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STATISTICAL CONVERGENCE AND STATISTICAL LIMIT POINTS OF DOUBLE SEQUENCES WITH RESPECT TO A POWER SERIES METHOD

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Abstract. Using the J_p^2-power series method, we redefine the sets of statistical limit points and statistical cluster points for double sequences of real numbers. We compare these sets with the set of usual limit points of double sequences and provide examples to demonstrate that the inclusions between these sets can be strict.

1. Introduction

Statistical convergence of sequences of real numbers was first discussed in [**20**] and [**47**]. Following these studies, statistical convergence has led to numerous new research topics in the field of mathematical analysis. Various properties of this notion of convergence have been studied and compared with other types of convergence by many authors (e.g., [**7–10**, **21–24**, **28**, **41–44**] etc.). The convergence of double sequences of real numbers was first given in [**39**]. Statistical convergence of double sequences was defined and some of its properties were investigated independently in [**35**] and [**36**]. Many more properties of this convergence were also investigated in [**13**, **29**, **40**, **50**]. Statistical convergence has also been extended to other types of convergence, for example, lacunary statistical convergence [**15**, **26**, **27**, **45**], A-statistical convergence [**14**, **19**, **25**], and λ-statistical convergence [**37**].

In summability theory, the power series method and especially the J_p-power series method was first used in Tauberian theorems (see [**1**, **5**, **6**, **30–34**, **46**, **48**]). Recently, statistical convergence has been extended to statistical convergence by the power series method in [**52**] (see also [**16–18**]). The J_p-statistical convergence and J_p-strong convergence with respect to the modulus function and the J_p-power series method were compared in [**4**]. The characterizations of statistical convergence via the J_p-power series method were explored by Sümbül et al. [**49**]. In [**3**] and [**51**]

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the concept of statistical convergence with respect to power series methods was discussed. Bayram [**2**] introduced the concepts of statistical limit superior and inferior with respect to the P_p-power series method. Yıldız et al. [**54**] used the same method and gave Korovkin-type theorems for double sequences via P_p^2-statistical relative modular convergence. Also in [**53**] a new definition of uniform integrability in the P_p^2-sense was given and was compared with the previously defined methods. Coşar et al. [**11**] studied some properties of J_p^2-statistical convergence for double sequences and compared J_p^2-strong convergence with J_p^2-statistical convergence and uniform integrability. Mursaleen et al. [**38**] compared J_p^2-statistical convergence and J_p^2-strong convergence for double sequences using a modulus function and the J_p^2-power series method. Coşar et al. [**12**] gave the definition of deferred Riesz statistical convergence for double sequences via the J_p^2-power series method and presented an application to a Korovkin type approximation theorem.

In this article, we introduce the definitions of J_p^2-statistical limit points and J_p^2-statistical cluster points for a double sequence. The set of ordinary limit points of double sequences, the set of J_p^2-statistical limit points, and the set of J_p^2-statistical cluster points are compared. Examples are given to demonstrate that the inclusions between these sets can be strict.

Let U be a nonempty set. A double sequence in U is a function $x\colon \mathbb{N}\times\mathbb{N}\to U$. The value of x at (j,k) is denoted by x_{jk}. A double sequence in U is denoted by $x=(x_{jk})$.

Throughout the article, the term double sequence will be understood as a double real sequence.

A double sequence $x=(x_{jk})$ is bounded if there exists a number $M>0$ such that $|x_{jk}|<M$ for all j and k, i.e., if $\|x\|_{(\infty,2)}=\sup_{j,k}|x_{j,k}|<\infty$. It is defined as

$$\ell_\infty^2 := \Big\{x=(x_{jk}) : \sup_{j,k}|x_{j,k}|<\infty\Big\}.$$

Let $x=(x_{mn})$ be a sequence of real or complex numbers. The sequence $x=(x_{mn})$ is said to be convergent to μ in the Pringsheim sense if for every $\varepsilon>0$, there exists a natural number $N_0(\varepsilon)$ such that $|x_{mn}-\mu|<\varepsilon$ for all $m,n>N_0$. This is denoted by $\text{P-lim}_{m,n\to\infty} x_{mn}=\mu$. It is important to note that if $\text{P-lim}\, x=\mu$, then $x=(x_{mn})$ does not necessarily belong to ℓ_∞^2 [**39**].

