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KANTOROVICH VERSION OF A GENERALIZED BERNSTEIN OPERATORS AND APPLICATIONS

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Abstract. We introduce generalized Bernstein–Kantorovich type operators with two shifted nodes and study their approximation properties. First, we calculate some estimates for these operators. Further, we discuss convergence theorems and order of approximation in terms of Korovkin theorem and first order modulus of smoothness respectively. Next, we study pointwise approximation results in terms of Peetre's K-functional, second order modulus of smoothness, Lipschitz type space and r^{th} order Lipschitz type maximal function. Lastly, weighted approximation results and statistical approximation theorems are proved.

1. Introduction and Preliminaries

Approximation theory indeed plays a crucial role across various disciplines, providing a framework to represent complex functions with simpler ones. Its applications span from mathematics to engineering, including fields like computational science, data analysis, and computer graphics. In computational aspects, approximation theory aids in describing geometric shapes and solving differential equations. It forms the backbone of numerical analysis, where it helps in devising efficient algorithms for solving mathematical problems numerically. Moreover, in applied mathematics, approximation theory contributes to areas like control theory, where control points and control nets are utilized to study parametric curves and surfaces. These concepts are fundamental in designing control systems for various engineering applications [**8**, **9**]. In recent years, with the rise of artificial intelligence, data science and machine learning, approximation theory has found new applications. Techniques from approximation theory are employed in developing algorithms for data analysis, pattern recognition and predictive modeling. They form the basis for constructing models that can approximate complex relationships within datasets. Furthermore, in computer graphics and computer algebra systems, approximation theory is indispensable. It enables the representation of curves and surfaces using simpler mathematical constructs, facilitating tasks like rendering realistic images

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and solving symbolic equations efficiently. Many scientists in medical sciences and others are also working in terms of these sequences of [**6**, **15**, **16**].

The first sequence of operators to support the above application part was introduced by Bernstein [**1**]. Although, his motive was to provide a short and elegant proof of the approximation theorem of Weierstrass with the assistance of binomial distribution as:

$$B_l(g;u)=\sum_{\nu=0}^{l}\binom{l}{\nu}u^{\nu}(1-u)^{l-\nu}g\left(\frac{\nu}{l}\right),\quad u\in[0,1], \tag{1.1}$$

where g belongs to $C[0,1]$. He proved that these operators approximate uniformly on $[0,1]$ to every continuous function $g\in C[0,1]$. The Bernstein operators have been one of the most extensively examined positive linear operators in the area of approximation theory. However, these operators are not applicable for discontinuous functions. Further, Kantorovich [**7**] introduced a sequence of operators which is a generalization of the operators (1.1) over the space of Lebesgue integrable functions $L[0,1]$ as:

$$K_l(g;u)=(l+1)\sum_{\nu=0}^{l}\binom{l}{\nu}u^{\nu}(1-u)^{l-\nu}\int_{\frac{u}{l+1}}^{\frac{\nu+1}{l+1}}g(t)\,dt,\quad u\in[0,1].$$

Recently, Usta [**14**] presented a new sequence of Bernstein operators for the function g, which are continuous and defined on $(0,1)$ with $u\in(0,1)$ as follows:

$$P_l^{*}(g;u)=\frac{1}{l}\sum_{\nu=0}^{l}\binom{l}{\nu}(\nu-lu)^2u^{\nu-1}(1-u)^{l-\nu-1}g\left(\frac{\nu}{l}\right),\quad l\in\mathbb{N}. \tag{1.2}$$

Remark 1.1. These operators given in (1.2) are restricted to the space of continuous functions only.

In addition of the above literature and to discuss approximation properties for Lebesgue integrable functions, we define generalized Bernstein–Kantorovich–Stancu type operators with two shifted nodes as follows

$$G_l^{\alpha,\beta}(g;u)=\sum_{\nu=0}^{l}Q_{l,\nu}(u)\int_{\frac{\nu+\alpha}{l+\beta+1}}^{\frac{\nu+\alpha+1}{l+\beta+1}}g(t)\,dt, \tag{1.3}$$

where $Q_{l,\nu}(u)=\frac{(l+\beta+1)}{l}\binom{l}{\nu}(\nu-lu)^2u^{\nu-1}(1-u)^{l-\nu-1}$, here α,β denote real parameters with $0\leqslant\alpha\leqslant\beta$. For $\alpha=\beta=0$, the operators (1.3) reduce to operators defined by Senapati [**12**].

Remark 1.2. For any $g,h\in C(0,1)$ and $a_1,a_2\in\mathbb{R}$, we have

$$\begin{aligned}G_l^{\alpha,\beta}(a_1g+a_2h;u)&=\sum_{\nu=0}^{l}Q_{l,\nu}(u)\int_{\frac{\nu+\alpha}{l+\beta+1}}^{\frac{\nu+\alpha+1}{l+\beta+1}}(a_1g+a_2h)(t)\,dt\\&=_1\sum_{\nu=0}^{l}Q_{l,\nu}(u)\int_{\frac{\nu+\alpha}{l+\beta+1}}^{\frac{\nu+\alpha+1}{l+\beta+1}}g(t)\,dt\end{aligned}$$

$$+ a_2 \sum_{\nu=0}^{l} Q_{l,\nu}(u) \int_{\frac{\nu+\alpha}{l+\beta+1}}^{\frac{\nu+\alpha+1}{l+\beta+1}} h(t)\,dt$$
$$= a_1 G_l^{\alpha,\beta}(g;u) + a_2 G_l^{\alpha,\beta}(h;u).$$

Which implies that the operator $G_l^{\alpha,\beta}(.;.)$ is a linear operator.

Remark 1.3. Also for any $g \geqslant 0$, we must have $G_l^{\alpha,\beta}(g;u) \geqslant 0$, which shows that the sequence of operators are positive.

The structure of our research work is organized as follows: Section 1 compute some estimates for the operators (1.3) in terms of test functions and central moments. In Section 2, we study the uniform convergence theorem and approximation order via the Korovkin theorem and first order modulus of continuity respectively. In Section 3, we discuss the local and global approximation results using first and second order modulus of continuity, Peetre's K-functional and weight functions in several functional spaces.

To discuss the existence and convergence of operators (1.3), we consider $e_i(t) = t^i$, $i = 0,1,2$. Then, in the following Lemmas (1.3) and (1.4), we estimate the operators introduced in terms of central moments and test functions.

