PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 119 (133) (2026), 49–60 DOI: https://doi.org/10.2298/PIM2633049O

ON GENERALIZED CESÀRO-BASED APOSTOL TYPE SPECIAL POLYNOMIALS

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ABSTRACT. We define a new and generalized family of polynomials based on a Cesàro-type generating function. This new class, called the generalized Cesàro-based special polynomials, includes several well-known families such as Bernoulli, Euler, and Genocchi polynomials as special cases. Although these classical polynomials are well studied, we provide new definitions for their Cesàro-based versions by using our general formulation. The generating function involves multiple parameters, allowing a wide range of flexibility and generalization. We study important properties of these polynomials, including recurrence relations, addition formulas, and generating function identities. Our results offer a unified approach to classical special polynomials and open new directions for further research in number theory, combinatorics, and approximation theory.

1. Introduction

Cesàro polynomials, named after Ernesto Cesàro, play a significant role in summability theory and approximation theory due to their connection with Cesàro means of orthogonal expansions. Originally introduced in the context of improving the convergence behavior of Fourier series, Cesàro means provide a smoothing technique for divergent or slowly convergent series, and the associated polynomials inherit these favorable approximation properties [**2**, **4**].

Over the years, numerous studies have investigated the analytical properties of Cesàro polynomials, including recurrence relations, generating functions, and orthogonality with respect to classical weight functions [**3**, **18**]. Their close connections with classical orthogonal polynomials—including Legendre, Chebyshev, and Jacobi polynomials—have enabled the application of powerful techniques from the theory of special functions. From an applied perspective, Cesàro polynomials have

2020 *Mathematics Subject Classification:* Primary 11B68; Secondary 33E20.
Key words and phrases: generalized Cesàro polynomials, Apostol type polynomials, generating function, recurrence relations.
Communicated by Gradimir Milovanović.

found relevance in signal processing, numerical approximation, and spectral methods for differential equations, due to their ability to produce stable approximations even in the presence of irregularities or discontinuities [**19**].

The generalized Cesàro polynomials $g_n^{(r)}(\lambda, x)$ are defined by the generating function relation (see [**8**])

$$\sum_{n=0}^{\infty} g_n^{(r)}(\lambda, x) s^n = (1-s)^{-r-1}(1-xs)^{-\lambda}. \tag{1.1}$$

It is noted that the special case $\lambda = 1$ of (1.1) reduces immediately to the second one of the generalized Cesàro polynomials $g_n^{(r)}(x)$ [**17**].

In recent years, researchers have incorporated Cesàro-type operators into the theory of special functions to define new families of polynomials with enriched structural features. In this context, several contributions by Özmen and Erkuş-Duman have expanded the theory of Cesàro polynomials significantly. In [**15**], they constructed Cesàro-type generating functions involving three variables. Later works extended these constructions to more general families, offering recurrence relations, generating identities, and combinatorial interpretations [**16**].

Motivated by these developments, we introduce in this paper a new class of generalized Cesàro-based special polynomials defined via the generating function. This general formulation encompasses Cesàro-type versions of Bernoulli, Euler, and Genocchi polynomials as special cases. By suitable parameter choices, we establish a variety of known and new identities, including recurrence relations, addition formulas, explicit representations and general families of generating functions. Recently, many authors have produced new works with similar extension [**6**, **7**, **12**, **13**].

This study adds to the research on polynomial generalizations by introducing a flexible and unified approach. It also opens new possibilities for further work in number theory, special functions, and approximation theory.

2. Generalized Cesàro-based special polynomials

Here, we first construct the generalized Cesàro-based special polynomials. The structure proposed in this definition unifies and extends well-known families such as the Bernoulli, Euler, and Genocchi polynomials by embedding them in a broader Cesàro-type approach with multiple parameters.

