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APPROXIMATION OF CONVOLUTION-TYPE INTEGRAL OPERATORS ON VARIABLE BOUNDED VARIATION SPACES OF HIGHER ORDER USING SUMMABILITY METHODS

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Abstract. We investigate a new class of function spaces called variable bounded variation spaces of higher order by generalizing the classical notion of bounded variation. Our approach incorporates concepts from higher-order variations, bounded variation in the sense of generalized norms, and variations defined through modular function spaces. We focus on analysing the approximation properties of convolution-type nonlinear integral operators within these newly defined function spaces. In particular, we study these operators on variable bounded variation spaces of a fixed order in the sense of higher variations. Several approximation results are obtained using regular summability methods, highlighting the effectiveness of the proposed approach in improving convergence behaviour.

1. Introduction

Integral operators of convolution types are used not only in the classical and constructive approximation theory [**2**,**5**,**6**,**9**,**11**], but also in applied studies, such as image processing, signal processing, ultrasound diagnostics etc. [**12**, **22**, **25**]. Some of them have linear structure [**13**, **16**, **28**] and others are nonlinear [**1**, **3**]. To increase the speed of convergence of sequences, certain summability methods such as subsequence matrix transforms are applied in the context of approximation theory [**7**, **8**, **10**, **19**]. This is one of the primary reasons for applying summability methods in approximation theory. For better approximation results, we use the Bell type summability methods [**14**].

Assume that $\mathcal{A} = \{A^v\} = \{[a^v_{nk}]\} : (n, k, v \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers. Let $x = \{x_k\}$ be a sequence, the $\mathcal{A}$-transform of x is a sequence $(\mathcal{A}x)^v_n$ which is defined by $(\mathcal{A}x)^v_n = \sum_{k=1}^{\infty} a^v_{nk} x_k$ $(n, v \in \mathbb{N})$ provided that

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the series converges for every n, v. Then x is called $\mathcal{A}$-summable to a number L whenever $\lim_{n\to\infty}(\mathcal{A}x)_n^v = L$ uniformly in $v \in \mathbb{N}$. Then we write $\mathcal{A} - \lim x = L$. A summability method $\mathcal{A} = \{A^v\} = \{[a_{nk}^v]\}$ is said to be regular if $\lim x_k = L$ implies $\mathcal{A} - \lim x = L$. We know that a method $\mathcal{A} = \{[a_{nk}^v]\}$ is regular if and only if the following conditions are satisfied:

(a) for each $n,\ v \in \mathbb{N}$, $\sum_{k=1}^{\infty} |a_{nk}^v| < \infty$, and there exist integer N, M such that $\sup_{n \geqslant N,\, v\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}^v| \leqslant M$,
(b) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}^v = 1$ uniformly in v,
(c) for every $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{nk}^v = 0$ uniformly in v [**15**].

The concept of variation of a given function was initially introduced by Jordan in [**20**]. He proposed the concept of a function with bounded variation in 1880. Subsequently, Wiener [**26**], and Love and Young [**21**], Musielak and Orlicz [**23**], and Tonelli [**24**] expanded the Jordan variation in other directions. Currently, function spaces with variable exponents are a very active area of research interest. In particular, Castillo et al. introduced the variable bounded variation spaces in the sense of Wiener in [**17**]. A generalization of the spaces of bounded p-variation is found in [**26**], which generalized the classical BV space in the sense of Jordan. The analysis of variable bounded variation spaces is progressing rapidly due to its fundamental significance as well as crucial applications in a variety of domains [**18**, **27**].

Here we introduce the variable bounded variation spaces of order $m > 1$ by generalizing the bounded variation spaces. We have taken care of the Wiener's [**26**] notion of higher variations, p-bounded variation by Love and Young [**21**] and Musielak–Orlicz φ-variations [**23**].

2. Preliminaries and Definitions

DEFINITION 2.1. By an admissible function $p(\cdot)$, we mean a function $p : \mathbb{R} \to [1, +\infty)$, such that $p_+ = \sup_{x\in\mathbb{R}} p(x)$ is finite. We use the notation $p_- = \inf_{x\in\mathbb{R}} p(x)$; obviously $p_- \geqslant 1$.

