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ON D-WEAK BASES, D-sn-NETWORKS AND TOPOLOGICAL GROUPS

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ABSTRACT. We discuss some properties of topological groups with D-*sn*-network and D-*weak*-base, respectively, for a basic order D. We mainly give the following results:

(1) Let κ be an infinite cardinal and let D and E be directed sets. If G is a topological group such that $G \leqslant_T D \times E$ and $t(G) = \lambda \leqslant \kappa$, then $G \leqslant_T E$.
(2) (CH) Let G be a topological group, H a normal closed subgroup of G such that $H \leqslant_T \omega_1$ and $G/H \leqslant_T \omega^\omega$. Then $G \leqslant_T \omega^\omega$.
(3) Every Fréchet Hausdorff paratopological group G having the property $(**)$ with a D-*sn*-network, for a basic order D, is first-countable.
(4) Let D be a basic order. If a topological space X has a D-*weak*-base, then it has countable cs*-character.

1. Introduction

In this note, all topological spaces and (para)topological groups are Hausdorff unless stated otherwise.

Tukey maps are typically used to compare cofinalities of various directed sets. Here we will use Tukey maps to compare topological groups and directed sets. Hence, we will say that a topological group G is Tukey reducible to directed set D if there is a Tukey map from a local base of the identity of G to directed set D, and we will write $G \leqslant_T D$. Also, Tukey maps can be used to compare topological groups and to analyse cofinal types of topological groups. There has been work on this topic, for example [**3**, **5**, **7**, **11**, **15**].

It is well known that some properties of topological groups do not necessarily hold for paratopological groups. So there is a natural question: if something holds for topological groups, does it hold for paratopological groups?

Topological group with ω^ω-sn-network and ω^ω-weak base were studied in [**4**,**10**]. Here we will make connection between basic orders and sn-networks and weak-bases.

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It is proved that every space with ω^ω-weak base has countable cs*-character and every almost metrizable topological group with countable cs*-character is metrizable. This improves result of Gabriyelyan, Kakol, and Leiderman in [**7**], which state that every almost metrizable topological group with an ω^ω-base is metrizable. In this note we will prove that every space with D-weak base, for a basic order D, has countable cs*-character.

This paper is organized in the following way. Section 2 is dedicated to preliminaries, all notions will be found there. In Section 3 we will improve [**11**, Theorem 4.9]. In Section 4 we will prove that under (CH), ω^ω is three-space property and we will see some results about exact sequences and metrizability. Section 5 is dedicated to Fréchet paratopological groups and D-sn-network, while Section 6 is dedicated to topological spaces with D-weak-base and first-countability.

2. Preliminaries

We use mostly standard terminology from set theory and topology. For topological groups, our main references is [**1**].

Whenever we consider a topological group, its identity will be denoted e. When there is a possibility for a confusion, the identity of a group G will be denoted e_G. In this paper, a topological group is a group $(G, \cdot)$ with Hausdorff topology τ such that both $\cdot$ and $^{-1}$ are continuous operations in τ. Typically, we will point out what is the operation in the group, and what is the topology on the group, only in cases when there is a danger of misunderstanding.

If A is a set, then $|A|$ denotes the cardinality of a set A. For a cardinal κ and a set A, we denote $[A]^{<\kappa} = \{X \subseteq A : |X| < \kappa\}$ and $[A]^{\kappa} = \{X \subseteq A : |X| = \kappa\}$. Note that all other variations on this notion, for example $[A]^{\leqslant\kappa}$, are defined analogously. For a function $f : X \to Y$, and $A \subseteq X$ and $B \subseteq Y$ we define the direct image of A via f by $f[A]$, and the preimage of B via f by $f^{-1}[B]$.