Definition 2.1. We define the generalized Cesàro-based Apostol-type polynomials $\mathfrak{C}_n^{\lambda,\mu,r}(x, y; m, c, d)$ by means of the following generating function:

$$\frac{2^{1-m} s^m (1-s)^{-r-1}(1-xs)^{-\lambda} e^{ys}}{\mu^d e^s - c^d} = \sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda,\mu,r}(x, y; m, c, d) \frac{s^n}{n!}, \tag{2.1}$$

where the parameters are chosen so that the generating function is well defined and analytic in a sufficiently small neighbourhood of the origin, and the above power series converges. Letting $x = y = 0$ in (2.1), the generalized Cesàro-based Apostol type numbers are defined by

$$\mathfrak{C}_n^{\lambda,\mu,r} := \mathfrak{C}_n^{\lambda,\mu,r}(0, 0; m, c, d), \qquad (n \in \mathbb{N}_0).$$

Using Definition 2.1, which introduces this new class of generalized Cesàro-based Apostol-type polynomials, we now derive specific cases that correspond to the Cesàro-based Apostol-type extensions of the well-known Bernoulli, Euler and Genocchi polynomials. These special cases are obtained by choosing appropriate values for the parameters in the general generating function. In what follows, we present the precise definitions of the generalized Cesàro-based Bernoulli, Euler and Genocchi polynomials by specializing the parameters in this definition.

Definition 2.2. The generalized Cesàro-based Apostol–Bernoulli polynomials ${}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y)$ are defined by means of the following generating function:

$$\frac{s(1-xs)^{-\lambda}(1-s)^{-r-1}e^{ys}}{\mu e^s-1}=\sum_{n=0}^{\infty}{}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y)\frac{s^n}{n!}, \tag{2.2}$$

where ${}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y)=\mathfrak{C}_n^{\lambda,\mu,r}(x,y;1,1,1)$.

Remark 2.1. Setting $\lambda=0$, $r=-1$, $y=x$ in (2.2), we get the Apostol-Bernoulli polynomials [**1**], that is, ${}_B\mathfrak{C}_n^{0,\mu,-1}(x,x)=B_n(x,\mu)$. The Apostol–Bernoulli numbers $B_n(\mu)$ can be obtained from the Apostol–Bernoulli polynomials $B_n(x,\mu)$ as $B_n(0,\mu)=B_n(\mu)$. The case $\mu=1$ in the above relations gives the classical Bernoulli polynomials $B_n(x)$ and the classical Bernoulli numbers B_n, respectively [**11**].

Remark 2.2. Letting $x=y=0$ in (2.2), the generalized Cesàro-based Apostol type Bernoulli numbers are defined by ${}_B\mathfrak{C}_n^{\lambda,\mu,r}:={}_B\mathfrak{C}_n^{\lambda,\mu,r}(0,0)$, $(n\in\mathbb{N}_0)$.

Definition 2.3. The generalized Cesàro-based Apostol–Euler polynomials, denoted by ${}_E\mathfrak{C}_n^{\lambda,\mu,r}(x,y)$, are defined by means of the following generating function:

$$\frac{2(1-xs)^{-\lambda}(1-s)^{-r-1}e^{ys}}{\mu e^s+1}=\sum_{n=0}^{\infty}{}_E\mathfrak{C}_n^{\lambda,\mu,r}(x,y)\frac{s^n}{n!}, \tag{2.3}$$

where ${}_E\mathfrak{C}_n^{\lambda,\mu,r}(x,y)=\mathfrak{C}_n^{\lambda,\mu,r}(x,y;0,-1,1)$.

Remark 2.3. Setting $\lambda=0$, $r=-1$, $y=x$ in (2.3), we get the Apostol–Euler polynomials [**9**] as ${}_E\mathfrak{C}_n^{0,\mu,-1}(x,x)=E_n(x,\mu)$. The Apostol–Euler numbers $E_n(\mu)$ can be obtained from the Apostol–Euler polynomials $E_n(x,\mu)$ as $E_n(0,\mu)=E_n(\mu)$. The case $\mu=1$ in the above relations gives the classical Euler polynomials $E_n(x)$ and the classical Euler numbers E_n, respectively [**11**].

Remark 2.4. When $x=y=0$ in (2.3), the generalized Cesàro-based Apostol type Euler numbers ${}_E\mathfrak{C}_n^{r}(\lambda,\mu)$ are given by ${}_E\mathfrak{C}_n^{\lambda,\mu,r}:={}_E\mathfrak{C}_n^{\lambda,\mu,r}(0,0)$, $(n\in\mathbb{N}_0)$.

Definition 2.4. The generalized Cesàro-based Apostol-Genocchi polynomials ${}_G\mathfrak{C}_n^{\lambda,\mu,r}(x,y)$ are defined by means of the following generating function:

$$\frac{2s(1-xs)^{-\lambda}(1-s)^{-r-1}e^{ys}}{\mu e^s+1}=\sum_{n=0}^{\infty}{}_G\mathfrak{C}_n^{\lambda,\mu,r}(x,y)\frac{s^n}{n!}, \tag{2.4}$$

where ${}_G\mathfrak{C}_n^{\lambda,\mu,r}(x,y)=\mathfrak{C}_n^{\lambda,\mu/2,r}(x,y;1,-\frac{1}{2},1)$.

