ccccccc}
x_{1,1}\,L & x_{1,2}\,L & x_{1,3}\,L & ... & & & \\
L & L & L & L & L & L & ... \\
x_{2,1}\,L & x_{2,2}\,L & x_{2,3}\,L & \cdots & & & \\
L & L & L & L & L & L & ... \\
x_{3,1}\,L & x_{3,2}\,L & x_{3,3}\,L & \cdots & & & \\
L & L & L & L & L & L & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is also P-convergent with P-limit L. Since any convergent double sequence is slowly oscillating, this sequence is slowly oscillating. So the transformed sequence of the sequence is slowly oscillating. Thus it follows that

$$\begin{array}{ccccccc}
f(x_{1,1})\,f(L) & f(x_{1,2})\,f(L) & f(x_{1,3})\,f(L) & ... & & & \\
f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
f(x_{2,1})\,f(L) & f(x_{2,2})\,f(L) & f(x_{2,3})\,f(L) & \cdots & & & \\
f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
f(x_{3,1})\,f(L) & f(x_{3,2})\,f(L) & f(x_{3,3})\,f(L) & \cdots & & & \\
f(L) & f(L) & f(L) & L & f(L) & f(L) & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is a slowly oscillating double sequence. Hence it is easy to see that the transformed sequence is slowly oscillating. $\square$

It is well known that uniform limit of a sequence of continuous functions is continuous. This is also true for two dimensional factorable real-valued functions that preserve slowly oscillating double sequences, i.e., uniform limit of a sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from $E \times E$ of $\mathbf{X}^2$ also preserves slowly oscillating double sequences from $E \times E$.

THEOREM 3.3. *If* (f_n) *is a sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from a double interval* $E \times E$ *of* $\mathbf{X}^2$ *and* (f_n) *is uniformly convergent to a function* f*, then* f *preserves slowly oscillating double sequences from* $E \times E$.

PROOF. Let (x_{nk}) be a slowly oscillating double sequence and $\varepsilon > 0$. Then there exists a positive integer N such that

$$d(f_n(x,y), f(\bar{x},\bar{y})) < \varepsilon/3$$

for all $(x,y), (\bar{x},\bar{y}) \in E \times E$ whenever $n \geqslant N$. As f_N preserves slowly oscillating double sequences from $E \times E$, there exist a $\delta > 0$ and a positive integer $N_1 = N_1(\varepsilon)$, greater than N, such that $d(f_N(x_{k,l}), f_N(x_{s,t})) < \frac{\varepsilon}{3}$ for $n \geqslant N_1$ and $k \leqslant s \leqslant (1+\delta)k$, $l \leqslant t \leqslant (1+\delta)l$. Now for $n \geqslant N_1$ and $k \leqslant s \leqslant (1+\delta)k$, $l \leqslant t \leqslant (1+\delta)l$. Thus for $n \geqslant N_1$ and $k \leqslant s \leqslant (1+\delta)k$, $l \leqslant t \leqslant (1+\delta)l$ we have

$$\begin{aligned}
d(f(x_{k,l}), f(x_{s,t})) &\leqslant d(f(x_{k,l}), f_N(x_{k,l})) + d(f_N(x_{k,l}), f_N(x_{s,t})) \\
&\quad + d(f_N(x_{s,t}), f(x_{s,t})) \leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \qquad \square
\end{aligned}$$

THEOREM 3.4. *If* $(f_{m,n})$ *is a double sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from a double interval* $E \times E$ *of* $\mathbf{X}^2$ *and* $(f_{m,n})$ *is uniformly* P*-convergent to a function* f*, then* f *preserves slowly oscillating double sequences from* $E \times E$.

The proof is similar to the last theorem and as of such it is omitted.