DEFINITION 2.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $m \geqslant 1$, we define the $p(\cdot)$-variation of f on $\mathbb{R}$ in the Wiener's sense as follows

$$V_{\varphi,\Delta^m}^{p(\cdot)}[f] = \sup_{\Pi_\delta^*} \sum_{i=1}^{n} [\varphi|\Delta^m f(t_i)|]^{p(x_{i-1})} = \sup_{\Pi_\delta^*} \sum_{i=1}^{n} \left[\varphi\left|\sum_{r=0}^{m} (-1)^r \binom{m}{r} f(t_{i-r})\right|\right]^{p(x_{i-1})},$$

where Π_δ^* is a tagged sequence, i.e., an increasing sequence $t_0 < t_1 < \cdots < t_n$, together with a finite sequence of numbers $x_0, x_1, \ldots, x_{n-1}$ subject to the condition $t_i \leqslant x_i \leqslant t_{i+1}$, for all $i = 0, \ldots, n-1$.

We consider the class of all convex functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that φ is a continuous, non-decreasing, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

DEFINITION 2.3. We define the space of functions of variable bounded variation of order m on $\mathbb{R}$ as

$$BV_{\varphi,\Delta^m}^{p(\cdot)}(\mathbb{R}) = \big\{f\colon \mathbb{R} \to \mathbb{R} \mid V_{\varphi,\Delta^m}^{p(\cdot)}[\lambda f] < \infty,\ \text{for some } \lambda > 0\big\}.$$

This space endowed with the norm

$$\|f\|_{BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})} = \inf\big\{\lambda > 0 \mid V^{p(\cdot)}_{\varphi,\Delta^m}[f/\lambda, \mathbb{R}]\big\},$$

which is the Luxemburg type modular norm. Therefore,

$$BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) = \big\{f\colon \mathbb{R} \to \mathbb{R} \mid \|f\|_{BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})} < +\infty\big\}.$$

The above variable bounded variation space of order m is a generalization of the bounded variation space. If $p(\cdot) = 1$, $\varphi = 1$, $m = 1$, the variable bounded variation of order m reduces to the Jordan variation [**20**], while this space coincides with the Wiener p-variation [**26**] for $p(\cdot) = p$, $p > 1$, $\varphi = 1$, $m = 1$ and finally, if $p(\cdot) = 1$, $m = 1$, the variable bounded variation of order m reduces to the Musielak–Orlicz φ-variation [**7**].

Definition 2.4. By modulus of $p(\cdot)$-continuity of the function $f\colon \mathbb{R} \to \mathbb{R}$ we mean

$$\omega^{p(\cdot)}_{\delta,\Delta^m}(f) = \sup_{\Pi^*_\delta} \sum_{i=1}^{n} |\Delta^m f(t_i)|^{p(x_{i-1})},$$

where Π^*_δ is a tagged partition of $\mathbb{R}$ with mesh not greater than δ.

Definition 2.5. The space of φ-absolutely $p(\cdot)$-continuous function of order m is denoted by $AC^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ and defined as

$$AC^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) = \Big\{f \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) \text{ and } \lim_{\delta\to 0} \omega^{p(\cdot)}_{\delta,\Delta^m}(\lambda f) = 0\Big\},$$

for some $\lambda > 0$, where Π^*_δ is a tagged partition of $\mathbb{R}$ with mesh, at most δ.

It is not hard to see that, the above definition reduces to the φ-absolute continuity, if $m = 1$, $\lambda = 1$ and $p(\cdot) = 1$ and it is denoted by $AC_\varphi(\mathbb{R})$, the space of φ-absolutely continuous functions on $\mathbb{R}$. We observe that $AC_\varphi(\mathbb{R}) \subset AC^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$.