Throughout the article, $x=(x_{jk})$ and $y=(y_{jk})$ will denote double sequences with real entries and the subsequence corresponding to the indices $(j,k)\in K$ of the sequence $x=(x_{jk})$ will be shown with $\{x\}_K$.

The statistical convergence of sequences of the form $x=(x_{mn})$ was introduced independently in [**36**] and [**35**]. Let us take $K\subset\mathbb{N}^2$ and let

$$K(u,v) := \{(m,n)\in K : m\leqslant u \text{ and } n\leqslant v\}.$$

If the limit

$$\delta_2(K)=\underset{u,v\to\infty}{\text{P-lim}}\,\frac{K(u,v)}{uv}$$

in the Pringsheim sense exists, $\delta_2(K)$ is called the double density of K.

Let $x = (x_{mn})$ be given. Using the concept of density, if

$$\delta_2(\{(m,n) \in \mathbb{N}^2 : |x_{mn} - \mu| \geqslant \varepsilon\}) = 0$$

for all $\varepsilon > 0$ or equivalently

$$\text{P-}\lim_{u,v} \frac{1}{uv} |\{(m,n), m \leqslant u, n \leqslant v : |x_{mn} - \mu| \geqslant \varepsilon\}| = 0$$

for each $\varepsilon > 0$, then the sequence $x = (x_{mn})$ is said to be statistically convergent to μ and denoted by $\text{st}_2\text{-}\lim_{m,n} x_{mn} = \mu$. The set of all statistical convergence double sequences is denoted by S_2 or st_2 [**36**].

The sets of statistical limit points Λ_x^2 and cluster points Γ_x^2 of a double sequence $x = (x_{mn})$ are defined as follows, respectively (see [**13**]):

$$\mu \in \Lambda_x^2 \iff \text{there exists } K \subset \mathbb{N}_0^2 \text{ such that } \{x\}_K \to \mu \text{ and } \delta_2(K) \neq 0,$$
$$\mu \in \Gamma_x^2 \iff \text{for } \forall \varepsilon > 0, \delta_2(\{(m,n) \in \mathbb{N}_0^2 : |x_{mn} - \mu| < \varepsilon\}) \neq 0.$$

2. J_p^2-statistical convergence in double sequences

Let (p_{mn}) be a sequence of nonnegative entries such that $p_{00} > 0$. Let

$$p(s,t) := \sum_{m,n=0}^{\infty} p_{mn} s^m t^n < \infty \quad \text{for} \quad s,t \in (0,1),$$

which is assumed to satisfy $p(t,s) \to \infty$ as $t, s \to 1^-$.

Note that the limit here is in the Pringsheim sense. If the limit

$$\lim_{t,s \to 1^-} \frac{p_x(t,s)}{p(t,s)} = L$$

exists for the double sequence $x = (x_{mn})$ such that

$$p_x(s,t) := \sum_{m,n=0}^{\infty} x_{mn} p_{mn} s^m t^n < \infty \quad \text{for} \quad s,t \in (0,1),$$

then $x = (x_{mn})$ is L-summable by the mean of the power series J_p^2 and we write $J_p^2\text{-}\lim x_{mn} = L$ [**1**].

Definition 2.1. [**11**] Let $A \subset \mathbb{N}_0^2$. If the limit

$$\delta_{J_p^2}(A) := \lim_{t,s \to 1^-} \frac{1}{p(s,t)} \sum_{(j,k) \in A} p_{jk} s^j t^k$$

exists, $\delta_{J_p^2}(A)$ is called J_p^2-density of A.

We present some main properties of the J_p^2 density.

i) Since

$$p(s,t) = \sum_{j=0}^{\infty} p_{jk} s^j t^k < \infty,$$

we get

$$\delta_{J_p^2}(\mathbb{N}^2) = \lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in\mathbb{N}^2} p_{jk} s^j t^k = 1.$$

ii) For any $A \subset \mathbb{N}_0^2$, if $\delta_{J_p^2}(A) = 0$, then $\delta_{J_p^2}(\mathbb{N}_0^2 \setminus A) = 1$. Indeed, since $\delta_{J_p^2}(A) = 0$,

$$\lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in A} p_{jk} s^j t^k = 0.$$