Lemma 1.1. [**12**] *For the operators $K_n^*(.;.)$, we have following identities are obtained*

$$\begin{aligned}
K_l^*(e_0;u) &= 1,\\
K_l^*(e_1;u) &= \left(\frac{l-2}{l+1}\right)u + \frac{3}{2(l+1)},\\
K_l^*(e_2;u) &= \left(\frac{l^2-7l+6}{(l+1)^2}\right)u^2 + \left(\frac{6l-8}{(l+1)^2}\right)u + \frac{7}{3(l+1)^2},\\
K_l^*(e_3;u) &= \left(\frac{l^3-15l^2-38l-24}{(l+1)^3}\right)u^3 + \left(\frac{\frac{27}{2}l^2-\frac{117}{2}l+45}{(l+1)^3}\right)u^2\\
&\quad + \left(\frac{\frac{43}{2}l-25}{(l+1)^3}\right)u + \frac{15}{4(l+1)^3}.
\end{aligned}$$

Lemma 1.2. *For $u \in (0,1)$ with $0 \leqslant \alpha \leqslant \beta$, we construct a recursive relation between the operators $G_l^{\alpha,\beta}(t^m;u)$, $m = 0,1,2,\dots$ and $K_l^*(g;u)$ where $g(t) = t^i$, $i = 0,1,2,\dots$ are the test functions as:*

$$G_l^{\alpha,\beta}(t^m;u) = \sum_{i=0}^{m} \binom{m}{i} \left(\frac{l+1}{l+\beta+1}\right)^i \left(\frac{\alpha}{l+\beta+1}\right)^{m-i} K_l^*(t^i;u).$$

Proof. From equation (1.3), we have

$$G_l^{\alpha,\beta}(g;u) = \frac{(l+\beta+1)}{l} \sum_{\nu=0}^{l} \binom{l}{\nu} (\nu - lu)^2 u^{\nu-1}(1-u)^{l-\nu-1} \int_{\frac{\nu+\alpha}{l+\beta+1}}^{\frac{\nu+\alpha+1}{l+\beta+1}} g(t)\,dt.$$

We can rearrange the above equation as

$$\begin{aligned}G_l^{\alpha,\beta}(g;u) &= \frac{(l+\beta+1)}{l}\sum_{\nu=0}^{l}\binom{l}{\nu}(\nu-lu)^2u^{\nu-1}(1-u)^{l-\nu-1}\\ &\int_{\frac{\nu}{l+1}}^{\frac{\nu+1}{l+1}} g\left(\frac{(l+1)t+\alpha}{l+\beta+1}\right)\frac{(l+1)}{l+\beta+1}dt\\ &= \frac{(l+1)}{l}\sum_{\nu=0}^{l}\left(\begin{array}{c} l\\ \nu\end{array}\right)(\nu-lu)^2u^{\nu-1}(1-u)^{l-\nu-1}\\ &\int_{\frac{\nu}{l+1}}^{\frac{\nu+1}{l+1}} g\left(\frac{(l+1)t+\alpha}{l+\beta+1}\right)dt.\end{aligned}$$

Now,

$$\begin{aligned}G_l^{\alpha,\beta}(t^i;u) &= \frac{(l+1)}{l}\sum_{\nu=0}^{l}\binom{l}{\nu}(\nu-lu)^2u^{\nu-1}(1-u)^{l-\nu-1}\\ &\int_{\frac{\nu}{l+1}}^{\frac{\nu+1}{l+1}} \left(\frac{(l+1)t+\alpha}{l+\beta+1}\right)^i dt\\ &= \frac{(l+1)}{l}\sum_{\nu=0}^{l}\binom{l}{\nu}(\nu-lu)^2u^{\nu-1}(1-u)^{l-\nu-1}\\ &\int_{\frac{\nu}{l+1}}^{\frac{\nu+1}{l+1}} \sum_{i=0}^{m}\binom{m}{i}\left(\frac{(l+1)t}{l+\beta+1}\right)^i\left(\frac{\alpha}{l+\beta+1}\right)^{m-i}dt\\ &= \sum_{i=0}^{m}\binom{m}{i}\left(\frac{l+1}{l+\beta+1}\right)^i\left(\frac{\alpha}{l+\beta+1}\right)^{m-i}\\ &\frac{(l+1)}{l}\sum_{\nu=0}^{l}\binom{l}{\nu}(\nu-lu)^2u^{\nu-1}(1-u)^{l-\nu-1}\int_{\frac{\nu}{l+1}}^{\frac{\nu+1}{l+1}} t^i dt\\ &= \sum_{i=0}^{m}\binom{m}{i}\left(\frac{l+1}{l+\beta+1}\right)^i\left(\frac{\alpha}{l+\beta+1}\right)^{m-i}K_l^*(t^i;u). \qquad \square\end{aligned}$$

LEMMA 1.3. *Let $G_l^{\alpha,\beta}(.;.)$ be presented in* (1.3)*. Then, the following identities are acquired*

$$\begin{aligned}G_l^{\alpha,\beta}(e_0;u) &= 1,\\ G_l^{\alpha,\beta}(e_1;u) &= \frac{1}{l+\beta+1}\left\{(l-2)u+\frac{1}{2}(3+2\alpha)\right\},\\ G_l^{\alpha,\beta}(e_2;u) &= \frac{1}{(l+\beta+1)^2}\left\{(l^2-7l-6)u^2+(6l-8+2l\alpha-4\alpha)u\right\}\\ &+\frac{(7+9\alpha+3\alpha^2)}{3(l+\beta+1)^2}.\end{aligned}$$

Proof. From the result of Lemma 1.1 and recursive relation in Lemma 1.2, we prove the above results of Lemma 1.3. ☐

Lemma 1.4. *Let* $\psi_u^i(t) = (t-u)^i$, $i = 0,1,2$. *Then, we have the central moments of generalized Bernstein Kantorovich Stancu operators* (1.3) *as follows:*

$$G_l^{\alpha,\beta}(\psi_u^0(t);u) = 1,$$

$$G_l^{\alpha,\beta}(\psi_u^1(t);u) = \frac{1}{l+\beta+1}\Big\{\frac{3}{2} - 3u + \alpha\Big\},$$

$$G_l^{\alpha,\beta}(\psi_u^2(t);u) = \frac{1}{(l+\beta+1)^2}\Big\{(11-3l)u^2 + (3l-11-6\alpha)u + \Big(\frac{7}{3} + \alpha^2 + 3\alpha\Big)\Big\}.$$

Proof. In the light of Lemma 1.3 and linearity properties, we can easily prove Lemma 1.4. ☐

Remark 1.4. If we take $\alpha = \beta = 0$, then the estimates of the operators (1.3) reduces to the estimates of the operators defined by Senapati [**12**].