REMARK 2.5. Setting $\lambda = 0$, $r = -1$, $y = x$ in (2.4), we get the Apostol–Genocchi polynomials [**10**] as ${}_{G}\mathfrak{C}_n^{0,\mu,-1}(x,x) = G_n(x,\mu)$. The Apostol–Genocchi numbers $G_n(\mu)$ can be obtained from the Apostol–Genocchi polynomials $G_n(x,\mu)$ as $G_n(0,\mu) = G_n(\mu)$. The case $\mu = 1$ in the above relations gives the classical Genocchi polynomials $G_n(x)$ and the classical Genocchi numbers G_n, respectively [**5**].

REMARK 2.6. When $x = y = 0$ in (2.4), the generalized Cesàro-based Apostol type Genocchi numbers ${}_{G}\mathfrak{C}_n^{r}(\lambda,\mu)$ are given by ${}_{G}\mathfrak{C}_n^{\lambda,\mu,r} := {}_{G}\mathfrak{C}_n^{\lambda,\mu,r}(0,0)$, $(n \in \mathbb{N}_0)$.

3. Explicit formulas and fundamental properties

In Section 2, the generalized Cesàro-based Apostol-type polynomials are originally defined via their generating functions. In this section, we derive their explicit representations directly from the definition. Subsequently, the corresponding addition formula is obtained, followed by a general theorem providing multilinear and multilateral generating functions. All of these results are also formulated as theorems for the Cesàro-based Apostol-type Bernoulli, Euler, and Genocchi polynomials.

THEOREM 3.1. *The generalized Cesàro-based Apostol type polynomials are explicitly given by*

$$\mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d) = n!\sum_{k=0}^{n} \frac{1}{(n-k)!} g_k^{(r)}(\lambda,x)y_{n-k,\mu}(y;m,c,d),$$

where $g_k^{(r)}(\lambda,x)$ *denotes the generalized Cesàro polynomials given by* (1.1), *and* $y_{n,\mu}(x;m,c,d)$ *is defined by* [**13**] *through the generating function relation*

$$\frac{2^{1-m}s^m e^{xs}}{\mu^d e^s - c^d} = \sum_{n=0}^{\infty} y_{n,\mu}(x;m,c,d)\frac{s^n}{n!}. \tag{3.1}$$

PROOF. By using the generating functions in (1.1) and (3.1), and arranging the resulting series expansions, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^n}{n!} &= \frac{2^{1-m}s^m(1-s)^{-r-1}(1-xs)^{-\lambda}e^{ys}}{\mu^d e^s - c^d} \\ &= \left(\sum_{k=0}^{\infty} g_k^{(r)}(\lambda,x)s^k\right)\left(\sum_{n=0}^{\infty} y_{n,\mu}(y;m,c,d)\frac{s^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} y_{n-k,\mu}(y;m,c,d)g_k^{(r)}(\lambda,x)\frac{s^n}{(n-k)!}.\end{aligned}$$

A term-by-term comparison of the coefficients of s^n in these expansions immediately establishes the claimed formula. □

THEOREM 3.2. *The generalized Cesàro-based Apostol type Bernoulli, Euler and Genocchi polynomials are explicitly given by*

$$
\begin{aligned}
{}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y) &= n!\sum_{k=0}^{n}\frac{1}{(n-k)!}B_{n-k}(y;\mu)g_k^{(r)}(\lambda,x),\\
{}_E\mathfrak{C}_n^{\lambda,\mu,r}(x,y) &= n!\sum_{k=0}^{n}\frac{1}{(n-k)!}E_{n-k}(y;\mu)g_k^{(r)}(\lambda,x),\\
{}_G\mathfrak{C}_n^{\lambda,\mu,r}(x,y) &= n!\sum_{k=0}^{n}\frac{1}{(n-k)!}G_{n-k}(y;\mu)g_k^{(r)}(\lambda,x),
\end{aligned}
$$

where $B_n(y;\mu)$, $E_n(y;\mu)$ and $G_n(y;\mu)$ are Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials respectively.