Hence

$$\begin{aligned}
\delta_{J_p^2}(\mathbb{N}^2) &= \lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in\mathbb{N}^2} p_{jk} s^j t^k \\
&= \lim_{s,t\to 1^-} \frac{1}{p(s,t)} \left(\sum_{(j,k)\in\mathbb{N}^2 - A} p_{jk} s^j t^k + \sum_{(j,k)\in A} p_{jk} s^j t^k \right) \\
&= \lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in\mathbb{N}^2 - A} p_{jk} s^j t^k = 1,
\end{aligned}$$

then we have $\delta_{J_p^2}(\mathbb{N}_0^2 \setminus A) = 1$.

iii) When $A \subset \mathbb{N}_0^2$ is a finite set, it is clear that

$$\delta_{J_p^2}(A) = \lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in A} p_{jk} s^j t^k = 0.$$

iv) If $A \subset B \subset \mathbb{N}_0^2$ then $\delta_{J_p^2}(A) \leqslant \delta_{J_p^2}(B)$. Indeed, since p_{jk} are positive and $s,t \in (0,1)$,

$$\sum_{(j,k)\in A} p_{jk} s^j t^k \leqslant \sum_{(j,k)\in B} p_{jk} s^j t^k.$$

If we take the limit for $s,t \to 1^-$ from both sides in the inequality

$$\frac{1}{p(s,t)} \sum_{(j,k)\in A} p_{jk} s^j t^k \leqslant \frac{1}{p(s,t)} \sum_{(j,k)\in B} p_{jk} s^j t^k,$$

we get $\delta_{J_p^2}(A) \leqslant \delta_{J_p^2}(B)$.

The double density and the density J_p^2 of an $A \subset \mathbb{N}_0^2$ set do not have to be equal:

Example 2.1. Let $A = \{(2j+1, 2k) : j,k \in \mathbb{N}_0\}$ and let p_{jk} factorize as $p_{jk} = p_j q_k$:

$$p_j = \begin{cases} 1, & j = 2l \\ 0, & j = 2l+1 \end{cases} \qquad q_k = \begin{cases} 1, & k = 2l+1 \\ 0, & k = 2l \end{cases} \qquad x_{jk} = \begin{cases} 1, & j = 2l \text{ and } k = 2l \\ 0, & \text{otherwise} \end{cases}$$

for $l \in \mathbb{N}_0$. Thus the sequence $x = (x_{jk})$ is not convergent in the Pringsheim sense, $\delta_{J_p^2}(A) = 0$ and

$$\delta^2(A) = \lim_{m,n} \frac{m.n}{a_{mn}} = \lim_{m,n} \frac{m.n}{(2m+1)(2n)} = \frac{1}{4}.$$

DEFINITION 2.2. [**11**] Let the sequence $x = (x_{jk})$ and $\mu \in \mathbb{R}$ be given and define $E_\varepsilon := \{(j,k) \in \mathbb{N}_0^2 : |x_{jk} - \mu| \geqslant \varepsilon\}$. If $\delta_{J_p^2}(E_\varepsilon) = 0$ for each $\varepsilon > 0$ i.e.,

$$\lim_{t,s\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in E_\varepsilon} p_{jk} t^j s^k = 0 \quad \text{for all } \varepsilon > 0,$$

then $x = (x_{jk})$ is said to be J_p^2-statistically convergent to μ and we write $st_{J_p^2}$-$\lim x = \mu$.

3. J_p^2-Statistical limit points and J_p^2-statistical cluster points

In this section, we introduce the definitions of J_p^2-statistical limit points and J_p^2-statistical cluster points for a double sequence and compare the set of ordinary limit points of a double sequence, the set of J_p^2-statistical limit points, and the set of J_p^2-statistical cluster points.

DEFINITION 3.1. If there exists $K \subset \mathbb{N}_0^2$ such that $\{x\}_K \to \mu$ and $\delta_{J_p^2}(K) \neq 0$, then μ is called a J_p^2-statistical limit point of $x = (x_{jk})$. The collection of all such points is denoted by $\Lambda_x^{J_p^2}$.

DEFINITION 3.2. We say that μ is a J_p^2-statistical cluster point of $x = (x_{jk})$ provided that the condition $\delta_{J_p^2}(\{(j,k) \in \mathbb{N}_0^2 : |x_{jk} - \mu| < \varepsilon\}) > 0$ holds for every $\varepsilon > 0$. The collection of all such points is denoted by $\Gamma_x^{J_p^2}$.

We shall use the notation L_x to represent the set of all classical limit points of $x = (x_{jk})$.