Remark 1.5. It is also noticed that $G_l^{\alpha,\beta}(\psi_u^2(t);u) \leqslant K_l^*(\psi_u^2(t);u)$, which shows that the rate of convergence of operators (1.3) is faster than Kantorovich operators [**12**].

2. Rapidity of convergence and order of approximation

Definition 2.1. Let $g \in C(0,1)$. Then, the modulus of continuity is defined as: $\omega(g;\tilde{\eta}) = \sup_{|u_1-u_2|\leqslant\tilde{\eta}} |g(u_1) - g(u_2)|$, $u_1, u_2 \in (0,1)$.

Theorem 2.1. *Let* $G_l^{\alpha,\beta}(.;.)$ *be given in* (1.3). *Then, for all* $g \in C_B(0,1)$, $G_l^{\alpha,\beta}(g;u) \rightrightarrows g$ *on each bounded and closed subset of* $(0,1)$, *where* $\rightrightarrows$ *symbol denotes uniform convergence.*

Proof. Using the Korovkin result which implies that the convergence is uniform, it is adequate to see that $\lim_{l\to\infty} G_l^{\alpha,\beta}(t^i;u) = u^i$, $i = 0,1,2$, uniformly on $(0,1)$. In view of Lemma 1.3, we can arrive at the desired result. ☐

In the light of Shisha et al. [**13**], one can show the order of approximation via the Ditzian–Totik modulus of continuity.

Theorem 2.2. *Let* $g \in C_B(0,1)$. *Then, operators* $G_l^{\alpha,\beta}(.;.)$ *given in* (1.3), *we have* $|G_l^{\alpha,\beta}(g;u) - g(u)| \leqslant 2\omega(g;\tilde{\eta})$, *where* $\tilde{\eta} = \sqrt{G_l^{\alpha,\beta}((t-u)^2;u)}$.

Proof. In accordance with Lemma 1.3, 1.4 and Cauchy–Schwartz inequality, we have

$$\begin{aligned}
|G_l^{\alpha,\beta}(g;u) - g(u)| &\leqslant G_l^{\alpha,\beta}|(g(t) - g(u)|;u) \\
&\leqslant G_l^{\alpha,\beta}\Big(\Big(1 + \frac{|t-u|}{\delta}\Big)\tilde{\omega}(g,\delta);u\Big) \\
&\leqslant \tilde{\omega}(g,\delta)\Big[1 + \frac{1}{\delta}G_l^{\alpha,\beta}(|t-u|;u)\Big]
\end{aligned}$$

$$\leqslant \tilde{\omega}(g,\delta)\Bigg[1+\frac{1}{\delta}\sqrt{G_l^{\alpha,\beta}((t-u)^2;u)}\Bigg]$$

By selecting $\delta=\sqrt{G_l^{\alpha,\beta}((t-u)^2;u)}$, we obtained the desired result. $\square$

3. Direct Results

We recall a functional space as: $C_B(0,1)$, where $C_B(0,1)$ denotes a space of continuous and bounded functions and Peetre's K-functional is as:

$$K_2(g,\tilde{\eta})=\inf_{h\in C_B^2(0,1)}\Big\{\|g-h\|_{C_B(0,1)}+\tilde{\eta}\|h''\|_{C_B^2(0,1)}\Big\},$$

where $C_B^2(0,1)=\{h\in C_B(0,1): h',h''\in C_B(0,1)\}$ endowed with the norm

$$\|g\|=\sup_{0<u<1}|g(u)|.$$

Further, we recall the second order Ditzian–Totik modulus of continuity is as:

$$\omega_2(g;\sqrt{\tilde{\eta}})=\sup_{0<\nu\leqslant\sqrt{\tilde{\eta}}}\ \sup_{u\in(0,1)}|g(u+2\nu)-2g(u+\nu)+g(u)|.$$

We also have a relation from [**2**, Theorem 2.4, p. 177], as follows:

$$K_2(g;\tilde{\eta})\leqslant\tilde{C}\omega_2(g;\sqrt{\tilde{\eta}}),\tag{3.1}$$

where $\tilde{C}$ is an absolute constant. Next, in order to discuss the approximation result, we consider the auxiliary sequence of operator as:

$$\widehat{G}_l^{\alpha,\beta}(g;u)=G_l^{\alpha,\beta}(g;u)+g(u)-g\Big(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}\Big),\tag{3.2}$$

where $g\in C_B(0,1)$, $u\geqslant 0$ and $l>2$. From (3.2), we get

$$\widehat{G}_l^{\alpha,\beta}(\psi_u^0(t);u)=1,\ \ \widehat{G}_l^{\alpha,\beta}(\psi_u^1(t);u)=0\text{ and }|\widehat{G}_l^{\alpha,\beta}(g;u)|\leqslant 3\|g\|.\tag{3.3}$$

Lemma 3.1. *For $u\geqslant 0$ and $l>2$, we have $|\widehat{G}_l^{\alpha,\beta}(g;u)-g(u)|\leqslant\theta(u)\|g''\|$, where $g\in C_B^2(0,1)$ and $\theta(u)=\widehat{G}_l^{\alpha,\beta}(\psi_u^2(t);u)+(\widehat{G}_l^{\alpha,\beta}(\psi_u^1(t);u))^2$.*

Proof. For $h\in C_B^2(0,1)$ and in the direction of Taylor's theorem, we have

$$g(t)=g(u)+(t-u)g'(u)+\int_u^t(t-v)g''(v)dv.\tag{3.4}$$

Now, by using the auxiliary operators $\widehat{G}_l^{\alpha,\beta}(.;.)$ introduced in (3.2) on both the sides in the above (3.4), we have

$$\widehat{G}_l^{\alpha,\beta}(g;u)-g(u)=g'(u)\widehat{G}_l^{\alpha,\beta}(\psi_u^1(t);u)+\widehat{G}_l^{\alpha,\beta}\bigg(\int_u^t(t-v)g''(v)dv;u\bigg).$$

In the light of (3.3) and (3.4), we have

$$\begin{aligned}\widehat{G}_l^{\alpha,\beta}(g;u)-g(u)&=\widehat{G}_l^{\alpha,\beta}\bigg(\int_u^t(t-v)g''(v)dv;u\bigg)\\&=G_l^{\alpha,\beta}\bigg(\int_u^t(t-v)g''(v)dv;u\bigg)\end{aligned}$$

$$-\int_u^{\frac{2(l-2)u+3+2\alpha}{l+\beta+1}}\left(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}-v\right)g''(v)dv,$$

$$\begin{aligned}|\widehat{G}_l^{\alpha,\beta}(g;u)-g(u)| &\leqslant \left|G_l^{\alpha,\beta}\Big(\int_u^t(t-v)g''(v)dv;u\Big)\right|\\ &+\left|\int_u^{\frac{2(l-2)u+3+2\alpha}{l+\beta+1}}\Big(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}-v\Big)g''(v)dv\right|. \qquad (3.5)\end{aligned}$$