PROOF. It is fairly straightforward to observe that from (1.1) and (2.2) we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}{}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y)\frac{s^n}{n!} &= \frac{s(1-xs)^{-\lambda}(1-s)^{-r-1}e^{ys}}{\mu e^s-1}\\
&= \left(\sum_{k=0}^{\infty}g_k^{(r)}(\lambda,x)s^k\right)\left(\sum_{n=0}^{\infty}B_n(y;\mu)\frac{s^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(n-k)!}B_{n-k}(y;\mu)g_k^{(r)}(\lambda,x)s^n,
\end{aligned}
$$

which readily yields the first assertion of Theorem 3.2. The other identities can likewise be proved in a similar manner by using (2.3) and (2.4). □

THEOREM 3.3. *The generalized Cesàro-based Apostol type polynomials have the following addition formula:*

$$
\begin{aligned}
&\mathfrak{C}_n^{\lambda_1+\lambda_2,\mu,r_1+r_2+1}(x,y_1+y_2;m_1+m_2,c,d)\\
&\qquad = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{C}_{n-k}^{\lambda_1,\mu,r_1}(x,y_1;m_1,c,d)\mathfrak{C}_k^{\lambda_2,0,r_2}\Big(x,y_2;m_2,\frac{-1}{2},1\Big).
\end{aligned}
$$

PROOF. Replacing λ by $\lambda_1+\lambda_2$, r by r_1+r_2+1, y by y_1+y_2 and m by m_1+m_2 in (2.1), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{C}_n^{\lambda_1+\lambda_2,\mu,r_1+r_2+1}(x,y_1+y_2;m_1+m_2,c,d)\frac{s^n}{n!}\\
&\qquad = \frac{2^{1-m_1-m_2}s^{m_1+m_2}(1-s)^{-r_1-r_2-2}(1-xs)^{-\lambda_1-\lambda_2}e^{(y_1+y_2)s}}{\mu^d e^s-c^d}\\
&= \frac{2^{1-m_1}s^{m_1}(1-s)^{-r_1-1}(1-xs)^{-\lambda_1}e^{y_1s}}{\mu^d e^s-c^d}\,\frac{2^{1-m_2}s^{m_2}(1-s)^{-r_2-1}(1-xs)^{-\lambda_2}e^{y_2s}}{2(0^1e^s-(-1/2)^1)}\\
&\qquad = \sum_{n=0}^{\infty}\mathfrak{C}_n^{\lambda_1,\mu,r_1}(x,y_1;m_1,c,d)\frac{s^n}{n!}\sum_{k=0}^{\infty}\mathfrak{C}_k^{\lambda_2,0,r_2}\Big(x,y_2;m_2,\frac{-1}{2},1\Big)\frac{s^k}{2k!}
\end{aligned}
$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathfrak{C}_{n-k}^{\lambda_1,\mu,r_1}(x,y_1;m_1,c,d)\mathfrak{C}_{k}^{\lambda_2,0,r_2}\Big(x,y_2;m_2,\frac{-1}{2},1\Big)\frac{s^n}{2(n-k)!k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}\mathfrak{C}_{n-k}^{\lambda_1,\mu,r_1}(x,y_1;m_1,c,d)\mathfrak{C}_{k}^{\lambda_2,0,r_2}\Big(x,y_2;m_2,\frac{-1}{2},1\Big)\frac{s^n}{2n!}.$$

From the coefficients of $\frac{s^n}{n!}$ both sides of the last equality, one can get the desired result. □

Corollary 3.1. *Using the generating function relations given by Definitions* 2.2, 2.3 *and* 2.4 *in Theorem* 3.3*, the following similar relations are obtained for the generalized Cesàro-based Apostol type Bernoulli, Euler and Genocchi polynomials:*

$${}_B\mathfrak{C}_{n}^{\lambda_1+\lambda_2,\mu,r_1+r_2+1}(x,y_1+y_2) = -\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}{}_B\mathfrak{C}_{n-k}^{\lambda_1,\mu,r_1}(x,y_1){}_E\mathfrak{C}_{k}^{\lambda_2,0,r_2}(x,y_2),$$

$${}_E\mathfrak{C}_{n}^{\lambda_1+\lambda_2,\mu,r_1+r_2+1}(x,y_1+y_2) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}{}_E\mathfrak{C}_{n-k}^{\lambda_1,\mu,r_1}(x,y_1){}_E\mathfrak{C}_{k}^{\lambda_2,0,r_2}(x,y_2),$$

$${}_G\mathfrak{C}_{n}^{\lambda_1+\lambda_2,\mu,r_1+r_2+1}(x,y_1+y_2) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}{}_G\mathfrak{C}_{n-k}^{\lambda_1,\mu,r_1}(x,y_1){}_E\mathfrak{C}_{k}^{\lambda_2,0,r_2}(x,y_2).$$