We shall study the approximation properties in $BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ of the classical nonlinear convolution integral operators defined in [**7**] (in the case of $\mathcal{A} = \{I\}$, the identity matrix, then we get the operators which were considered in [**1**, **3**]).

$$\mathcal{T}_{n,v}(f;s) = \sum_{k=1}^{\infty} a^v_{nk} T_k(f;s), \quad \text{where } T_k(f;s) = \int_{\mathbb{R}} K_k(t, f(s-t))dt.$$

i.e.,

$$\mathcal{T}_{n,v}(f;s) = \sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} K_k(t, f(s-t))dt, \; n, v \in \mathbb{N}.$$

Here the kernel $K_k\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a family of measurable functions such that $K_k(t,s) = L_k(t)H_k(s)$ for every $t, s \in \mathbb{R}$, where $L_k \in L^1(\mathbb{R})$ and $\{H_k\}$ is a family of functions such that $H_k(0) = 0$ having the Lipschitz property,

$$|H_k(x) - H_k(y)| \leqslant C|x - y|$$

for some $C > 0$ and for every $k \in \mathbb{N}$.

To avoid repetition, we present the following conditions here for use in subsequent parts of the paper:

(i) There exists a constant $D > 0$, such that $\|L_k\|_1 \leqslant D < \infty$, $\forall\, k \in \mathbb{N}$.

(ii) $\mathcal{A} - \lim \left(\int_{\mathbb{R}} L_k(t)dt\right) = 1$,

(iii) for any fixed $\delta > 0$, $\mathcal{A} - \lim \left(\int_{|t|\geqslant\delta} |L_k(t)|\, dt\right) = 0$.

(iv) Let $G_k(u) = H_k(u) - u$, $u \in \mathbb{R}$, for every $\gamma > 0$, there exists a $\lambda > 0$, such that $\lim_{k\to\infty} \frac{V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda G_k]}{\varphi(\gamma m(J))} = 0$ uniformly with respect to every bounded interval $J \subseteq \mathbb{R}$, which means that for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k \geqslant k_0$, $\frac{V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda G_k]}{\varphi(\gamma m(J))} < \epsilon$, where $m(J)$ is the length of the interval J.

3. Convergence Results

PROPOSITION 3.1. *Let p and q be admissible functions. If $p(\cdot) \leqslant q(\cdot)$, then $V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f] \geqslant V^{q(\cdot)}_{\varphi,\Delta^m}[\lambda f]$ for some $\lambda > 0$ and therefore $BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) \subset BV^{q(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$.*

PROOF. Let $f \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$, then by the definition, for some $\lambda > 0$

$$V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f] = \sup_{\Pi^*_\delta} \sum_{i=1}^{n} \left[\varphi|\Delta^m(\lambda f)(t_i)|\right]^{p(x_{i-1})} < \infty$$

where Π^*_δ is a tagged sequence $t_0 < t_1 < \cdots < t_n$, $x_0 < \cdots < x_{n-1}$ on $\mathbb{R}$. We take

$$\|f\|_{BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})} = \inf\left\{\lambda > 0 \mid V^{p(\cdot)}_{\varphi,\Delta^m}[f/\lambda] \leqslant 1\right\} = 1.$$

Since $p(x_{i-1}) \leqslant q(x_{i-1})$ and the function $t \to t^r$ is increasing for $t \geqslant 0$ and $r > 0$, we obtain

$$\sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{q(x_{i-1})} \leqslant \sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{p(x_{i-1})}.$$

Now, taking supremum on both sides, we get

$$\sup_{\Pi^*_\delta}\sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{q(x_{i-1})} \leqslant \sup_{\Pi^*_\delta}\sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{p(x_{i-1})}$$

$$V^{q(\cdot)}_{\varphi,\Delta^m}[\lambda f] \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f] < \infty.$$

Hence, $f \in BV^{q(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ and therefore $BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) \subset BV^{q(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$. □

PROPOSITION 3.2. *If $f_1, \ldots, f_r \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$, $r \in \mathbb{N}$, then $f_1 + \cdots + f_r \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ and for some $\lambda > 0$ we have*

$$V^{p(\cdot)}_{\varphi,\Delta^m}\left[\lambda\left(\sum_{j=1}^{r} f_j\right)\right] \leqslant r^{p_+-1}\sum_{j=1}^{r} V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_j].$$