Since,

$$\left|\int_u^t(t-v)g''(v)dv\right|\leqslant(t-u)^2\|g''\|, \qquad (3.6)$$

therefore

$$\begin{aligned}\left|\int_u^{\frac{2(l-2)u+3+2\alpha}{l+\beta+1}}\Big(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}-v\Big)g''(v)dv\right| \qquad (3.7)\\ \leqslant\left(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}-u\right)^2\|g''\|.\end{aligned}$$

In account of (3.5), (3.6) and (3.7), we find

$$\begin{aligned}|\widehat{G}_l^{\alpha,\beta}(g;u)-g(u)| &\leqslant\left\{\widehat{G}_l^{\alpha,\beta}(\psi_u^2(t);u)+\left(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}-u\right)^2\right\}\|g''\|\\ &=\theta(u)\|g''\|.\end{aligned}$$

We arrive at the required result. ☐

THEOREM 3.1. *For $g\in C_B^2(0,1)$, there exists a constant $\tilde{C}>0$ such that*

$$\left|G_l^{\alpha,\beta}(g;u)-g(u)\right|\leqslant\tilde{C}\omega_2\big(g;\sqrt{\theta(u)}\big)+\omega(g;G_l^{\alpha,\beta}(\psi_u^1(t);u)),$$

where $\theta(u)$ is in Lemma 3.1.

PROOF. For $g\in C_B(0,1)$ and $h\in C_B^2(0,1)$ and in account of the definition of $\widehat{G}_l^{\alpha,\beta}(.;.)$, we yield

$$\begin{aligned}|G_l^{\alpha,\beta}(g;u)-g(u)| &\leqslant|\widehat{G}_l^{\alpha,\beta}(g-h;u)|+|(g-h)(u)|+|\widehat{G}_l^{\alpha,\beta}(h;u)-h(u)|\\ &+\left|g\Big(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}\Big)-g(u)\right|.\end{aligned}$$

In the direction of Lemma 3.1 and inequalities in (3.3), we yield

$$\begin{aligned}|G_l^{\alpha,\beta}(g;u)-g(u)| &\leqslant 4\|g-h\|+|\widehat{G}_l^{\alpha,\beta}(h;u)-h(u)|+\left|g\Big(\frac{2(l-2)u+3+2\alpha}{l+\beta+1}\Big)-g(u)\right|\\ &\leqslant 4\|g-h\|+\theta(u)\|h''\|+\omega\big(g;G_l^{\alpha,\beta}((t-u);u)\big).\end{aligned}$$

In view of (3.1), we arrived at required result. ☐

We recall the fillowing result in Lipschitz type space [**11**], which is given as

$$\mathrm{Lip}_{\tilde{M}}^{\zeta_1,\zeta_2}(\gamma) := \Big\{ g \in C_B(0,1) : |g(t)-g(u)| \leqslant \tilde{M} \frac{|t-u|^\gamma}{(t+\zeta_1 u+\zeta_2 u^2)^{\frac{\gamma}{2}}} : u, t \in (0,1) \Big\},$$

where $0 < \gamma \leqslant 1$, $\tilde{M} > 0$ and $\zeta_1, \zeta_2 > 0$.

THEOREM 3.2. *Let* $g \in \mathrm{Lip}_{\tilde{M}}^{\zeta_1,\zeta_2}(\gamma)$ *and the operators* $G_l^{\alpha,\beta}(.;.)$ *given in* (1.3). *Then, we have*

$$|G_l^{\alpha,\beta}(g;u) - g(u)| \leqslant \tilde{M} \bigg(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2} \bigg)^{\frac{\gamma}{2}},$$

where $0 < \gamma \leqslant 1$, $\zeta_1, \zeta_2 \in (0,1)$ *and* $\lambda(u) = G_l^{\alpha,\beta}(\psi_u^2(t); u)$.

PROOF. For $\gamma = 1$ and $u > 0$, we yield

$$|G_l^{\alpha,\beta}(g;u) - g(u)| \leqslant G_l^{\alpha,\beta}(|g(t) - g(u)|; u) \leqslant \tilde{M} G_l^{\alpha,\beta} \bigg(\frac{|t-u|}{(t+\zeta_1 u + \zeta_2 u^2)^{\frac{1}{2}}}; u \bigg).$$

Since $\dfrac{1}{t+\zeta_1 u+\zeta_2 u^2} < \dfrac{1}{\zeta_1 u + \zeta_2 u^2}$, for all $u \in (0,1)$, we get

$$|G_l^{\alpha,\beta}(g;u) - g(u)| \leqslant \frac{\tilde{M}}{(\zeta_1 u + \zeta_2 u^2)^{\frac{1}{2}}} (G_l^{\alpha,\beta}(\psi_u^2(t); u))^{\frac{1}{2}} \leqslant \tilde{M} \bigg(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2} \bigg)^{\frac{1}{2}},$$

which proves the result of Theorem 3.2 holds for $\gamma = 1$. Further, we consider for $\gamma \in (0,1)$ with the account of Hölder's inequality via $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, one get

$$\begin{aligned} |G_l^{\alpha,\beta}(g;u) - g(u)| &\leqslant \big(G_l^{\alpha,\beta}(|g(t) - g(u)|^{\frac{2}{\gamma}}; u) \big)^{\frac{\gamma}{2}} \\ &\leqslant \tilde{M} \bigg(G_l^{\alpha,\beta} \bigg(\frac{|t-u|^2}{(t+\zeta_1 u + \zeta_2 u^2)}; u \bigg) \bigg)^{\frac{\gamma}{2}}. \end{aligned}$$

Since $\dfrac{1}{t+\zeta_1 u+\zeta_2 u^2} < \dfrac{1}{\zeta_1 u + \zeta_2 u^2}$, for all $u \in (0,1)$, we have

$$|G_l^{\alpha,\beta}(g;u) - g(u)| \leqslant \tilde{M} \bigg(\frac{G_l^{\alpha,\beta}(|t-u|^2; u)}{\zeta_1 u + \zeta_2 u^2} \bigg)^{\frac{\gamma}{2}} \leqslant \tilde{M} \Big(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2} \Big)^{\frac{\gamma}{2}}.$$