Theorem 3.4. *Let $\Omega_\mu(z_1,\ldots,z_q)$ be a function of q complex arguments which does not vanish identically, and let its order be denoted by ϖ, ψ. Associated with this function, define the series*

$$\Lambda_{\varpi,\psi}[z_1,\ldots,z_q;\zeta] := \sum_{i=0}^{\infty} a_i\Omega_{\varpi+\psi i}(z_1,\ldots,z_q)\zeta^i, \qquad (a_i \neq 0),$$

and, for integers p, introduce

$$\Theta_{n,\varpi,\psi}^{\lambda,\mu,r}(x,y;m,c,d;z_1,\ldots,z_q;\zeta) := \sum_{j=0}^{[n/p]} \frac{a_j}{(n-pj)!}\mathfrak{C}_{n-pj}^{\lambda,\mu,r}(x,y;m,c,d)\Omega_{\varpi+\psi j}(z_1,\ldots,z_q)\zeta^j.$$

Here $[n/p]$ denotes the integer part of n/p. Under these definitions, one obtains the identity

$$\sum_{n=0}^{\infty}\Theta_{n,\varpi,\psi}^{\lambda,\mu,r}\Big(x,y;m,c,d;z_1,\ldots,z_q;\frac{\eta}{s^p}\Big)s^n = \frac{2^{1-m}s^m(1-xs)^{-\lambda}(1-s)^{-r-1}}{\mu^d e^s - c^d}e^{ys}\Lambda_{\varpi,\psi}[z_1,\ldots,z_q;\eta], \tag{3.2}$$

whenever both sides of (3.2) *converge.*

Proof. For brevity, let S denote the left–hand side of identity (3.2). By expanding the definition of the polynomial family involved, we have

$$S=\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]}\frac{a_j}{(n-pj)!}\mathfrak{C}_{n-pj}^{\lambda,\mu,r}(x,y;m,c,d)\Omega_{\varpi+\psi j}(z_1,\dots,z_q)\eta^j s^{n-pj}.$$

A change of index, namely replacing n with $n+pj$, transforms the expression into

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}a_j\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y;m,c,d)\Omega_{\varpi+\psi j}(z_1,\dots,z_q)\eta^j\frac{s^n}{n!}\\&=\sum_{n=0}^{\infty}\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^n}{n!}\sum_{j=0}^{\infty}a_j\Omega_{\varpi+\psi j}(z_1,\dots,z_q)\eta^j.\end{aligned}$$

Recognizing the inner sum as the function $\Lambda_{\varpi,\psi}[z_1,\dots,z_q;\zeta]$, and invoking the generating function of the polynomials $\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y;m,c,d)$, we arrive at

$$S=\frac{2^{1-m}s^m(1-xs)^{-\lambda}(1-s)^{-r-1}}{\mu^d e^s-c^d}e^{ys}\Lambda_{\varpi,\psi}[z_1,\dots,z_q;\eta].$$

This verifies the stated identity, and the proof is complete. □

Example 3.1. Consider the specialization of Theorem 3.4 obtained by taking $q=1$ and choosing $\Omega_{\varpi+\psi j}(z)=L_{\alpha,\beta,k,\varpi+\psi j}(z)$, $a_j=1$, $\varpi=0$, $\psi=1$. Here $L_{\alpha,\beta,k,n}(z)$ denotes the modified Laguerre polynomials introduced in [**14**] which are characterized by the generating function

$$(1-\beta t)^{-k}\exp\left(\frac{\alpha z t}{\beta t-1}\right)=\sum_{n=0}^{\infty}L_{\alpha,\beta,k,n}(z)t^n\qquad(|\beta t|<1).$$

Under these assumptions, Theorem 3.4 yields a bilateral generating function that couples two families of the modified Laguerre polynomials with the generalized Cesàro-based Apostol-type polynomials. So, we obtain

$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]}\frac{\eta^j}{(n-pj)!}\mathfrak{C}_{n-pj}^{\lambda,\mu,r}(x,y;m,c,d)L_{\alpha,\beta,k,j}(z)s^{n-pj}=\frac{2^{1-m}s^m(1-xs)^{-\lambda}(1-s)^{-r-1}}{(\mu^d e^s-c^d)(1-\beta\eta)^k}e^{ys+\frac{\alpha z\eta}{\beta\eta-1}}.$$

