

## THE PROBLEM OF CONVERGENCE OF GENERAL FOURIER SERIES FOR FUNCTIONS OF CLASS $\text{Lip } \alpha$ ( $0 < \alpha < 1$ )

Giorgi Tutberidze, Vakhtang Tsagareishvili,  
and Giorgi Cagareishvili

**ABSTRACT.** Banach demonstrated that strong differentiability conditions on a function do not necessarily ensure the almost everywhere (a.e.) convergence of its Fourier series concerning arbitrary orthonormal systems (ONS). Conversely, it is well established that the Menshov–Rademacher theorem provides a sufficient criterion for a.e. convergence of an orthonormal series.

We investigate the convergence of the general Fourier series  $\text{Lip } \alpha$ , when  $0 < \alpha < 1$ . In particular, we found special conditions for functions of the orthonormal system, for which the Fourier series of functions of class  $\text{Lip } \alpha$  converges. It is established that the resulting conditions are the best possible in a certain sense.

### 1. Introduction

In the monographs [1, 5, 6] and the papers [4, 7, 10], problems of the theory of convergence of orthonormal series are studied. Interestingly, according to Menchov's and Banach's results, the convergence of general orthonormal series and the convergence of general Fourier series for functions from some differential classes are different problems. In the first case, the coefficients of the orthonormal series play a decisive role. In the second case, the fact that a function  $f$  ( $f \neq 0$ ) belongs to any differential class does not guarantee the convergence of its Fourier series for general Orthonormal systems (ONS). Therefore, for the Fourier series, concerning an ONS, to be convergent, one should impose additional conditions on the functions  $\varphi_n$  of the ONS ( $\varphi_n$ ). Our main problem is finding these conditions for the functions  $\varphi_n$  of an ONS ( $\varphi_n$ ) for the Fourier series of every function of  $\text{Lip } \alpha$  classes to converge.

Similar problems are studied in the papers [4, 8, 9, 11–15].

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The main results of Theorems 3.1 and 3.2 are proved in Section 3. The new applications concerning the convergence of the general Fourier series can be found in Section 4, see Theorems 4.1, and 4.2.

## 2. Preliminaries

DEFINITION 2.1. Let  $f(x)$  be defined on an interval  $[0, 1]$  and suppose that  $x, x + h \in [0, 1]$ . If  $\alpha \in (0, 1]$  and  $\|f(x + h) - f(x)\|_C < M|h|^\alpha$  ( $M$ -absolute constant), then we say that  $f \in \text{Lip } \alpha$ .

Let  $(\varphi_n)$  be an orthonormal system (ONS) on  $[0, 1]$  and

$$C_n(f) = \int_0^1 f(x)\varphi_n(x) dx, \quad n = 1, 2$$

be the Fourier coefficients of a function  $f \in L_2(0, 1)$ .

THEOREM OF MENSHOV–RADEMACHER. [6, ch. 9, p. 332] *If  $(\varphi_n)$  is an ONS on  $[0, 1]$  and a number sequence  $(c_n)$  satisfies the condition  $\sum_{n=1}^{\infty} c_n^2 \log^2 n < +\infty$ , then the series  $\sum_{n=1}^{\infty} c_n \varphi_n(x)$  converges a.e. on  $[0, 1]$ .*

LEMMA 2.1. [4] *If  $f \in L_2(0, 1)$  takes only finite values on  $[0, 1]$  and  $g \in L_2(0, 1)$  is an arbitrary function, then*

$$(2.1) \quad \int_0^1 f(x)g(x) dx = \sum_{i=1}^{n-1} \left( f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{i/n} g(x) dx \\ + \sum_{i=1}^n \int_{(i-1)/n}^{i/n} \left( f(x) - f\left(\frac{i}{n}\right) \right) g(x) dx + f(1) \int_0^1 g(x) dx.$$

Firstly, we must establish the expression

$$B_n(a, \alpha) := 2^{-\alpha n} \sum_{i=1}^{2^n-1} \left| \int_0^{\frac{i}{2^n}} P_n(a, x) dx \right|,$$

where

$$P_n(a, x) := \sum_{k=2^n}^{2^{n+1}} a_k \log k \varphi_k(x).$$

Also, let us note that

$$u_n(a, \alpha) = 2^{(1-\alpha)n} \int_0^1 \left| \int_0^t \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx \right| dt.$$

LEMMA 2.2. *If  $(a_n) \in l_2$ , then  $B_n(a, \alpha) \leq u_n(a, \alpha) + O(1)n2^{-\alpha n}$ .*