Hence, we get the proof of Theorem 3.2. □

Further, we investigate approximation result locally in the direction of r^{th} order modulus of continuity. Lenze [**10**] gave the Lipschitz-type maximal function as:

$$\widetilde{\omega}_r(g;u) = \sup_{t \neq u, t \in (0,\infty)} \frac{|g(t) - g(u)|}{|t-u|^r}, \ u \in (0,1) \text{ and } r \in (0,1]. \tag{3.8}$$

THEOREM 3.3. *For* $g \in C_B(0,1)$ *and* $r \in (0,1]$ *and for all* $u \in (0,1)$, *one get*

$$|G_l^{\alpha,\beta}(g;u) - g(u)| \leqslant \widetilde{\omega}_r(g;u) \big(\lambda(u)\big)^{\frac{r}{2}}.$$

Proof. It is found that $|G_l^{\alpha,\beta}(g;u)-g(u)| \leqslant G_l^{\alpha,\beta}(|g(t)-g(u)|;u)$. In view of (3.8), we have $|G_l^{\alpha,\beta}(g;u)-g(u)| \leqslant \widetilde{\omega}_s(g;u)G_l^{\alpha,\beta}(|t-u|^r;u)$. In account of Hölder's inequality with the aid of $p=\frac{2}{r}$ and $q=\frac{2}{2-r}$, we have

$$|G_l^{\alpha,\beta}(g;u)-g(u)| \leqslant \widetilde{\omega}_r(g;u)\big(G_l^{\alpha,\beta}(|t-u|^2;u)\big)^{\frac{r}{2}}. \qquad \square$$

4. Approximation Properties Globally

Suppose that $\nu(u)=1+u^2, 0<u<1$ is a weight function. Then, $B_\nu(0,1)=\{g(u):|g(u)|\leqslant \tilde{M}_g(1+u^2)$, here $\tilde{M}_g$ is a constant based on g and $C_\nu(0,1)$ is the space of continuous function in $B_\nu(0,1)$ equipped with $\|g(u)\|_\nu=\sup_{u\in(0,1)}\frac{|g(u)|}{\nu(u)}$ and

$$C_\nu^{\tilde{k}}(0,1)=\Big\{g\in C_\nu(0,1):\lim_{u\to\infty}\frac{g(u)}{\nu(u)}=\tilde{k},\ \text{where constant } \tilde{k} \text{ is depending on } g\Big\}.$$

The Ditzian–Totik modulus of smoothness for the function g defined on the closed interval $[a,b]$ with $a,b>0$ is defined by

$$\omega_b(g,\tilde{\eta})=\sup_{|t-u|\leqslant\tilde{\eta}}\ \sup_{u,t\in[a,b]}|g(t)-g(u)|. \tag{4.1}$$

One can easily note that for any $g\in C_\nu(0,1)$, the modulus of smoothness given in (4.1) tends to zero.

Theorem 4.1. *Let $g\in C_\nu(0,1)$ and the modulus of continuity $\omega_{b+1}(g;\tilde{\eta})$ be given on $[a,b+1]\subset(0,1)$. Then, for any $u\in[a,b]$, we get*

$$\|G_l^{\alpha,\beta}(.;.)-g\|_{C[a,b]}\leqslant 4\tilde{M}_g(1+b^2)\tilde{\eta}_s(b)+2\omega_{b+1}\big(g;\sqrt{\tilde{\eta}_s(b)}\big),$$

where $\tilde{\eta}_s(b)=\max_{u\in[a,b]}G_l^{\alpha,\beta}(\psi_u^2(t);u)$.

Proof. For $u\in[a,b]$ and $t\in(0,1)$, we get

$$|g(t)-g(u)|\leqslant 4\tilde{M}_g(1+b^2)(t-u)^2+\left(1+\frac{|t-u|}{\tilde{\eta}}\right)\omega_{b+1}(g;\tilde{\eta}). \tag{4.2}$$

Using operator $G_l^{\alpha,\beta}(.;.)$ in (4.2), we get

$$\begin{aligned}|G_l^{\alpha,\beta}(g;u)-g(u)|\leqslant{}& 4\tilde{M}_g(1+b^2)G_l^{\alpha,\beta}((\psi_u^2(t);u)\\ &+\left(1+\frac{G_l^{\alpha,\beta}(|t-u|;u)}{\tilde{\eta}}\right)\omega_{b+1}(g;\tilde{\eta}).\end{aligned}$$

Further, in the light of Lemma 1.4 and $u\in[a,b]$, we have

$$|G_l^{\alpha,\beta}(.;.)-g|\leqslant 4\tilde{M}_g(1+b^2)\tilde{\eta}_l(b)+\left(1+\frac{\sqrt{\tilde{\eta}_l(b)}}{\tilde{\eta}}\right)\omega_{b+1}(g;\tilde{\eta}).$$

We take $\tilde{\eta}=\tilde{\eta}_l(b)$. Then, one can easily arrive at the desired result. $\square$

Remark 4.1. In what follows, we consider test function as $e_i(t)=t^i$, $i=0,1,2$.

Theorem 4.2. *[3, 4] Let the sequence of positive linear operators $(L_n)_{n\geqslant1}$ acting from $C_\nu(0,1)$ to $B_\nu(0,1)$ satisfy the conditions $\lim_{n\to\infty}||L_n(e_i;.)-e_i||_\nu=0$, for $i=0,1,2$. Then, for $g\in C_\nu^{\tilde{k}}(0,1)$, we have $\lim_{n\to\infty}||L_ng-g||_\nu=0$.*

THEOREM 4.3. *For $g \in C_{\nu}^{\tilde{k}}(0,1)$, we have* $\lim_{l\to\infty} \|G_l^{\alpha,\beta}(g;u) - g\|_{\nu} = 0$.