Proof. Let $t_0 < t_1 < t_2 < \cdots < t_n$, $x_0 < \cdots < x_{n-1}$ be a tagged sequence. Let $\lambda > 0$ be such that $V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_j] < \infty$ for every $j = 1, \dots, r$. By the monotonicity and convexity of the power function $u^{p(\cdot)}$, $u \geqslant 0$ we have

$$\sum_{i=1}^{n}\left[\varphi\left|\Delta^m\lambda\sum_{j=1}^{r}f_j(t_i)\right|\right]^{p(x_{i-1})} \leqslant \sum_{i=1}^{n}\left[\varphi\sum_{j=1}^{r}|\Delta^m\lambda f_j(t_i)|\right]^{p(x_{i-1})}$$
$$\leqslant \sum_{i=1}^{n}\left\{\frac{1}{r}[r\lambda\varphi|\Delta^m f_1(t_i)|]^{p(x_{i-1})}+\cdots+\frac{1}{r}[r\lambda\varphi|\Delta^m f_r(t_i)|]^{p(x_{i-1})}\right\}$$
$$\leqslant r^{p(x_{i-1})-1}\sum_{i=1}^{n}\{[\lambda\varphi|\Delta^m f_1(t_i)|]^{p(x_{i-1})}+\cdots+[\lambda\varphi|\Delta^m f_r(t_i)|]^{p(x_{i-1})}\}.$$

Since $p(x_{i-1}) \leqslant p_+$ and passing supremum over all the tagged sequences in $\mathbb{R}$

$$V^{p(\cdot)}_{\varphi,\Delta^m}\left[\lambda\left(\sum_{j=1}^{r}f_j\right)\right] \leqslant r^{p_+-1}\left\{V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_1(t_i)]+\cdots+V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_r(t_i)]\right\}$$
$$\leqslant r^{p_+-1}\sum_{j=1}^{r}V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_j]. \qquad \square$$

In order to operate using the convolution integral operators, we establish a relationship between $V^{p(\cdot)}_{\varphi,\Delta^m}[f]$ and its shifted form $\tau_t f(u) = f(u-t),\ \ t, u \in \mathbb{R}$.

Proposition 3.3. *For every* $t \in \mathbb{R}$, $\tau_t f \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ *if and only if* $f \in BV^{\tau-tp(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$.

Proof. Let $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ be a tagged sequence in $\mathbb{R}$; then for $t \in \mathbb{R}$, $s_0 - t < s_1 - t < \cdots < s_n - t$, $x_0 - t < \cdots < x_{n-1} - t$ is again a tagged sequence. Therefore

$$\sum_{i=1}^{n}[\varphi(|\Delta^m f(s_i-t)|)]^{p(x_{i-1})} = \sum_{i=1}^{n}[\varphi(|\Delta^m f(s_i-t)|)]^{p((x_{i-1}-t)+t)} \leqslant V^{p(\cdot+t)}_{\varphi,\Delta^m}[f]$$

and passing to the supremum over all the tagged sequences in $\mathbb{R}$

$$V^{p(\cdot)}_{\varphi,\Delta^m}[\tau_t f] \leqslant V^{\tau-tp(\cdot)}_{\varphi,\Delta^m}[f].$$

Again, if $s_0 < s_1 < \cdots < s_n$, $x_0 < x_1 < \cdots < x_{n-1}$ is a tagged sequence in $\mathbb{R}$, then for $t \in \mathbb{R}$, $s_0 + t < s_1 + t < \cdots < s_n + t,\ x_0 + t < \cdots < x_{n-1} + t$, is also a tagged sequence. Then,

$$\sum_{i=1}^{n}[\varphi(|\Delta^m f(s_i)|]^{p(x_{i-1}+t)} = \sum_{i=1}^{n}[\varphi(|\Delta^m f((s_i+t)-t)]^{p(x_{i-1}+t)} \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\tau_t f].$$

Now, passing the supremum over all the tagged sequences, we have

$$V^{\tau-tp(\cdot)}_{\varphi,\Delta^m}[f] \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\tau_t f]. \qquad \square$$