EXAMPLE 3.1. Let p_{jk} factorize as $p_{jk} = p_j q_k$ and let

$$p_j = \begin{cases} 1, & j = 2l+1 \\ 0, & \text{otherwise} \end{cases}, \quad q_k = \begin{cases} 1, & k = 2l+1 \\ 0, & \text{otherwise} \end{cases}, \quad \text{for all } l \in \mathbb{N}_0$$

and define

$$(x_{jk}) := \begin{cases} 1, & \text{if } j = 2m+1,\ k = 2n+1 \\ 0, & \text{otherwise} \end{cases}, \quad m,n \in \mathbb{N}_0,$$

and sets

$$E := \{(j,k) \in \mathbb{N}^2 : j = 2m+1, k = 2n+1,\ m,n \in \mathbb{N}_0\}.$$

In this case we have

$$\begin{aligned} p(t,s) &= \sum_{j,k=0}^{\infty} p_j q_k t^j s^k = \sum_{j,k=0}^{\infty} p_{2j+1} q_{2k+1} t^{2j+1} s^{2k+1} + \sum_{j,k=0}^{\infty} p_{2j} q_{2k} t^{2j} s^{2k} \\ &\quad + \sum_{j,k=0}^{\infty} p_{2j+1} q_{2k} t^{2j+1} s^{2k} + \sum_{j,k=0}^{\infty} p_{2j} q_{2k+1} t^{2j} s^{2k+1} \\ &= \sum_{j,k=0}^{\infty} p_{2j+1} q_{2k+1} t^{2j+1} s^{2k+1} = \left(\sum_{j=0}^{\infty} t^{2j+1}\right)\left(\sum_{k=0}^{\infty} s^{2k+1}\right) \end{aligned}$$

$$= \frac{t}{1-t^2}\frac{s}{1-s^2}, \quad |t|<1 \text{ and } |s|<1,$$

$$\delta_{J_p^2}(E) = \lim_{t,s\to 1^-} \frac{1}{p(t,s)} \sum_{(j,k)\in E} p_j q_k t^j s^k = \lim_{t,s\to 1^-} \frac{1}{p(t,s)} \sum_{j,k=0}^{\infty} 1.1.t^{2j+1}s^{2k+1} = 1,$$

Even though $L_x = \{0,1\}$, since the subsequence $\{x\}_E$ converges to 1 and the J_p^2-density of E is nonzero, it follows that $\Lambda_x^{J_p^2} = \{1\}$.

It is possible to provide an example illustrating that the sets Λ_x^2 and $\Lambda_x^{J_p^2}$ are incomparable.

Example 3.2. Let p_{jk} factorize as $p_{jk} = p_j q_k$ and let

$$p_j = \begin{cases} 1, & \text{if } j = m^2 \\ 0, & \text{if } j \neq m^2 \end{cases} \text{ and } q_k = \begin{cases} 1, & \text{if } k = n^2 \\ 0, & \text{if } k \neq n^2 \end{cases}, \quad m,n \in \mathbb{N}_0.$$

Let's define $x = (x_{jk})$ multiplicatively as $x_{jk} = x_j y_k$, where

$$x_j = \begin{cases} 2, & \text{if } j = m^2 \\ 1, & \text{if } j \neq m^2 \text{ and } j \text{ is odd} \\ 0, & \text{if } j \neq m^2 \text{ and } j \text{ is even} \end{cases} \quad y_k = \begin{cases} 2, & \text{if } k = n^2 \\ 1, & \text{if } k \neq n^2 \text{ and } k \text{ is odd} \\ 0, & \text{if } k \neq n^2 \text{ and } k \text{ is even} \end{cases} \quad m,n \in \mathbb{N}_0.$$

Then, since $\{x\}_{E_1} \to 1$ and $\{x\}_{E_2} \to 0$ where

$$E_1 = \{(j,k) \in \mathbb{N}_0^2 : j,k \text{ are odd nonsquare}\},$$
$$E_2 = \{(j,k) \in \mathbb{N}_0^2 : j,k \text{ are even nonsquare}\},$$
$$E_3 = \{(j,k) \in \mathbb{N}_0^2 : j,k \text{ are square}\},$$

we have $L_x = \{0,1,2\}$. We can also see that

$$\delta_{J_p^2}(E_3) = \lim_{t,s\to 1^-} \frac{1}{p(t,s)} \sum_{(j,k)\in E} p_{jk} t^j s^k \neq 0.$$

So we get $\Lambda_x^{J_p^2} = \{2\}$. But let us note that $\Lambda_x^2 = \{0,1\}$.