PROOF. We have to show that $\lim_{l\to\infty} \|G_l^{\alpha,\beta}(e_i;.) - e_i\|_{\nu} = 0$, for $i = 0,1,2$. In account of Lemma 1.3, it is obvious $\|G_l^{\alpha,\beta}(e_0;.) - 1\|_{\nu} = 0$, also

$$\begin{aligned}\|G_l^{\alpha,\beta}(e_1;.) - e_1\|_{\nu(u)} &= \sup_{u\in(0,1)} \frac{1}{\nu(u)}\left|\frac{2(l-2)u+3+2\alpha}{l+\beta+1} - u\right| \\ &= \frac{1}{l+\beta+1}\sup_{u\in(0,1)}\frac{(l-5-\beta)u}{1+u^2} + \frac{1}{l+\beta+1}\sup_{u\in(0,1)}\frac{3+2\alpha}{1+u^2}.\end{aligned}$$

For a large value of l, we get $\|G_l^{\alpha,\beta}(e_1;.) - e_1\|_{\nu} \to 0$. Also,

$$\begin{aligned}\|G_l^{\alpha,\beta}(e_2;.) - e_2\|_{\nu} \leqslant{}& \left(\frac{1}{(l+\beta+1)^2}\right)\sup_{u\in(0,1)}\frac{(l^2-7l-6)u^2}{1+u^2} \\ &+ \left(\frac{1}{(l+\beta+1)^2}\right)\sup_{u\in(0,1)}\frac{\{6l-8+(2l-4)\alpha\}u}{1+u^2} \\ &+ \left(\frac{1}{(l+\beta+1)^2}\right)\sup_{u\in(0,1)}\frac{7+9\alpha+3\alpha^2}{1+u^2}.\end{aligned}$$

This implies that $\|G_l^{\alpha,\beta}(e_2;.) - e_2\|_{\nu} \to 0$ as $l \to \infty$. □

THEOREM 4.4. *Let $g \in C_{\nu}^{\tilde{k}}(0,1)$ and $\zeta > 0$. Then,*

$$\lim_{n\to\infty}\sup_{u\in(0,1)}\frac{|G_l^{\alpha,\beta}(g;u) - g(u)|}{(1+u^2)^{1+\zeta}} = 0.$$

PROOF. Since $|g(u)| \leqslant \|g\|_{\nu}(1+u^2)$, for any real fixed number $u_0 > 0$, we get

$$\begin{aligned}\sup_{u\in(0,1)}\frac{|G_l^{\alpha,\beta}(g;u)-g(u)|}{(1+u^2)^{1+\zeta}} &\leqslant \sup_{u\leqslant u_0}\frac{|G_l^{\alpha,\beta}(g;u)-g(u)|}{(1+u^2)^{1+\zeta}} + \sup_{u\geqslant u_0}\frac{|G_l^{\alpha,\beta}(g;u)-g(u)|}{(1+u^2)^{1+\zeta}} \\ &\leqslant \|G_l^{\alpha,\beta}(g;u) - g(u)|\|_{C_{\nu}^{\tilde{k}}(0,1)} \\ &\quad + \|g\|_{\nu}\sup_{u\geqslant u_0}\frac{|G_l^{\alpha,\beta}(1+t^2;u)|}{(1+u^2)^{1+\zeta}} + \sup_{u\geqslant u_0}\frac{|g(u)|}{(1+u^2)^{1+\zeta}} \\ &= \tilde{W_1} + \tilde{W_2} + \tilde{W_3}, \quad \text{say.}\end{aligned} \tag{4.3}$$

Now,

$$\tilde{W_3} = \sup_{u\geqslant u_0}\frac{|g(u)|}{(1+u^2)^{1+\zeta}} \leqslant \sup_{u\geqslant u_0}\frac{\|g\|_{\nu}(1+u^2)}{(1+u^2)^{1+\zeta}} \leqslant \frac{\|g\|_{\nu}}{(1+u_0^2)^{\zeta}}.$$

In view of Lemma 1.3, it gives

$$\lim_{l\to\infty}\sup_{u\in[u_0,1)}\frac{G_l^{\alpha,\beta}(1+t^2;u)}{1+u^2} = 1.$$

Therefore, for any arbitrary $\epsilon > 0$, there corresponds $n_1 \in \mathbb{N}$ with

$$\sup_{u\in[u_0,1)}\frac{G_l^{\alpha,\beta}(1+t^2;u)}{1+u^2} \leqslant \frac{(1+u_0^2)^{\zeta}}{\|g\|_{\nu}}\frac{\epsilon}{3} + 1, \text{ for all } l \geqslant l_1,$$

that is

$$\tilde{W_2} = ||g||_\nu \sup_{u\in[u_0,1)} \frac{G_l^{\alpha,\beta}(1+t^2;u)}{(1+u^2)^{1+\zeta}} \leqslant \frac{||g||_\nu}{(1+u_0^2)^\zeta} + \frac{\epsilon}{3}, \text{ for all } l \geqslant l_1.$$

Hence, we get

$$\tilde{W_2} + \tilde{W_3} < 2\frac{\|g\|_\nu}{(1+u^2)^\zeta} + \frac{\epsilon}{3}.$$

If we take u_0 to be so large that $\frac{\|g\|_\nu}{(1+u^2)^\zeta} < \frac{\epsilon}{6}$, then we have

$$\tilde{W_2} + \tilde{W_3} < \frac{2\epsilon}{3} \text{ for all } l \geqslant l_1. \tag{4.4}$$

Now, from Theorem 4.1, there corresponds $l_2 > l$ with

$$\tilde{W_1} = \|G_l^{\alpha,\beta}(g;\cdot) - g\|_{C[0,u_0]} < \frac{\epsilon}{3} \text{ for all } l_2 \geqslant l. \tag{4.5}$$

Let $l_3 = \max(l_1, l_2)$. Then, with the aid of (4.3), (4.4) and (4.5), we get

$$\sup_{u\in(0,1)} \frac{|G_l^{\alpha,\beta}(g;u) - g(u)|}{(1+u^2)^{1+\zeta}} < \epsilon,$$

which completes the desired result. □

5. A-Statistical Approximation

Here, we recall a few abbreviations and notation from [**5**]. Let $A = (a_{l\nu})$ be non-negative infinite summability matrix. Then, a sequence $z := (z_\nu)$ is said to be A-statistically convergent to $\tilde{L}$, that is $st_A - \lim z = \tilde{L}$, if for all $\epsilon > 0$

$$\lim_l \sum_{\nu:|z_\nu \tilde{L}|\geqslant\epsilon} a_{l\nu} = 0.$$

Consider $q = \{q_l\}$ to be a sequence with the following conditions

$$\text{st}_A\text{-}\lim_l q_l = 1 \text{ and } \text{st}_A\text{-}\lim_l q_l^l = a,\ 0 \leqslant a < 1. \tag{5.1}$$

THEOREM 5.1. *Consider $A = (a_{l\nu})$ to be a nonnegative regular summability matrix and $q = \{q_l\}$ sequence with* (5.1) *and $q_l \in (0,1)$, $l \in \mathbb{N}$. Then, for every $g \in C_\nu^0(0,1)$, $\text{st}_A\text{-}\lim_l \|G_l^{\alpha,\beta}(g;.) - g\|_\nu = 0$.*