THEOREM 3.4. *Let $\mathcal{A} = \{[a^v_{nk}]\}$ be a non-negative regular summability method and condition* (i) *holds. Then for $f \in BV^{p(\cdot)}_{\varphi,\Delta^m}$ there exists a $\mu > 0$ such that*

$$V^{p_+/p_-p(\cdot)}_{\varphi,\Delta^m}[\mu \mathfrak{T}_{n,v}(f)] \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f];$$

$\mathfrak{T}_{n,v}$ maps from $BV^{p(\cdot)}_{\varphi,\Delta^m}$ into $BV^{p_+/p_-p(\cdot)}_{\varphi,\Delta^m}$, and $\lambda > 0$ is such that $V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f] < +\infty$.

PROOF. Let $s_0 < s_1 < \cdots < s_n,\ ,x_0 < \cdots < x_{n-1}$ be a tagged sequence in $\mathbb{R}$. Then for $\mu > 0$, we have

$$\begin{aligned} U &= \sum_{i=1}^{n} [\varphi\left(|\Delta^m \mu \mathfrak{T}_{n,v} f(s_i)|\right)]^{p_+/p_-p(x_{i-1})} \\ &= \sum_{i=1}^{n} \left[\varphi\bigg(\mu\bigg|\sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} L_k(t)(\Delta^m H_k(f(s_i - t)))dt\bigg|\bigg)\right]^{p_+/p_-p(x_{i-1})} \end{aligned}$$

Now, using Jensen's inequality and convexity property of the power function $u^{p_+/p_-p(\cdot)}$, $u \geqslant 0$, and condition (i), we get

$$\begin{aligned} &\leqslant \frac{1}{D}\sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} |L_k(t)| \sum_{i=1}^{n} [\varphi\mu|\Delta^m H_k(f(s_i - t))|]^{p_+/p_-p(x_{i-1})} dt \\ &\qquad \leqslant \frac{1}{D}\sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} |L_k(t)| V^{p_+/p_-p(x_{i-1})}_{\varphi,\Delta^m}[\mu D C f(s_i - t)]\, dt \\ &\qquad \leqslant \frac{1}{D}\sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} |L_k(t)| V^{p_+/p_-p(x_{i-1})}_{\varphi,\Delta^m}[\mu D C \tau_t f]\, dt \end{aligned}$$

where $\tau_t f(u) = f(u - t)$, $t, u \in \mathbb{R}$ and by Proposition 3.3

$$U \leqslant \frac{1}{D}\sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} |L_k(t)| V^{p_+/p_-p(\cdot+t)}_{\varphi,\Delta^m}[\mu D C f]\, dt.$$

Since for every $t \in \mathbb{R}$, $p_+/p_-p(\cdot + t) \geqslant p(\cdot)/p_-p(\cdot + t) \geqslant p(\cdot)$, then by Proposition 3.1 and condition (i) we have

$$U \leqslant \frac{1}{D}\sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} |L_k(t)| V^{p(\cdot)}_{\varphi,\Delta^m}[\mu D C f]\, dt \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\mu D C f].$$

Therefore, $0 < \mu < \frac{\lambda}{DC}$, passing to the supremum over all the possible tagged sequences in $\mathbb{R}$, we conclude that $V^{p_+/p_-p(\cdot)}_{\varphi,\Delta^m}[\mu \mathfrak{T}_{n,v}(f)] \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f]$. □

THEOREM 3.5. *If $f \in AC^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$, then for some $\lambda > 0$, there holds*

$$\lim_{\delta \to 0} \omega^{p(\cdot)p_+^2/p_-^2}_{\delta,\Delta^m}(\lambda f, \delta) = 0$$

or equivalently,

$$\lim_{t \to 0} V^{p(\cdot)p_+^2/p_-^2}_{\varphi,\Delta^m}[\lambda(f(\cdot - t) - f)] = 0.$$