4. Conclusion

It is easy to see that Cauchy double sequences are slowly oscillating double in a metric space. The converse is easily seen to be false as in the single dimensional case [**2**, **6**, **20**]. One should also note that there are nice connections between slowly oscillating double sequences and uniform continuity of two-dimensional functions. This is illustrated through the following result. Suppose that $E \times E$ is any two dimensional totally bounded subset of a product metric space $X \times X$. Then a two dimensional factorable function is uniformly continuous on $E \times E$ if and only if it preserves factorable slowly oscillating double sequences from $E \times E$. Extensions and variations of the above theorem are also presented. For a further study, we suggest to investigate slowly oscillating double sequences of fuzzy points in metric spaces, and slowly oscillating continuity for the factorable fuzzy functions in metric spaces (see [**8**, **11**]). As a problem, we suggest to investigate a theory in dynamical systems by introducing the following concept: two dynamical systems are called pseudo conjugate if there is one-to-one, onto, pseudo continuous h, h^{-1} is pseudo continuous, and h commutes the mappings at each point x.

References

1. A. Alotaibi, M. Mursaleen, M. A. Alghamdi, *Invariant and absolute invariant means of double sequences*, J. Funct. Spaces Appl. 2012, Article ID 465364, 9 pp.
2. H. Çakalli, *Slowly oscillating continuity*, Abstr. Appl. Anal. 2008, Article ID 485706, 5 pp.
3. ———, *Sequential definitions of compactness*, Appl. Math. Lett. **21** (2008), 594–598.
4. ———, *On G-continuity*, Comput. Math. Appl. **61**(2) (2011) 313–318.
5. ———, *New kinds of continuities*, Comput. Math. Appl. **61**(4) (2011), 960–965.
6. M. Dik, I. Canak, *New types of continuities*, Abstr. Appl. Anal. 2010, Article ID 258980, 6 pp.
7. D. Djurcic, Lj. D. R. Kocinac, M. R. Zizovic, *Double sequences and elections*, Abstr. Appl. Anal. 2012, Article ID 497594, 6 pp.
8. H. Dutta, *A characterization of the class of statistically pre-Cauchy double sequences of fuzzy numbers*, Appl. Math. Inf. Sci. **7**(4) (2013), 1437–1440.
9. H. J. Hamilton, *Transformations of multiple sequences*, Duke Math. J. **2** (1936), 29–60.
10. G. H. Hardy, *On the convergence of certain multiple series*, London M. S. Proc. (2) **1** (1903), 124–125.
11. S. A. Mohiuddine, A. Alotaibi, M. Mursaleen, *Statistical convergence of double sequences in locally solid Riesz spaces*, Abstr. Appl. Anal. 2012, Article ID 863483, 20 pp.
12. M. Mursaleen, S. A. Mohiuddine, *Banach limit and some new spaces of double sequences*, Turk. J. Math. **36**(1), (2012), 121–130.
13. R. F. Patterson, *A theorem on entire four dimensional summability methods*, Appl. Math. Comput. **219**(14) (2013), 7777–7782.
14. ———, *Four dimensional matrix characterization* P*-convergence fields of summability methods*, Appl. Math. Comput. **219**(12) (2013), 6783–6791.
15. ———, *RH-regular transformations which sums a given double sequence*, Filomat **27**(4) (2013), 625–627.

16. ______, *Analogues of some fundamental theorems of summability theory*, Internat. J. Math. Math. Sci. **23**(1) (2000), 1–9.
17. R. F. Patterson, E. Savas, *Asymptotic equivalence of double sequences*, Hacet. J. Math. Stat. **41**(4) (2012), 487–497.
18. A. Pringsheim, *Zur theorie der zweifach unendlichen zahlenfolgen*, Math. Ann. **53**(3) (1900), 289–321.
19. G. M. Robison, *Divergent double sequences and series*, Trans. Am. Math. Soc. **28**(1) (1926), 50–73.
20. R. W. Vallin, *Creating slowly oscillating sequences and slowly oscillating continuous functions*, Acta Math. Univ. Comen., New Ser. **80**(1) (2011), 71–78.

(Received 30 06 2025)

Professor Emeritus
Department of Mathematics
Istanbul University
Istanbul
Turkey
`hcakalli@istanbul.edu.tr`
`hcakalli@gmail.com`

Ordu University
Mathematics Department
Ordu
Turkey
`sebnemyildizyar@odu.edu.tr`