For $x = (x_{jk})$, the inclusion $\Lambda_x^{J_p^2} \subset L_x$ may be very large.

Example 3.3. Let $\{r_{jk}\}_{j,k=1}^{\infty}$ be a sequence whose range is the set of all rational numbers and define

$$x_{jk} := \begin{cases} jk, & \text{if } j = 2m+1,\ k = 2n+1 \\ r_{jk}, & \text{otherwise} \end{cases}, \quad m,n \in \mathbb{N}_0,$$

$$p_j := \begin{cases} 1, & \text{if } j = 2m+1 \\ 0, & \text{if } j = 2m \end{cases}, \quad m \in \mathbb{N}_0, \quad q_k := \begin{cases} 1, & \text{if } k = 2n+1 \\ 0, & \text{if } k = 2n \end{cases}, \quad n \in \mathbb{N}_0,$$

$$E_1 := \{(2j+1, 2k+1) : j,k \in \mathbb{N}_0\}, E_2 := \{(2j, 2k) : j,k \in \mathbb{N}_0\},$$
$$E_3 := \{(2j, 2k+1) : j,k \in \mathbb{N}_0\}, \quad E_4 := \{(2j+1, 2k) : j,k \in \mathbb{N}_0\}.$$

Hence, we have $\delta_{J_p^2}(E_1) = 1$, $\delta_{J_p^2}(E_2) = \delta_{J_p^2}(E_3) = \delta_{J_p^2}(E_4) = 0$. Since $\{x\}_{E_1}$ does not converge, we get $\Lambda_x^{J_p^2} = \emptyset$. However, the set L_x, which is the of limit points $x = (x_{jk})$, is equal to $\mathbb{R}$, because the sequence $\{r_{jk}\}_{j,k=1}^{\infty}$ is dense in the real numbers.

The following theorem shows that for a double sequence $x = (x_{jk})$, the set of points of the J_p^2-cluster is contained in the set of J_p^2-statistical limit points and the set of points of the J_p^2-statistical cluster is contained in the set of limit points.

Theorem 3.1. *Let $x = (x_{jk})$ be any double sequence. Then $\Lambda_x^{J_p^2} \subset \Gamma_x^{J_p^2} \subset L_x$.*

Proof. Let us take any $\mu \in \Lambda_x^{J_p^2}$. Then, there exists a subsequence $\{x\}_K$ of x such that P-lim$\{x\}_K = \mu$ where $K = \{(j_n, k_m) \in \mathbb{N}_0^2 : n, m \in \mathbb{N}_0\} \subset \mathbb{N}_0^2$ and $\delta_{J_p^2}(K) \neq 0$. Since $\{x\}_K \to \mu$, for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{j_n k_m} - \mu| < \varepsilon$ for $j_n, k_m > N$. Also since

$$\{(j_n, k_m) \in K : |x_{j_n k_m} - \mu| < \varepsilon\} \subset \{(j, k) \in \mathbb{N}_0^2 : |x_{jk} - \mu| < \varepsilon\},$$

we have

$$\delta_{J_p^2}(\{(j_n, k_m) \in K : |x_{j_n k_m} - \mu| < \varepsilon\}) \leqslant \delta_{J_p^2}(\{(j, k) \in \mathbb{N}_0^2 : |x_{jk} - \mu| < \varepsilon\});$$

that is, there exists a number $\alpha \neq 0$ such that $\delta_{J_p^2}(\{(j, k) \in \mathbb{N}_0^2 : |x_{jk} - \mu| < \varepsilon\}) = \alpha$. Therefore, $\mu \in \Gamma_x^{J_p^2}$. Hence we find $\Lambda_x^{J_p^2} \subset \Gamma_x^{J_p^2}$. Now let us take any $\mu \in \Gamma_x^{J_p^2}$. Then for all $\varepsilon > 0$, there exists a number $\alpha \neq 0$ such that

$$\alpha = \delta_{J_p^2}(\{(j, k) \in \mathbb{N}^2 : |x_{jk} - \mu| < \varepsilon\}) = \lim_{t,s \to 1^-} \frac{1}{p(t, s)} \sum_{(j,k):|x_{jk} - \mu| < \varepsilon} p_{jk} t^j s^k.$$

Thus, we obtain

$$\alpha p(t, s) \sim \sum_{(j,k):|x_{jk} - \mu| < \varepsilon} p_{jk} t^j s^k \text{ as } t, s \to 1^-.$$

Therefore, there are infinitely many x_{jk} terms in the interval $(\mu - \varepsilon, \mu + \varepsilon)$. That is, $\mu \in L_x$. Hence $\Gamma_x^{J_p^2} \subset L_x$. Thus, $\Lambda_x^{J_p^2} \subset \Gamma_x^{J_p^2} \subset L_x$. □

To see that the inclusion in Theorem 3.1 is strict, let us examine the following example.