Proof. Let $\lambda > 0$ be fixed. For a fixed tagged sequence $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$

$$V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(f(\cdot - t) - f)(s_i)] = \sup_{\Pi_\delta^*} \sum_{i=1}^{n} [\varphi(|\Delta^m(\lambda f)(s_i - t) - \Delta^m(\lambda f)(s_i)|)]^{p(\cdot)p_+^2/p_-^2}$$

Since, Δ^m is a linear operator, so

$$\begin{aligned} \Delta^m[f(\cdot - t) - f](s_i) &= \Delta^m f(s_i - t) - \Delta^m f(s_i), \\ |\Delta^m[f(\cdot - t) - f](s_i) &\leqslant |\Delta^m f(s_i - t) - \Delta^m f(s_i)|, \\ V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(f(\cdot - t) - f)(s_i)] &\leqslant \sup_{\Pi_\delta^*} \sum_{i=1}^{n} [\varphi(\lambda|\Delta^m f(s_i - t) - \Delta^m f(s_i)|)]^{p(\cdot)p_+^2/p_-^2}. \end{aligned}$$

Since $f \in AC_{\varphi,\Delta^m}^{p(\cdot)}(\mathbb{R})$, it follows that $\Delta^m f$ uniformly continuous on $\mathbb{R}$, hence as $t \to 0$ we have $|\Delta^m f(s_i - t) - \Delta^m f(s_i)| \to 0$. Moreover, φ is continuous, increasing, $\varphi(0) = 0$ and for all $x \in \mathbb{R}$, $p(\cdot)p_+^2/p_-^2 \geqslant p(\cdot) \geqslant 1$, each term in the summation tends to zero as $t \to 0$. Thus, we have $\lim_{t\to 0} V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(f(\cdot - t) - f)] = 0$. $\square$

Theorem 3.6. *Let* $\mathcal{A} = \{[a_{nk}^v]\}$ *be a nonnegative regular summability method. Assume that conditions* (i)–(iv) *hold. If* $f \in AC_{\varphi,\Delta^m}^{p(\cdot)}(\mathbb{R})$*, then there exists* $\lambda > 0$*, such that* $\lim_{n\to\infty} V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(\mathcal{T}_{n,v}(f) - f)] = 0$ *uniformly in* $v \in \mathbb{N}$.

Proof. For a fixed tagged sequence $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ and $\lambda > 0$, there holds

$$\begin{aligned} U &= \sum_{i=1}^{n} [\varphi(\lambda|\Delta^m(\mathcal{T}_{n,v} f - f)(s_i)|)]^{p(\cdot)p_+^2/p_-^2} \\ &= \sum_{i=1}^{n} \Bigg[\varphi\Bigg|\Delta^m \lambda\Bigg(\sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} L_k(t)\big(H_k(f(s_i - t)) - f(s_i - t)\big)\, dt \\ &\qquad + \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} L_k(t)\big(f(s_i - t) - f(s_i)\big)\, dt \\ &\qquad + \Bigg(\sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} L_k(t)\, dt - 1\Bigg) f(s_i)\Bigg)\Bigg|\Bigg]^{p(\cdot)p_+^2/p_-^2} \\ &\leqslant \frac{1}{3} \sum_{i=1}^{n} \Bigg[\varphi\Bigg(3\lambda \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} |L_k(t)| \Big|\Delta^m\big(H_k(f(s_i - t)) - f(s_i - t)\big)\Big|\, dt\Bigg)\Bigg]^{p(\cdot)p_+^2/p_-^2} \\ &\quad + \frac{1}{3} \sum_{i=1}^{n} \Bigg[\varphi\Bigg(3\lambda \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} |L_k(t)| \Big|\Delta^m\big(f(s_i - t) - f(s_i)\big)\Big|\, dt\Bigg)\Bigg]^{p(\cdot)p_+^2/p_-^2} \\ &\quad + \frac{1}{3} \sum_{i=1}^{n} \Bigg[\varphi\Bigg(3\lambda \Bigg|\sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} L_k(t) dt - 1\Bigg| |\Delta^m f(s_i)|\Bigg)\Bigg]^{p(\cdot)p_+^2/p_-^2} = U_1 + U_2 + U_3. \end{aligned}$$