Example 3.2. Let us consider the specialization of Theorem 3.4 corresponding to the bivariate case $q=2$. In this setting, choose

$$\Omega_{\varpi+\psi j}(z_1,z_2)=\mathfrak{C}_{\varpi+\psi j}^{\lambda_1,\mu_1,r_1}(z_1,z_2;k,a,b),$$

where $a_j=\frac{1}{j!}$, $\varpi=0$, $\psi=1$. With these parameter selections, Theorem 3.4 leads to a bilinear generating function connecting the univariate extension of generalized Cesàro-based Apostol-type polynomials as follows:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]}\frac{\eta^j}{j!(n-pj)!}\mathfrak{C}_{n-pj}^{\lambda,\mu,r}(x,y;m,c,d)\mathfrak{C}_{\varpi+\psi j}^{\lambda_1,\mu_1,r_1}(z_1,z_2;k,a,b)s^{n-pj}$$
$$=\frac{2^{2-m-k}s^m\eta^k(1-xs)^{-\lambda}(1-s)^{-r-1}(1-z_1\eta)^{-\lambda_1}(1-\eta)^{-r_1-1}}{(\mu^d e^s-c^d)(\mu_1^b e^\eta-a^b)}e^{ys+z_2\eta}.$$

In what follows, families of multilinear and multilateral generating functions for the generalized Cesàro-based Bernoulli, Euler, and Genocchi polynomials are presented. Since the proofs follow along similar lines, they are omitted here.

Theorem 3.5. *Let $\Omega_\mu(z_1,\dots,z_q)$ be a function of q complex arguments which does not vanish identically, and let its order be denoted by ϖ, ψ. Associated with this function, define the series*

$$\Lambda_{\varpi,\psi}[z_1,\dots,z_q;\zeta]:=\sum_{i=0}^{\infty}a_i\Omega_{\varpi+\psi i}(z_1,\dots,z_q)\zeta^i,\qquad (a_i\neq 0).$$

Then, the following generating function relations hold:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]}\frac{a_j}{(n-pj)!}{}_B\mathfrak{C}_{n-pj}^{\lambda,\mu,r}(x,y)\Omega_{\varpi+\psi j}(z_1,\dots,z_q)\eta^j s^{n-pj}$$
$$=\frac{s(1-xs)^{-\lambda}(1-s)^{-r-1}}{\mu e^s-1}e^{ys}\Lambda_{\varpi,\psi}[z_1,\dots,z_q;\eta],$$
$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]}\frac{a_j}{(n-pj)!}{}_E\mathfrak{C}_{n-pj}^{\lambda,\mu,r}(x,y)\Omega_{\varpi+\psi j}(z_1,\dots,z_q)\eta^j s^{n-pj}$$
$$=\frac{2(1-xs)^{-\lambda}(1-s)^{-r-1}}{\mu e^s+1}e^{ys}\Lambda_{\varpi,\psi}[z_1,\dots,z_q;\eta],$$
$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]}\frac{a_j}{(n-pj)!}{}_G\mathfrak{C}_{n-pj}^{\lambda,\mu,r}(x,y)\Omega_{\varpi+\psi j}(z_1,\dots,z_q)\eta^j s^{n-pj}$$
$$=\frac{2s(1-xs)^{-\lambda}(1-s)^{-r-1}}{\mu e^s+1}e^{ys}\Lambda_{\varpi,\psi}[z_1,\dots,z_q;\eta].$$

4. Recursive formulas for generalized Cesàro-based Apostol-type polynomials

This section is devoted to presenting recurrence relations for the generalized Cesàro-based Apostol-type polynomials, together with the generalized Cesàro-based Apostol-type Bernoulli, Euler, and Genocchi polynomials.