Example 3.4. Let $p_{jk} = 1$ for all $j, k \in \mathbb{N}_0$. Then, we have

$$p(t, s) = \sum_{j,k=0}^{\infty} p_{jk} t^j s^k = \left(\sum_{j=0}^{\infty} t^j\right)\left(\sum_{k=0}^{\infty} s^k\right) = \frac{1}{1-t}\frac{1}{1-s} \text{ for } |t| < 1 \text{and } |s| < 1.$$

Also, let us define $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} 0, & \text{if } j = k = 0 \\ \frac{1}{r}, & \text{if } j = k = 2^{r-1}(2q+1). \\ jk, & \text{otherwise.} \end{cases}$$

In this case we can determine the densities of the sets as follows:

$$\begin{aligned}\delta_{J_p^2}(\{(j,k):x_{jk}=1\}) &= \delta_{J_p^2}(\{j=k=2l+1:l\in\mathbb{N}_0\})\\ &= \lim_{t,s\to1^-}\frac{1}{p(t,s)}\sum_{j,k=0}^{\infty}p_{jk}t^js^k=\lim_{t,s\to1^-}\sum_{j,k=0}^{\infty}t^{2k+1}s^{2k+1}\\ &= \lim_{t,s\to1^-}\frac{ts(1-t)(1-s)}{(1-t^2)(1-s^2)}=4^{-1},\end{aligned}$$

$$\begin{aligned}\delta_{J_p^2}(\{(j,k):x_{jk}=1/2\}) &= \delta_{J_p^2}(\{j=k=4l+2:l\in\mathbb{N}_0\})\\ &= \lim_{t,s\to1^-}(1-t)(1-s)\sum_{j,k=0}^{\infty}t^{4k+2}s^{4k+2}=4^{-2},\end{aligned}$$

$$\begin{aligned}\delta_{J_p^2}(\{(j,k):x_{jk}=1/3\}) &= \delta_{J_p^2}(\{j=k=8l+4:l\in\mathbb{N}_0\})\\ &= \lim_{t,s\to1^-}(1-t)(1-s)\sum_{j,k=0}^{\infty}t^{8k+4}s^{8k+4}=4^{-3},\end{aligned}$$

$$\vdots$$

Therefore, $\delta_{J_p^2}(\{(j,k):x_{jk}=1/r\})=4^{-r}>0$ for each r, that is, $\frac{1}{r}\in\Lambda_x^{J_p^2}$. Similarly, it can be seen that

$$\delta_{J_p^2}\Big(\Big\{(j,k):|x_{jk}|<\frac{1}{r}\Big\}\Big)=\delta_{J_p^2}\Big(\Big\{(j,k):0<x_{jk}<\frac{1}{r}\Big\}\Big)=4^{-r}.$$

Therefore, we obtain that $0\in\Gamma_x^{J_p^2}$, i.e., $\Gamma_x^{J_p^2}=\{0\}\cup\{\frac{1}{r}\}_{r=1}^{\infty}$. Let us now show that $0\notin\Lambda_x^{J_p^2}$. Suppose, to the contrary, that $0\in\Lambda_x^{J_p^2}$. Then there exists a $K\subset\mathbb{N}_0^2$ such that $\{x\}_K\to0$ and $\delta_{J_p^2}(K)\neq0$. Therefore, we see that

$$\begin{aligned}\delta_{J_p^2}(K) &= \lim_{t,s\to1^-}\frac{1}{p(t,s)}\sum_{(j,k)\in K:x_{jk}<1/r}p_{jk}t^js^k\\ &\quad+\lim_{t,s\to1^-}\frac{1}{p(t,s)}\sum_{(j,k)\in K:x_{jk}\geqslant1/r}p_{jk}t^js^k\leqslant4^{-r}+o(1).\end{aligned}$$

for all r. Hence, $\delta_{J_p^2}(K)=0$, which is a contradiction. Thus $0\notin\Lambda_x^{J_p^2}$.