PROOF. From Lemma 1.3, we get $\text{st}_A\text{-}\lim_l \|G_l^{\alpha,\beta}(e_0;u) - e_0\|_\nu = 0$ and

$$\begin{aligned}\|G_l^{\alpha,\beta}(e_1;.) - u\|_\nu &= \sup_{u\in(0,1)} \frac{1}{1+u^2}\left|\frac{2(l-2)u+3+2\alpha}{l+\beta+1} - u\right| \\ &= \frac{1}{1+u^2}\sup_{u\in(0,1)} \frac{(l-5-\beta)u}{l+\beta+1} + \frac{1}{1+u^2}\sup_{u\in(0,1)} \frac{3+2\alpha}{l+\beta+1}.\end{aligned}$$

Now

$$\tilde{I_1} := \big\{l : \|G_l^{\alpha,\beta}(e_1;u) - u\| \geqslant \epsilon\big\},$$

$$\tilde{I_2} := \left\{l : \frac{(l-5-\beta)}{l+\beta+1} \geqslant \frac{\epsilon}{2}\right\}, \quad \tilde{I_3} := \left\{l : \frac{3+2\alpha}{l+\beta+1} \geqslant \frac{\epsilon}{2}\right\}.$$

This implies that $\tilde{I_1} \subseteq \tilde{I_2} \cup \tilde{I_3}$ and this shows that

$$\sum_{\nu \in \tilde{I_1}} a_{l\nu} \leqslant \sum_{\nu \in \tilde{I_2}} a_{l\nu} + \sum_{\nu \in \tilde{I_3}} a_{l\nu}.$$

Therefore, we get

$$\mathrm{st}_A\text{-}\lim_l \|G_l^{\alpha,\beta}(e_1;u) - u\|_\nu = 0. \tag{5.2}$$

Now, in the light of Lemma 1.3, we get

$$\|G_l^{\alpha,\beta}(e_2;u) - u^2\|_{1+u^2} \leqslant \sup_{u\in(0,1)} \frac{1}{\nu(u)} \bigg| \frac{1}{(l+\beta+1)^2} \bigg\{ (l^2 - 7l - 6)u^2 + (6l - 8 + 2l\alpha - 4\alpha)u + \Big(\frac{7}{3} + 3\alpha + \alpha^2\Big) \bigg\} - u^2 \bigg|.$$

For $\varepsilon > 0$, we have the following sets

$$\tilde{G_1} := \Big\{ l : \big\|G_l^{\alpha,\beta}(e_2;u) - u^2\big\|_\nu \geqslant \epsilon \Big\}, \quad \tilde{G_2} := \Big\{ l : \frac{l^2 - 7l - 7}{(l+\beta+1)^2} \geqslant \frac{\epsilon}{3} \Big\},$$
$$\tilde{G_3} := \Big\{ l : \frac{6l - 8 + 2l\alpha - 4\alpha}{(l+\beta+1)^2} \geqslant \frac{\epsilon}{3} \Big\}, \quad \tilde{G_4} := \Big\{ l : \frac{7 + 9\alpha + 3\alpha^2}{3(l+\beta+1)^2} \geqslant \frac{\epsilon}{3} \Big\}.$$

We note that $\tilde{G_1} \subseteq \tilde{G_2} \bigcup \tilde{G_3} \bigcup \tilde{G_4}$. Therefore, we get

$$\sum_{\nu \in \tilde{G_1}} a_{l\nu} \leqslant \sum_{\nu \in \tilde{G_2}} a_{l\nu} + \sum_{\nu \in \tilde{G_3}} a_{l\nu} + \sum_{\nu \in \tilde{G_4}} a_{l\nu}.$$

As $l \to \infty$, we have

$$\mathrm{st}_A\text{-}\lim_l \|G_l^{\alpha,\beta}(e_2;.) - e_2\|_\nu = 0. \tag{5.3}$$ □

Further, we examine the rate of A-statistical approximation in the account of Peetre's K-functional for $G_l^{\alpha,\beta}(.;.)$. Peetre's K-functional of $g \in C_B(0,1)$ is

$$K(g;\tilde{\eta}) = \inf_{h\in C_B^2(0,1)} \Big\{ \|g - h\|_{C_B(0,1)} + \tilde{\eta}\|h\|_{C_B^2(0,1)} \Big\},$$

where $\tilde{\eta} > 0$ and $C_B^2(0,1) = \{g \in C_B(0,1) : g', g'' \in C_B(0,1)\}$, endowed with the following relation of norm $\|g\|_{C_B^2(0,1)} = \|g\|_{C_B(0,1)} + \|g'\|_{C_B(0,1)} + \|g''\|_{C_B(0,1)}$.

THEOREM 5.2. *If* $g \in C_B^2(0,1)$, *then* $\mathrm{st}_A\text{-}\lim_l \|G_l^{\alpha,\beta}(g;\cdot) - g\|_{C_B(0,1)} = 0$.

PROOF. In the light of Taylor's theorem, we have

$$g(t) = g(u) + g'(u)(t-u) + \frac{1}{2}g''(\psi_u^0(t))(t-u)^2,$$

where $t \leqslant \psi \leqslant u$. On operating the operators $G_l^{\alpha,\beta}(.;.)$, both sides in the above equation, one get

$$G_l^{\alpha,\beta}(g;u) - g(u) = g'(u)G_l^{\alpha,\beta}(\psi_u^1(t);u) + \frac{1}{2}g''(\psi G_l^{\alpha,\beta}(\psi_u^2(t);u),$$

which yields that

$$\begin{aligned}
(5.4)\qquad \|G_l^{\alpha,\beta}(g;\cdot)-g\|_{C_B(0,1)} &\leqslant \|g'\|_{C_B(0,1)}\|G_l^{\alpha,\beta}(e_1-,.)\|_{C_B(0,1)}\\
&\quad+\|g''\|_{C_B(0,1)}\|G_l^{\alpha,\beta}(e_1-,.)^2\|_{C_B(0,1)}\\
&=\tilde{V}_1+\tilde{V}_2,\quad \text{say.}
\end{aligned}$$

From (5.2) and (5.3), we have

$$\lim_l \sum_{\nu\in\mathbb{N}:\tilde{V}_1\geqslant\frac{\epsilon}{2}} a_{l\nu}=0,\quad \lim_l \sum_{\nu\in\mathbb{N}:\tilde{V}_2\geqslant\frac{\epsilon}{2}} a_{l\nu}=0.$$