Let $p(\cdot)p_+^2/p_-^2 = \alpha$; using Jensen's inequality in U_1 with (i) and (b), there holds

$$\begin{aligned} U_1 &\leqslant \frac{1}{3M}\sum_{i=1}^{n}\left[\varphi\Big(3M\lambda\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)|\big|\Delta^m\big(H_k\big(f(s_i-t)\big)-f(s_i-t)\big)\big|\Big)\right]^{\alpha}dt \\ &\leqslant \frac{1}{3MD}\sum_{k=1}^{n}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)|\sum_{i=1}^{n}\big[\varphi\big(3MD\lambda\big|\Delta^m\big(H_k\big(f(s_i-t)\big)-f(s_i-t)\big)\big|\big)\big]^{\alpha}dt \\ &\leqslant \frac{1}{3MD}\sum_{k=1}^{n}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)|V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k\big(f(\cdot-t)\big)-f(\cdot-t))]dt. \end{aligned}$$

Considering that

$$V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k\,(f(\cdot-t))-f(\cdot-t))] = V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k\,(f)-f)]$$

for every $t \in \mathbb{R}$, we get

$$U_1 \leqslant \frac{1}{3M}\sum_{k=1}^{\infty}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)].$$

Now, considering condition (iv) in Lemma 1 of [**4**], there exists a $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, $V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)] < \varepsilon V_{\varphi,\Delta^m}^{p(\cdot)}[\gamma f]$ for some $\lambda > 0$. Then, divide the sum given above into two parts as follows

$$\begin{aligned} U_1 &\leqslant \frac{1}{3M}\bigg(\sum_{k=1}^{k_0-1}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)] \\ &\qquad\qquad + \sum_{k=k_0}^{\infty}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)]\bigg) \\ &\leqslant \frac{1}{3M}\bigg(\sum_{k=1}^{k_0-1}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)] + \varepsilon V_{\varphi,\Delta^m}^{p(\cdot)}[\gamma f]\sum_{k=k_0}^{\infty}a_{nk}^{v}\bigg) \\ &= U_1^1 + U_1^2. \end{aligned}$$

Now, defined $P = \max_{k\in\{1,2,\ldots,k_0-1\}} V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)]$, from (c) we get

$$U_1^1 \leqslant \frac{P}{3M}\sum_{k=1}^{k_0-1}a_{nk}^{v} < \frac{P}{3M}(k_0-1)\varepsilon.$$

In U_1^2, from the regularity property (a), we get $U_1^2 \leqslant \frac{\epsilon}{3}V_{\varphi,\Delta^m}^{p(\cdot)}[\gamma f]$. For U_2, by the Jensen's inequality we observe that

$$\begin{aligned} U_2 &\leqslant \frac{1}{3M}\sum_{i=1}^{n}\left[\varphi\Big(3M\lambda\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)||\Delta^m(f(s_i-t)-f(s_i))|\Big)\right]^{\alpha}dt \\ &\leqslant \frac{\lambda}{3M}\sum_{i=1}^{n}\left[\Big(3M\sum_{k=1}^{\infty}a_{nk}^{v}\Big\{\int_{|t|<\delta}+\int_{|t|\geqslant\delta}\Big\}|L_k(t)||\Delta^m(f(s_i-t)-f(s_i))|\Big)\right]^{\alpha}dt \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{3MD}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\sum_{i=1}^{n}[\varphi(3MD\lambda|\Delta^m(f(s_i-t)-f(s_i))|)]^{\alpha}dt\\
&\quad+\frac{1}{3MD}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\sum_{i=1}^{n}[\varphi(3MD\lambda|\Delta^m(f(s_i-t)-f(s_i))|)]^{\alpha}dt\\
&\leqslant \frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(f(s_i-t)-f(s_i))]\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\,dt\\
&\quad+\frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(f(s_i-t)-f(s_i))]\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt\leqslant U_2^1+U_2^2.
\end{aligned}
$$