The example above shows that $\Lambda_x^{J_p^2}$ need not be a closed set of points, but the following theorem says that $\Gamma_x^{J_p^2}$ is closed, as is L_x.

THEOREM 3.2. *For a sequence $x=(x_{jk})$, the set $\Gamma_x^{J_p^2}$ is a closed set.*

PROOF. Let $\mu\in\overline{(\Gamma_x^{J_p^2})}$. Since $(\mu-\varepsilon,\mu+\varepsilon)\cap\Gamma_x^{J_p^2}\neq\emptyset$, for every $\varepsilon>0$, there is at least one $\lambda\in(\mu-\varepsilon,\mu+\varepsilon)\cap\Gamma_x^{J_p^2}$. Let us take ε' as $(\lambda-\varepsilon',\lambda+\varepsilon')\subset(\mu-\varepsilon,\mu+\varepsilon)$.

Now, since $\lambda \in \Gamma_x^{J_p^2}$, the density $\delta_{J_p^2}$ of the set $\{(j,k) : x_{jk} \in (\lambda - \varepsilon', \lambda + \varepsilon')\}$ is nonzero. Thus, the set $\{(j,k) : x_{jk} \in (\mu - \varepsilon, \mu + \varepsilon)\}$ also has a nonzero density $\delta_{J_p^2}$. This gives $\mu \in \Gamma_x^{J_p^2}$, meaning that $\Gamma_x^{J_p^2}$ is a closed set. □

Let $x = (x_{jk})$ and $y = (y_{jk})$ be double sequences. If $\delta_{J_p^2}(\{(j,k) : x_{jk} \neq y_{jk}\}) = 0$, then we say that $x_{jk} = y_{jk}$ for almost all j, k in the sense of J_p^2.

The following theorem says that the sets of J_p^2-statistical limit points of the sequences $x = (x_{jk})$ and $y = (y_{jk})$ such that $x_{jk} = y_{jk}$ for almost all j, k in the sense of J_p^2 are equal.

Theorem 3.3. *Let $x = (x_{jk})$ and $y = (y_{jk})$ be given as sequences such that $\delta_{J_p^2}(\{(j,k) : x_{jk} \neq y_{jk}\}) = 0$. Then $\Lambda_x^{J_p^2} = \Lambda_y^{J_p^2}$ and $\Gamma_x^{J_p^2} = \Gamma_y^{J_p^2}$.*

Proof. Let $\delta_{J_p^2}(\{(j,k) : x_{jk} \neq y_{jk}\}) = 0$. Let us take any $\mu \in \Lambda_x^{J_p^2}$. Then there exists $K \subset \mathbb{N}_0^2$ such that $\{x\}_K \to \mu$ and $\delta_{J_p^2}(K) \neq 0$. We know that

$$\delta_{J_p^2}(\{(j,k) \in \mathbb{N}_0^2 : x_{jk} \neq y_{jk}\}) = 0, \quad \delta_{J_p^2}(\{(j,k) \in \mathbb{N}_0^2 : x_{jk} = y_{jk}\}) \neq 0.$$

If we say $M := \{(j,k) \in \mathbb{N}_0^2 : x_{jk} = y_{jk}\}$, we find a subsequence $\{y\}_M$ of $\{y\}$ such that $\delta_{J_p^2}(M) \neq 0$ and $\{y\}_M \to \mu$. This means that $\mu \in \Lambda_y^{J_p^2}$. That is, it follows that $\Lambda_x^{J_p^2} \subset \Lambda_y^{J_p^2}$. Since the inclusion $\Lambda_y^{J_p^2} \subset \Lambda_x^{J_p^2}$ can be shown in the same way, we obtain $\Lambda_y^{J_p^2} = \Lambda_x^{J_p^2}$.