Theorem 4.1. *The following recurrence relation holds true for $n\in\mathbb{N}_0$:*

$$\mathfrak{C}_{n+1}^{\lambda,\mu,r}(x,y;m,c,d)=\lambda x\mathfrak{C}_n^{\lambda+1,\mu,r}(x,y;m,c,d)+(r+1)\mathfrak{C}_n^{\lambda,\mu,r+1}(x,y;m,c,d)$$
$$+\frac{m}{2}\mathfrak{C}_n^{\lambda,\mu,r}(x,y;m-1,c,d)$$
$$-\frac{\mu^d}{2}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{C}_{n-k}^{\lambda,\mu,r}(x,y+1;0,c,d)\mathfrak{C}_k^{0,\mu,-1}$$
$$+y\mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d).$$

PROOF. Taking the derivative of (2.1) with respect to s, we proceed by comparing both sides term by term. The right-hand side can be written as

$$\sum_{n=1}^{\infty} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \mathfrak{C}_{n+1}^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^n}{n!}.$$

After straightforward simplifications, the left-hand side takes the form

$$\frac{2^{1-m}s^m(1-s)^{-r-1}(1-xs)^{-\lambda}e^{ys}}{\mu^d e^s - c^d} \times \Big\{\lambda x(1-xs)^{-1} + (r+1)(1-s)^{-1} + ms^{-1} - \frac{\mu^d e^s}{\mu^d e^s - c^d} + y\Big\}.$$

Using the definition (2.1) and applying the Cauchy product formula, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{C}_{n+1}^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^n}{n!} &= \lambda x \sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda+1,\mu,r}(x,y;m,c,d)\frac{s^n}{n!} \\ &+ (r+1)\sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda,\mu,r+1}(x,y;m,c,d)\frac{s^n}{n!} \\ &+ \frac{m}{2}\sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m-1,c,d)\frac{s^n}{n!} \\ &- \frac{\mu^d}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathfrak{C}_{n-k}^{\lambda,\mu,r}(x,y+1;0,c,d)\mathfrak{C}_k^{0,\mu,-1}\frac{s^n}{(n-k)!\,k!} \\ &+ y\sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{s^n}{n!}$ on both sides yields the stated recurrence relation. □

THEOREM 4.2. *The following differential–recurrence relation with respect to x holds, for $n \in \mathbb{N}$:*

$$\frac{\partial}{\partial x}\mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d) = \lambda n \mathfrak{C}_{n-1}^{\lambda+1,\mu,r}(x,y;m,c,d).$$

PROOF. Taking the partial derivative of both sides of (2.1) with respect to x, we observe that the dependence on x appears only through the factor $(1-xs)^{-\lambda}$. Thus, $\frac{\partial}{\partial x}(1-xs)^{-\lambda} = \lambda s(1-xs)^{-\lambda-1}$. Consequently, the derivative of the left-hand side of (2.1) becomes

$$\begin{aligned}\lambda s\frac{2^{1-m}s^m(1-s)^{-r-1}(1-xs)^{-\lambda-1}e^{ys}}{\mu^d e^s - c^d} &= \lambda s\sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda+1,\mu,r}(x,y;m,c,d)\frac{s^n}{n!} \\ &= \lambda\sum_{n=1}^{\infty} \mathfrak{C}_{n-1}^{\lambda+1,\mu,r}(x,y;m,c,d)\frac{s^n}{(n-1)!}.\end{aligned}$$

On the other hand, differentiating the right-hand side of (2.1) term by term yields

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial x} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d) \frac{s^n}{n!}$$

whereas $\mathfrak{C}_0^{\lambda,\mu,r}(x,y;m,c,d)$ is independent of x its partial derivative with respect to x vanishes. Equating the coefficients of like powers of s on both sides yields the desired differential–recurrence relation. □

THEOREM 4.3. *The following differential–recurrence relation with respect to y holds, for $n \in \mathbb{N}$:*

$$\frac{\partial}{\partial y} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d) = n\mathfrak{C}_{n-1}^{\lambda,\mu,r}(x,y;m,c,d).$$

PROOF. Taking the partial derivative of both sides of (2.1) with respect to y, we note that the dependence on y occurs only through the exponential factor e^{ys}. Hence, $\frac{\partial}{\partial y} e^{ys} = se^{ys}$. Therefore, the derivative of the left-hand side of (2.1) becomes

$$\begin{aligned} s\frac{2^{1-m}s^m(1-s)^{-r-1}(1-xs)^{-\lambda}e^{ys}}{\mu^d e^s - c^d} &= s\sum_{n=0}^{\infty} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^n}{n!} \\ &= \sum_{n=1}^{\infty} \mathfrak{C}_{n-1}^{\lambda,\mu,r}(x,y;m,c,d)\frac{s^n}{(n-1)!}. \end{aligned}$$

On the other hand, differentiating the right-hand side of (2.1) term by term yields

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial y} \mathfrak{C}_n^{\lambda,\mu,r}(x,y;m,c,d) \frac{s^n}{n!},$$

whereas $\mathfrak{C}_0^{\lambda,\mu,r}(x,y;m,c,d)$ is independent of y its partial derivative with respect to y vanishes. Finally, equating the coefficients of like powers of s on both sides yields the desired differential–recurrence relation. □

The proofs of the following theorems that provide recurrence relations for the generalized Cesàro-based Bernoulli, Euler, and Genocchi polynomials are omitted, since they can be carried out in a manner similar to that of the previous theorems.