Consider (i) and Theorem 3.5, we get

$$
\begin{aligned}
U_2^1&\leqslant \frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(f(s_i-t)-f(s_i))]\sum_{k=1}^{n} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\,dt\\
&\leqslant \frac{1}{3MD}\omega^{\alpha}_{\delta,\Delta^m}(3MD\lambda f,\delta)\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\,dt<\frac{\epsilon}{3},
\end{aligned}
$$

and consider U_2^2, by Propositions 3.1 and 3.2

$$
\begin{aligned}
U_2^2&\leqslant \frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(\tau_t f-f)]\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt\\
&\leqslant \frac{1}{3MD}2^{p_+-1}\left\{V^{\alpha}_{\varphi,\Delta^m}[6MD\lambda\tau_t f]+V^{\alpha}_{\varphi,\Delta^m}[6MD\lambda f]\right\}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt\\
&\leqslant \frac{1}{3MD}2^{p_+}\left\{V^{p(\cdot)p_+/p_-}_{\varphi,\Delta^m}[6MD\lambda f]\right\}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt\\
&\leqslant \frac{1}{3MD}2^{p_+}\left\{V^{p(\cdot)}_{\varphi,\Delta^m}[\mu f]\right\}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt
\end{aligned}
$$

for $0<\lambda<\frac{\mu}{6MD}$, where μ is such that $V^{p(\cdot)}_{\varphi,\Delta^m}[\mu f]<\infty$. By assumption (iii), there exists $k>0$, such that

$$
\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt<\frac{3MD\epsilon}{2^{(p_++1)}V^{p(\cdot)}_{\varphi,\Delta^m}[\mu f]}.
$$

Therefore, $U_2^2<\epsilon/2$.

Now, from (iii), for every $0<\epsilon<1$, there exists n_0, such that, for all $n\geqslant n_0$

$$
\left|\sum_{k=1}^{n} a_{nk}^{v}\int_{\mathbb{R}}L_k(t)\,dt-1\right|<\epsilon<1.
$$

We consider this in U_3, by the convexity of φ we observe that

$$\begin{aligned}U_3 &\leqslant \frac{1}{3}\sum_{i=1}^{n}\left|\varphi\left(3\lambda\left[\sum_{k=1}^{n}a_{nk}^{v}\int_{\mathbb{R}}L_k(t)dt-1\right]\right)\Delta^m f(s_i)\right|^{\alpha}\\ &\leqslant \frac{1}{3}\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}}L_k(t)dt-1\right|\sum_{i=1}^{n}[\varphi(3\lambda|\Delta^m f(s_i)|)]^{\alpha}\leqslant \frac{\epsilon}{3}V_{\varphi,\Delta^m}^{p(\cdot)}[3\lambda f]\end{aligned}$$

where $V_{\varphi,\Delta^m}^{p(\cdot)}[3\lambda f]<\infty$ for some $\lambda>0$. Therefore, the proof is complete. □

Corollary 3.7. *For $m=1$ and $p(\cdot)=1$, the space of φ-absolutely $p(\cdot)$ continuous functions of order m becomes the space of φ-absolutely continuous functions and above Theorem 3.6 reduces to Theorem 3.2 in [**7**]. (i.e., if $f\in AC_{\varphi}(\mathbb{R})$ then there exists $\mu>0$ such that $\lim_{n\to\infty}V_{\varphi}[\mu(\mathcal{T}_{n,v}f-f)]=0$, uniformly in $v\in\mathbb{N}$.)*

4. Conclusion

We observe that classical bounded variation spaces are mapped into themselves by convolution type nonlinear integral operators with regular summability methods. They map $BV_{\varphi,\Delta^m}^{p(\cdot)}$ into $BV_{\varphi,\Delta^m}^{p+/p-p(\cdot)}$ however, in the generalised context of variable bounded variations of higher order, capturing more complex function behaviour while maintaining the modular framework. This pattern opens up new paths for the study of integral operators under more adaptable and realistic functional contexts, in addition to extending classical approximation theory. Adding the variable exponent $p(\cdot)$ and a higher order difference operator greatly enhances the theory and increases its applicability in a wide range of fields.

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