Recall that $(P, \leqslant_P)$ is a *partial order* if P is a set and $\leqslant_P$ is a reflexive, antisymmetric and transitive binary relation on P. If $(P, \leqslant_P)$ is a partial order, we say that $X \subseteq P$ is *unbounded* in P if there is no $z \in P$ such that $x \leqslant_P z$ for all $x \in X$, while we say that $Y \subseteq P$ is *cofinal* in P if for every $x \in P$ there is some $y \in Y$ such that $x \leqslant_P y$.

Definition 2.1. We say that a partially ordered set $(D, \leqslant_D)$ is a *directed set* if for every x and y in D there is some $z \in D$ such that $x \leqslant_D z$ and $y \leqslant_D z$.

We will always denote the ordering of a directed set D by $\leqslant_D$, so we will be able to write D in place of $(D, \leqslant_D)$.

Definition 2.2. [**16, 17**] Let D and E be directed sets. We say that D is *Tukey reducible* to E if there is a map $f : D \to E$ such that for every unbounded set $X \subseteq D$ the set $f[X]$ is unbounded in E. We write this $D \leqslant_T E$, and we call such a map *Tukey map*.

Note that this is equivalent to saying that the preimage of every set bounded in E is also bounded in D.

DEFINITION 2.3. [**16**] Let D and E be directed sets. We say that a map $g : E \to D$ is *cofinal* if for every $x \in D$ there is $y \in E$ such that $x \leqslant_D g(z)$ whenever $y \leqslant_E z$.

LEMMA 2.1. [**16**] *Let D and E be directed sets. Then $D \leqslant_T E$ if and only if there is a cofinal map $g : E \to D$.*

LEMMA 2.2. [**16**] *Let D and E be directed sets. Then $f : D \to E$ is a cofinal map if and only if $f[A]$ is cofinal in E for every A which is cofinal in D.*

DEFINITION 2.4. [**16**] Let D and E be directed sets. If $D \leqslant_T E$ and $E \leqslant_T D$, then we say that D and E are *cofinally similar* or that D and E have the same *cofinal type*. We write this $D \equiv_T E$.

THEOREM 2.1. [**16**] *Let D and E be directed sets. Then $D \equiv_T E$ if and only if there is a poset P such that both D and E can be embedded as cofinal subsets of P.*

REMARK 2.1. [**16**] Note that for directed sets D and E, if D is a cofinal subset of E, then $D \equiv_T E$.

REMARK 2.2. Note that if D is a countable directed set, then $D \leqslant_T \omega$. In particular, either $D \equiv_T 1$ or $D \equiv_T \omega$.

DEFINITION 2.5. [**13**] Let D be a separable metric space and let $\leqslant$ be a partial order on D. We say that $(D, \leqslant)$ is *basic* if

(1) each pair of elements of D has the least upper bound with respect to $\leqslant$ and the binary operation of of the smallest upper bound from $D \times D$ to D is continuous;
(2) each bounded sequence has a convergent subsequence;
(3) each convergent sequence has a bounded subsequence.

EXAMPLE 2.1. [**15**] Here is a list of some of the examples of basic orders:

(i) $1, \mathbb{N}$, and $\mathbb{N}^{\mathbb{N}}$.
(ii) P-point ultrafilters on $\mathbb{N}$.
(iii) $\mathcal{K}(X)$ for X a separable metric space, where $\mathcal{K}(X)$ is set of all compact subsets of X.
(iv) Compact nowhere dense subsets of the Cantor set.
(v) Compact subsets of the unit interval of measure zero.

We say that a point x in a topological space X is *basic* if its neighborhood filter, or equivalently, its ideal $\mathcal{I}_x$ is Tukey reducible to some basic order.

3. Tightness and topological groups

In this section we will give generalization of Theorem 3.1.

DEFINITION 3.1. [**15**] For a topological space X and a point $x \in X$ we define

$$\mathcal{I}_x = \{A \subseteq X \setminus \{x\} : x \notin \overline{A}\} \text{ and } \mathcal{I}_x^{\perp} = \{A \subseteq X \setminus \{x\} : A \to x\}.$$

Since we work with topological groups, we will denote $\mathcal{I}_x$ with $\mathcal{I}_e$ and $\mathcal{I}_x^{\perp}$ with $\mathcal{I}_e^{\perp}$, for identity e.