PROOF. For some  $(t_{i_n} \in (\frac{i}{2^n}, \frac{i+1}{2^n}), i = 1, 2, \dots, 2^n)$ , we have

$$u_n(a, \alpha) = 2^{(1-\alpha)n} \int_0^1 \left| \int_0^t \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx \right| dt$$

$$\begin{aligned}
&= 2^{(1-\alpha)n} \sum_{i=1}^{2^n} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left| \int_0^t \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx \right| dt \\
&= 2^{(1-\alpha)n} 2^{-n} \sum_{i=1}^{2^n} \left| \int_0^{t_{i_n}} \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx \right| \\
&= 2^{-\alpha n} \sum_{i=1}^{2^n} \left| \int_0^{\frac{i}{2^n}} \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx + \int_{\frac{i}{2^n}}^{t_{i_n}} \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx \right|.
\end{aligned}$$

Next, using the Cauchy inequality,  $(a_n) \in l_2$ , we get

$$\begin{aligned}
u_n(a, \alpha) &\geq B_n(a, \alpha) - 2 \times 2^{-\alpha n} \int_0^1 \left| \int_0^1 \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx \right| \\
&\geq B_n(a, \alpha) - 2^{-\alpha n} n \left( \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}} \geq B_n(a, \alpha) - T(a) n 2^{-\alpha n},
\end{aligned}$$

where  $T(a) = \left( \sum_{k=1}^{\infty} a_k^2 \right)^{1/2} = O(1)$ . □

### 3. The main results

**THEOREM 3.1.** *Let  $(\varphi_k)$  be an ONS on the  $[0, 1]$  and*

$$(3.1) \quad \sum_{k=1}^{\infty} C_k^2(h) \log^2 k < +\infty.$$

where  $h(x) = 1$ ,  $x \in [0, 1]$ . If for every,  $(a_n) \in l_2$

$$(3.2) \quad \sum_{n=1}^{\infty} u_n(a, \alpha) < +\infty,$$

then for any  $f \in \text{Lip } \alpha$  ( $0 < \alpha < 1$ ) we have  $\sum_{k=1}^{\infty} C_k^2(f) \log^2 k < +\infty$ .

**PROOF.** In (2.1), we substitute  $g(x) = P_n(a, x)$ .

$$\begin{aligned}
\int_0^1 f(x) P_n(a, x) dx &= \sum_{i=1}^{2^n-1} \left( f\left(\frac{i}{2^n}\right) - f\left(\frac{i+1}{2^n}\right) \right) \int_0^{\frac{i}{2^n}} P_n(a, x) dx \\
(3.3) \quad &+ \sum_{i=1}^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left( f(x) - f\left(\frac{i}{2^n}\right) \right) P_n(a, x) dx \\
&+ f(1) \int_0^1 P_n(a, x) dx = A_1 + A_2 + A_3.
\end{aligned}$$

According to the condition of Theorem 3.1, we get

$$(3.4) \quad |A_1| = O(1) 2^{-\alpha n} \sum_{i=1}^{2^n-1} \left| \int_0^{\frac{i}{2^n}} P_n(a, x) dx \right| = O(1) B_n(a, \alpha).$$

Using the Cauchy inequality for  $A_2$  we have

$$|A_2| = O(1)2^{-\alpha n} \sum_{i=1}^{2^n} \left| \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} P_n(a, x) dx \right| = O(1)2^{-\alpha n} (n+1) \left( \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}}.$$

For  $h(x) = 1$

$$\begin{aligned} \left| \int_0^1 P_n(a, x) dx \right| &= \left| \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \int_0^1 \varphi_k(x) dx \right| = \left| \sum_{k=2^n}^{2^{n+1}-1} a_k C_k(h) \log k \right| \\ (3.5) \quad &\leq \left( \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=2^n}^{2^{n+1}-1} C_k^2(h) \log^2 k \right)^{\frac{1}{2}} \\ &= O(1) \left( \sum_{k=2^n}^{2^{n+1}-1} C_k^2(h) \log^2 k \right)^{\frac{1}{2}}. \end{aligned}$$

Let us consider the equation

$$\sum_{k=2^n}^{2^{n+1}-1} C_k(f) a_k \log k = \int_0^1 \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx = \int_0^1 f(x) P_n(a, x) dx.$$

By using (3.4), (3.5) and (3.5) in (3.3), we get

$$\begin{aligned} \left| \sum_{k=2^n}^{2^{n+1}-1} C_k(f) a_k \log k \right| &\leq B_n(a, \alpha) + O(1)2^{-\alpha n} (n+1) \left( \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}} \\ &\quad + \left( \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=2^n}^{2^{n+1}-1} C_k^2(h) \log^2 k \right)^{\frac{1}{2}}. \end{aligned}$$