THEOREM 4.4. *The generalized Cesàro-based Apostol–Bernoulli polynomials have the following recurrence relations for $n \in \mathbb{N}$:*

$$\begin{aligned} {}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y) = \lambda x {}_B\mathfrak{C}_{n-1}^{\lambda+1,\mu,r}(x,y) + (r+1){}_B\mathfrak{C}_{n-1}^{\lambda,\mu,r+1}(x,y) + \frac{1}{n}{}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y) \\ - \frac{\mu}{n}\sum_{k=0}^{n}\binom{n}{k} {}_B\mathfrak{C}_{n-k}^{\lambda,\mu,r}(x,y+1){}_B\mathfrak{C}_k^{\lambda,\mu,-1} + y{}_B\mathfrak{C}_{n-1}^{\lambda,\mu,r}(x,y), \quad \mu \neq 1. \end{aligned}$$

$$\frac{\partial}{\partial x}{}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y) = \lambda n {}_B\mathfrak{C}_{n-1}^{\lambda+1,\mu,r}(x,y).$$

$$\frac{\partial}{\partial y}{}_B\mathfrak{C}_n^{\lambda,\mu,r}(x,y) = n {}_B\mathfrak{C}_{n-1}^{\lambda,\mu,r}(x,y).$$

THEOREM 4.5. *The generalized Cesàro-based Apostol-Euler polynomials have the following recurrence relations for $n \in \mathbb{N}$:*

$$
\begin{aligned}
{}_E\mathfrak{C}_{n+1}^{\lambda,\mu,r}(x,y) &= \lambda x {}_E\mathfrak{C}_{n}^{\lambda+1,\mu,r}(x,y) + (r+1) {}_E\mathfrak{C}_{n}^{\lambda,\mu,r+1}(x,y) \\
&\quad - \frac{\mu}{2}\sum_{k=0}^{n}\binom{n}{k} {}_E\mathfrak{C}_{n-k}^{\lambda,\mu,r}(x,y+1) {}_E\mathfrak{C}_{k}^{\lambda,\mu,-1} + y {}_E\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y).
\end{aligned}
$$

$$\frac{\partial}{\partial x} {}_E\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y) = \lambda n {}_E\mathfrak{C}_{n-1}^{\lambda+1,\mu,r}(x,y).$$

$$\frac{\partial}{\partial y} {}_E\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y) = n {}_E\mathfrak{C}_{n-1}^{\lambda,\mu,r}(x,y).$$

THEOREM 4.6. *The generalized Cesàro-based Apostol–Genocchi polynomials have the following recurrence relations for $n \in \mathbb{N}$:*

$$
\begin{aligned}
{}_G\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y) &= \lambda x {}_G\mathfrak{C}_{n-1}^{\lambda+1,\mu,r}(x,y) + (r+1) {}_G\mathfrak{C}_{n-1}^{\lambda,\mu,r+1}(x,y) + \frac{1}{n} {}_G\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y) \\
&\quad - \frac{\mu}{2n}\sum_{k=0}^{n}\binom{n}{k} {}_G\mathfrak{C}_{n-k}^{\lambda,\mu,r}(x,y+1) {}_G\mathfrak{C}_{k}^{\lambda,\mu,-1} + y {}_G\mathfrak{C}_{n-1}^{\lambda,\mu,r}(x,y), \quad \mu \neq 1.
\end{aligned}
$$

$$\frac{\partial}{\partial x} {}_G\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y) = \lambda n {}_G\mathfrak{C}_{n-1}^{\lambda+1,\mu,r}(x,y).$$

$$\frac{\partial}{\partial y} {}_G\mathfrak{C}_{n}^{\lambda,\mu,r}(x,y) = n {}_G\mathfrak{C}_{n-1}^{\lambda,\mu,r}(x,y).$$

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(Received 10 01 2026)

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