Let G be a topological group with the identity e and let $\mathcal{N}_e$ be its local base. It is easy to see that $(\mathcal{I}_e, \subseteq) \equiv_T (\mathcal{N}_e, \supseteq)$.

Definition 3.2. For a topological space X, the *tightness* of X is the minimal cardinal $\kappa \geqslant \omega$ with the property that for every set $A \subseteq X$ and every point $x \in \overline{A}$, there is $C \subseteq A$ such that $|C| \leqslant \kappa$ and $x \in \overline{C}$.

The tightness of a space X is denoted by $t(X)$.

Let us connect the tightness of a topological group G with its metrizability.

Theorem 3.1. [**11**] *Let κ, λ be regular infinite cardinals and G a topological group with $t(G) = \kappa$ and such that $G \leqslant_T \lambda \times \kappa^+$. Then $G \leqslant_T \lambda$.*

Corollary 3.1. [**11**] *Let κ be an infinite regular cardinal and G a topological group with $t(G) \leqslant \kappa$ and such that $G \leqslant_T \omega \times \kappa^+$. Then G is metrizable.*

Theorem 3.2. *Let κ be an infinite cardinal and let D and E be directed sets. If G is a topological group such that $G \leqslant_T D \times E$ and $t(G) = \lambda \leqslant \kappa$, then $G \leqslant_T E$.*

Proof. Let e be identity of topological group G and let $\mathcal{I}_e$ be ideal as in Definition 3.1.

Let $f : \mathcal{I}_e \to D \times E$ be a Tukey map. For each $(d, a) \in D \times E$, define

$$A_d^a = \{b \in \mathcal{I}_e : f(b) \leqslant (d, a)\}.$$

Set $A_d^a = f^{-1}[\{(c, f) \in D \times E : (c, f) \leqslant (d, a)\}]$. Since f is a Tukey map and since $\{(c, f) \in D \times E : (c, f) \leqslant (d, a)\}$ is bounded, then is A_d^a bounded as inverse image under f. Also, for fixed $a \in E$, if $d_1 \leqslant d_2$, for $d_1, d_2 \in D$, then $A_{d_1}^a \subseteq A_{d_2}^a$.

Now, for each $a \in E$, let us define $B_a = \bigcup(\bigcup_{d \in D} A_d^a)$. It is obvious that each B_a is a subset of G.

First show that $B_a \in \mathcal{I}_e$ for each $a \in E$. Suppose there is $a \in E$ such that $B_a \notin \mathcal{I}_e$. This means that $e \in \overline{B_a}$. Since $t(G) = \lambda$, there is $X \subseteq B_a$, $|X| = \lambda$ and $e \in \overline{X}$. Let us enumerate $X = \{x_\alpha : \alpha < \lambda\}$. Since $B_a = \bigcup(\bigcup_{d \in D} A_d^a)$, for each $\alpha < \lambda$ there is some $b_\alpha \in \bigcup_{d \in D} A_d^a$ such that $x_\alpha \in b_\alpha$. Hence, for each $\alpha < \kappa$ there is some $d_\alpha \in D$ such that $b_\alpha \in A_{d_\alpha}^a$. Let $D_1 = \{d_\alpha : \alpha < \lambda\}$. Then $D_1 \in [D]^\lambda$. This means that there is a $d^* \in D$ such that $d_\alpha \leqslant_D d^*$ for each $\alpha < \lambda$. Since $A_{d_\alpha}^a \subseteq A_{d^*}^a$ for each $\alpha < \lambda$, then $b_\alpha \in A_{d^*}^a$. Since $A_{d^*}^a$ is bounded, there is a $B \in \mathcal{I}_e$ such that $b_\alpha \subseteq B$ for each $\alpha < \kappa$. Thus $X \subseteq B$, which imply that $e \in \overline{B}$. Contradiction.