Taking into consideration (3.5),

$$\begin{aligned} \left| \sum_{k=2^n}^{2^{n+1}-1} C_k(f) a_k \log k \right| &\leq O(1)u_n(a, \alpha) + O(1)2^{-\alpha n} n \\ &\quad + \left( \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=2^n}^{2^{n+1}-1} C_k^2(h) \log^2 k \right)^{\frac{1}{2}}. \end{aligned}$$

Finally using (3.5) and the Cauchy inequality we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \sum_{k=2^n}^{2^{n+1}-1} C_k(f) a_k \log k \right| &\leq O(1) \sum_{n=0}^{\infty} u_n(a, \alpha) + O(1) \sum_{n=0}^{\infty} 2^{-\alpha n} n \\ &\quad + \left( \sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} C_k^2(h) \log^2 k \right)^{\frac{1}{2}} < +\infty. \end{aligned}$$

The corollary is that for every  $(a_k) \in l_2$

$$\sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} C_k(f) a_k \log k < \infty.$$

Therefore  $(C_k(f) a_k \log k) \in l_2$  i.e., for every  $f \in \text{Lip } \alpha$   $\sum_{k=1}^{\infty} C_k^2(f) \log^2 k < \infty$ .  $\square$

**THEOREM 3.2.** *Every ONS contains a subsystem such that for every  $f \in \text{Lip } \alpha$*

$$\sum_{k=1}^{\infty} C_{n_k}^2(f) \log^2 k < \infty.$$

**PROOF.** Without loss of generality, we can assume that the system  $(\varphi_k)$  is complete. Then using Parseval's equality, we obtain  $\sum_{n=1}^{\infty} \left( \int_0^x \varphi_n(t) dt \right)^2 = x$ ,  $x \in [0, 1]$ . From here, by virtue of Dini's theorem [3], we get that the series  $\sum_{n=1}^{\infty} \left( \int_0^x \varphi_n(t) dt \right)^2$  is uniformly convergent on  $[0, 1]$ .

Therefore, for any natural number  $k$  there exists a natural number  $n_k$  such that, uniformly for  $x \in [0, 1]$ , we have  $\sum_{n=n_k}^{\infty} \left( \int_0^x \varphi_n(t) dt \right)^2 \leq \frac{1}{k^4}$ .

Consequently, for any natural  $k$ ,

$$(3.6) \quad \left| \int_0^x \varphi_{n_k}(t) dt \right| \leq \frac{1}{k^2}, \quad x \in [0, 1].$$

Now, without loss of generality, we can suppose that for any ONS inequality (3.6) is valid. By using (3.6) and the Cauchy inequality,

$$\begin{aligned} u_m(a, \alpha) &= 2^{-\alpha m} \int_0^1 \left| \int_0^t \sum_{k=2^m}^{2^{m+1}-1} a_k \log k \varphi_{n_k}(x) dx \right| dt \\ &= 2^{(1-\alpha)m} \int_0^1 \left| \sum_{k=2^m}^{2^{m+1}-1} a_k \log k \int_0^t \varphi_{n_k}(x) dx \right| dt \\ &= O(1) 2^{m(1-\alpha)} \sum_{k=2^m}^{2^{m+1}-1} |a_k| \log k \frac{1}{k^2} \\ &= O(1) 2^{m(1-\alpha)} m \left( \sum_{k=2^m}^{2^{m+1}-1} a_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=2^m}^{2^{m+1}-1} \frac{1}{k^4} \right)^{\frac{1}{2}} \\ &= O(1) 2^{m(1-\alpha)} 2^{-m} \cdot m = O(1) 2^{-\alpha m} \cdot m. \end{aligned}$$

In the final analysis, we have that condition (3.2) holds.

From Theorem 3.1 follows the validity of Theorem 3.2.  $\square$

Now we are going to show that the condition of Theorem 3.1 (see condition (3.1)) does not cause the convergence of the series  $\sum_{k=1}^{\infty} C_n^2(f) \log^2 n < \infty$  for any  $f \in \text{Lip } \alpha$ .

**THEOREM 3.3.** *There exists a function  $v \in \text{Lip } 1$  and orthonormal system  $(\psi_n)$  on  $[0, 1]$  such that*

$$(a) \int_0^1 \psi_n(x) dx = 0, \quad n = 1, 2, \dots \quad (b) \sum_{n=1}^{\infty} C_n^2(v) \log^2 n = +\infty.$$

**PROOF.** Let us introduce the function  $v(x) = 1 - \cos 4\pi x$  and the numbers  $a_n = \frac{1}{\sqrt{n \log(n+1)}}$ ,  $n = 1, 2, \dots$ . It is evident that  $v \in \text{Lip } 1$  and  $\sum_{n=1}^{\infty} a_n^2 < +\infty$ . From here, according to the Olevsky theorem, we conclude that there exists an ONS  $(\varphi_n)$  on  $[0, 1]$  such that  $\int_0^1 v(x)\varphi_n(x) dx = a_n$ ,  $n = 1, 2, \dots$ .