From (5.4) one has

$$\lim_l \sum_{\nu\in\mathbb{N}:\|G_l^{\alpha,\beta}(g;\cdot)-g\|_{C_B(0,1)}\geqslant\epsilon} a_{l\nu}\leqslant \lim_l \sum_{\nu\in\mathbb{N}:\tilde{V}_1\geqslant\frac{\epsilon}{2}} a_{l\nu}+\lim_l \sum_{\nu\in N:\tilde{V}_2\geqslant\frac{\epsilon}{2}} a_{l\nu}.$$

Therefore $\mathrm{st}_A\text{-}\lim_l \|G_l^{\alpha,\beta}(h;\cdot)-h\|_{C_B(0,1)}\to 0$ as $l\to\infty$. □

Theorem 5.3. *Let* $g\in C_B^2(0,1)$. *Then* $\|G_l^{\alpha,\beta}(g;\cdot)-g\|_{C_B(0,1)}\leqslant M\omega_2(g;\sqrt{\tilde{\eta}})$, *where*

$$\tilde{\eta}=\|G_l^{\alpha,\beta}(e_1-\cdot;\cdot)\|_{C_B(0,1)}+\|G_l^{\alpha,\beta}((e_1-\cdot)^2;\cdot)\|_{C_B(0,1)},$$

and $\|g\|_{C_B^2(0,1)}=\|g\|_{C_B(0,1)}+\|g'\|_{C_B(0,1)}+\|g''\|_{C_B(0,1)}$.

Proof. Let $h\in C_B^2(0,1)$. Using (5.4), we obtain that

$$\begin{aligned}
(5.5)\qquad \|G_l^{\alpha,\beta}(h)-h\|_{C_B(0,1)} &\leqslant \|h'\|_{C_B(0,1)}\|G_l^{\alpha,\beta}((e_1-\cdot);\cdot)\|_{C_B(0,1)}\\
&\quad+\frac{1}{2}\|h''\|_{C_B(0,1)}G_l^{\alpha,\beta}((e_1-\cdot)^2;\cdot)\|_{C_B(0,1)}\\
&\leqslant \tilde{\eta}\|h\|_{C_B^2(0,1)}.
\end{aligned}$$

Now, for every $g\in C_B(0,1)$ and $h\in C_B^2(0,1)$, from (5.5), one get

$$\begin{aligned}
\|G_l^{\alpha,\beta}(g;\cdot)-g\|_{C_B(0,1)} &\leqslant \|G_l^{\alpha,\beta}(g;\cdot)-G_l^{\alpha,\beta}(h;\cdot)\|_{C_B(0,1)}\\
&\quad+\|G_l^{\alpha,\beta}(h;\cdot)-h\|_{C_B(0,1)}+\|g-h\|_{C_B(0,1)}\\
&\leqslant 2\|h-g\|_{C_B(0,1)}+\|G_l^{\alpha,\beta}(h;\cdot)-h\|_{C_B(0,1)}\\
&\leqslant 2\|h-g\|_{C_B(0,1)}+\tilde{\eta}\|g\|_{C_B^2(0,1)}.
\end{aligned}$$

In view of Peetre's K-functional, we have $\|G_l^{\alpha,\beta}(g;\cdot)-g\|_{C_B(0,1)}\leqslant 2K_2(g;\tilde{\eta})$ and

$$\|G_l^{\alpha,\beta}(g;\cdot)-g\|_{C_B(0,1)}\leqslant \tilde{M}\big\{\omega_2(g;\sqrt{\tilde{\eta}})+\min(1,\tilde{\eta})\|g\|_{C_B(0,1)}\big\}.$$

In view of (5.3), we have $\mathrm{st}_A\text{-}\lim_l \tilde{\eta}=0$, therefore $\mathrm{st}_A\text{-}\lim_l \omega(h;\sqrt{\tilde{\eta}})=0$. □

References

1. S. N. Bernstein, *Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités*, Commun. Kharkov Math. Soc. **13**(1–2) (1912–1913).
2. R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Grundlehren Math. Wiss. **303**, Springer-Verlag, Berlin, 1993.
3. A. D. Gadjiev, *The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P. P. Korovkin*, Dokl. Akad. Nauk SSSR **218**(5) (1974).
4. ———, *On P. P. Korovkin type theorems*, Mat. Zametki **20** (1976), 781–786; Transl. in Math. Notes **(5–6)** (1978), 995–998.
5. A. D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32**(1) (2007), 129–138.
6. A. Izadbakhsh, A. A. Kalat, S. Khorashadizadeh, *Observer-based adaptive control for HIV infection therapy using the Baskakov operator*, Biomed. Signal Process. Control **65** (2021), 102343.
7. L. V. Kantorovich, *Sur certain developpments suivant les polynomes de la forme de S. Bernstein, I, II*, C. R. Acad. URSS (1930), 563–568.
8. K. Khan, D. K. Lobiyal, *Bèzier curves based on Lupaş (p,q)-analogue of Bernstein functions in CAGD*, Comput. Appl. Math. **317** (2017), 458–477.
9. K. Khan, D. K. Lobiyal, A. Kilicman, *Bézier curves and surfaces based on modified Bernstein polynomials*, Azerb. J. Math. **9**(1) (2019).
10. B. Lenze, *On Lipschitz type maximal functions and their smoothness spaces*, Nederl. Akad. Indag. Math. **50** (1988), 53–63.
11. M. A. Özarslan, H. Aktuğlu, *Local approximation for certain King type operators*, Filomat **27** (2013), 173–181.
12. A. Senapati, A. Kumar, T. Som, *Convergence analysis of modified Bernstein–Kantorovich type operators*, Rend. Circ. Mat. Palermo, Ser. 2 **72**(7), (2023), 3749–3764.
13. O. Shisha, B. Mond, *The degree of convergence of linear positive operators*, Proc. Nat. Acad. Sci. USA **60** (1968), 1196–1200.
14. F. Usta, *On new modification of Bernstein operators: theory and applications*, Iran. J. Sci. Technol. Trans. **44**(7) (2020), 1119–1124.
15. H. Uyan, A. O. Aslan, S. Karateke, İ. Büyükyazıcı, *Interpolation for neural network operators activated with a generalized logistic-type function*, J. Inequal. Appl. (2024), Paper No. 125, 18 pp.
16. Q. Zhang, M. Mu, X. Wang, *A modified robotic manipulator controller based on Bernstein–Kantorovich–Stancu operator*, Micromachines **14**(1) (2022).

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