Now let us show the equality $\Gamma_x^{J_p^2} = \Gamma_y^{J_p^2}$. Let $\mu \in \Gamma_x^{J_p^2}$. For every $\varepsilon > 0$,

$$\delta_{J_p^2}(\{(j,k) \in \mathbb{N}_0^2 : |x_{jk} - \mu| < \varepsilon\}) \neq 0.$$

Now let us denote the set $\{(j,k) \in \mathbb{N}_0^2 : |x_{jk} - \mu| < \varepsilon\}$ by E and consider the following decomposition of this set:

$$E_1 := \{(j,k) \in \mathbb{N}_0^2 : x_{jk} \neq y_{jk} \text{ and } |x_{jk} - \mu| < \varepsilon\},$$
$$E_2 := \{(j,k) \in \mathbb{N}_0^2 : x_{jk} = y_{jk} \text{ and } |x_{jk} - \mu| < \varepsilon\}.$$

Since

$$\frac{1}{p(t,s)} \sum_{(j,k)\in E} p_j t^j q_k s^k = \frac{1}{p(t,s)} \sum_{(j,k)\in E_1} p_j t^j q_k s^k + \frac{1}{p(t,s)}$$

and $\delta_{J_p^2}(E_1) = 0$, we find

$$0 \neq \lim_{t,s\to 1^-} \frac{1}{p(t,s)} \sum_{(j,k)\in E} p_{jk} t^j s^k = \frac{1}{p(t,s)} \sum_{(j,k)\in E_2} p_{jk} t^j s^k,$$

in this case we get $\delta_{J_p^2}(\{(j,k) \in \mathbb{N}_0^2 : |y_{jk} - \mu| < \varepsilon\}) \neq 0$ for each $\varepsilon > 0$. This means that $\mu \in \Gamma_y^{J_p^2}$. That is, it is valid to include $\Gamma_y^{J_p^2} \subset \Gamma_x^{J_p^2}$. Since the inclusion $\Gamma_x^{J_p^2} \subset \Gamma_y^{J_p^2}$ can be shown in the same way, we obtain $\Gamma_y^{J_p^2} = \Gamma_x^{J_p^2}$. □

Now we can give the following result as in Theorem 2 of [**24**].

THEOREM 3.4. *For $x = (x_{jk})$, there exists another $y = (y_{jk})$ satisfying $L_y = \Gamma_x^{J_p^2}$ and $\delta_{J_p^2}(\{(j,k) \in \mathbb{N}_0^2 : x_{jk} \neq y_{jk}\}) = 0$. Furthermore, the sequence $y = (y_{jk})$ is a subsequence of the sequence $x = (x_{jk})$.*

PROOF. Let $\Gamma_x^{J_p^2} \subsetneq L_x$. For each $\alpha \in L_x - \Gamma_x^{J_p^2} =: A$, choose an open ball I_α centered at α such that $\delta_{J_p^2}(\{(j,k) : x_{jk} \in I_\alpha\}) = 0$. Then $L_x - \Gamma_x^{J_p^2} \subset \bigcup_{\alpha \in A} I_\alpha$. By the Lindelöf covering property (every open cover of a closed set in a Lindelöf space has a countable subcover), $(I_\alpha)_{\alpha \in A}$ has a countable subcover. Let this subcover be $\{I_m\}_{m=1}^{\infty}$, where $\delta_{J_p^2}(\{(j,k) : x_{jk} \in I_m\}) = 0$. Thus, each I_m contains a subsequence of $x = (x_{jk})$ with density zero in the J_p^2-sense. By [**10**, Corollary 9], this gives a unique set Ω such that $\delta_{J_p^2}(\Omega) = 0$ and for each m, $(\{(j,k) : x_{jk} \in I_m\}) - \Omega$ is a finite set. Let

$$\mathbb{N}_0^2 - \Omega := \{(m(j), m(k)) : j,k \in \mathbb{N}_0\}$$

and define the sequence y by

$$y_{jk} := \begin{cases} x_{m(j),m(k)}, & (j,k) \in \Omega \\ x_{jk}, & (j,k) \in \mathbb{N}_0^2 - \Omega. \end{cases}$$

Then $\delta_{J_p^2}(\{(j,k) : y_{jk} \neq x_{jk}\}) = 0$. Moreover, by Theorem 3.3, $\Gamma_x^{J_p^2} = \Gamma_y^{J_p^2}$. Since the subsequence $\{y\}_\Omega$ has no ordinary limit points, it can have no J_p^2-statistical limit points either. Hence, $L_y = \Gamma_y^{J_p^2}$ and thus $L_y = \Gamma_x^{J_p^2}$. □

Note that although L_x is always a closed set, $\Lambda_x^{J_p^2}$ may not be closed. Therefore, in the statement of Theorem 3.3, $\Gamma_x^{J_p^2}$ and $\Lambda_x^{J_p^2}$ cannot be interchanged.

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