Now let us discuss the ONS  $(\psi_n)$  on  $[0, 1]$

$$\psi_n(x) = \begin{cases} \varphi_n(2x), & x \in [0, \frac{1}{2}], \\ -\varphi_n(2x-1), & x \in (\frac{1}{2}, 1]. \end{cases}$$

It is easy to prove that  $(\psi_n)$  is an ONS on  $[0, 1]$ .

Let

$$u(x) = \begin{cases} v(2x), & x \in [0, \frac{1}{2}], \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

We get

$$\begin{aligned} C_n(u, \psi) &= \int_0^1 u(x)\psi_n(x) dx = \int_0^{\frac{1}{2}} u(x)\psi_n(x) dx \\ &= \frac{1}{2} \int_0^1 v(t)\varphi_n(t) dt = \frac{1}{2} C_n(v). \end{aligned}$$

From here, we conclude that

$$\sum_{n=1}^{\infty} C_n^2(u, \psi) \log^2 n = \frac{1}{4} \sum_{n=1}^{\infty} C_n^2(v) \log^2 n = \frac{1}{4} \sum_{n=1}^{\infty} a_n^2 \log^2 n = +\infty. \quad \square$$

#### 4. Efficiency

**THEOREM 4.1.** *The trigonometric system  $(\sqrt{2} \cos 2\pi nx)$  for each  $(a_n) \in l_2$  satisfies condition (3.2).*

**PROOF.** In our case

$$\begin{aligned} u_n(a, \alpha) &= 2^{(1-\alpha)n} \int_0^1 \left| \sum_{k=2^n}^{2^{n+1}} \frac{a_k}{2\pi k} \log k \sin 2\pi kt \right| dt \\ &= O(1) 2^{(1-\alpha)n} \left( \sum_{k=2^n}^{2^{n+1}} \frac{a_k^2}{k^2} \log k \right)^{\frac{1}{2}} \\ &= O(1) 2^{(1-\alpha)n} 2^{-n} n \left( \sum_{k=1}^{2^{n+1}} a_k^2 \right)^{\frac{1}{2}} = O(1) n 2^{-n\alpha}. \quad \square \end{aligned}$$

**THEOREM 4.2.** *Let  $(\varphi_k)$  be the Haar system. Then condition (3.2) is satisfied.*

PROOF. As it is known

$$\begin{aligned} \left| \int_0^t \varphi_k(x) dx \right| &\leq 2^{-\frac{n}{2}}, \quad \text{when } k = 2^n + s, t \in \left[ \frac{s-1}{2^n}, \frac{s}{2^n} \right], \\ \left| \int_0^t \varphi_k(x) dx \right| &= 0, \quad \text{when } t \notin \left[ \frac{s-1}{2^n}, \frac{s}{2^n} \right], \end{aligned}$$

so that

$$\left| \int_0^t \varphi_k(x) dx \int_0^t \varphi_m(x) dx \right| = 0, k \neq m, \quad \int_0^1 \left( \int_0^t \varphi_k(x) dx \right)^2 dt \leq 2^{-2n}.$$

Finally

$$\begin{aligned} u_n(a, \alpha) &= 2^{(1-\alpha)n} \int_0^1 \left| \int_0^t \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \varphi_k(x) dx \right| dt \\ &= 2^{(1-\alpha)n} \int_0^1 \left| \sum_{k=2^n}^{2^{n+1}-1} a_k \log k \int_0^t \varphi_k(x) dx \right| dt \\ &\leq 2^{(1-\alpha)n} \left( \int_0^1 \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \log^2 k \left( \int_0^t \varphi_k(x) dx \right)^2 dt \right)^{\frac{1}{2}} \\ &\leq 2^{-n\alpha} n \left( \sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right)^{\frac{1}{2}}. \quad \square \end{aligned}$$

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Institute of Mathematics  
The University of Georgia  
Tbilisi  
Georgia

and

Faculty of Exact and Natural Sciences  
Ivane Javakhishvili Tbilisi State University  
Tbilisi  
Georgia

`g.tutberidze@ug.edu.ge` and `giorgi.tutberidze@tsu.ge`

Department of Mathematics  
Ivane Javakhishvili Tbilisi State University  
Tbilisi  
Georgia  
`cagare@ymail.com`

Faculty of Exact and Natural Sciences  
Ivane Javakhishvili Tbilisi State University  
Tbilisi  
Georgia  
`giorgicagareishvili7@gmail.com`

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