Now we will prove that $\{B_a : a \in E\}$ is cofinal subset of $\mathcal{I}_e$. Let $S \in \mathcal{I}_e$. Then there is a $(d', a) \in D \times E$ such that $f(S) = (d', a)$. This means that $S \in A_{d'}^a \subseteq \bigcup_{d \in D} A_d^a$. This implies that $S \subseteq \bigcup(\bigcup_{d \in D} A_d^a) = B_a \in \mathcal{I}_e$.

Note that the mapping $a \mapsto B_a$, $a \in E$, is increasing mapping. By Lemma 2.1 $\mathcal{I}_e \leqslant_T E$, i.e. $G \leqslant_T E$. □

Corollary 3.2. *Let κ be a regular infinite cardinal and let D be a directed set such that each subset of D of size at most κ is bounded. If G is a topological group such that $G \leqslant_T D$ and $t(G) = \kappa$, then G is discrete.*

PROOF. For each directed set D it is easy to see that $D \equiv_T D \times 1$. By Theorem 3.2, $G \leqslant_T 1$. Thus G is discrete. □

EXAMPLE 3.1. Let G be a topological group such that $G \leqslant_T [\omega_2]^{\leqslant\omega} \times [\omega_1]^{<\omega}$ and $t(G) = \omega$. Then $G \leqslant_T [\omega_1]^{<\omega}$.

LEMMA 3.1. *Let κ, λ be infinite cardinals and let D, E be directed sets such that each subset of D of size at most κ is bounded and each subset of E of size at most λ is bounded. If G is a topological group such that $t(G) < \min\{\kappa, \lambda\}$ and $G \leqslant_T D \times E$, then G is discrete.*

PROOF. By Theorem 3.2, $G \leqslant_T E$ and by Corollary 3.2, G is discrete. □

COROLLARY 3.3. *Let κ, λ be regular uncountable cardinals. If G is a topological group such that $G \leqslant_T \lambda \times \kappa$ and $t(G) < \min\{\lambda, \kappa\}$, then G is discrete.*

COROLLARY 3.4. *Let κ, λ be regular uncountable cardinals. Let G be a topological group such that $G \leqslant_T \lambda \times [\kappa]^{\leqslant\lambda}$ and $t(G) = \lambda$. Then $G \leqslant_T \lambda$.*

COROLLARY 3.5. *Let κ, λ, μ be regular infinite cardinals and G a topological group such that $t(G) \leqslant \kappa$. If $G \leqslant_T \lambda \times \mu \times \kappa^+$, then $G \leqslant_T \lambda \times \mu$.*

4. Three-space property

A property P is said to be a *three-space property* if for every topological group G and a closed normal subgroup H of G both H and G/H having property P implies that G has property P. A classical result says that metrizability is a three-space property (see [**9**, 5.38]). In particular, it was shown that

$$P = \{\text{each compact subset is metrizable}\}$$

is a three-space property.

DEFINITION 4.1. [**3**] Let P be a partially ordered set (poset). A topological space X is defined to have a *neighborhood P-base* at $x \in X$ if there exists a neighborhood base $\{U_p(x) : p \in P\}$ at x such that $U_p(x) \subseteq U_{p'}(x)$ for all $p \geqslant p'$ in P. We say that a topological space X has a *P-base* if it has a neighborhood P-base at each x in X.

THEOREM 4.1. [**5**] *Let G be a topological group and H a closed normal subgroup of G such that (G, H) is a good pair. Suppose that P, Q and R are Dedekind-complete posets such that $P \times Q \leqslant_T R$. If H has a P-base and G/H has a Q-base, then G has an R-base.*

Thus, as a corollary, in [**5**], Feng proved that if H is a metrizable closed normal subgroup of G, M is noncompact separable metric space, and G/H has a $\mathcal{K}(M)$-base, then G has a $\mathcal{K}(M)$-base.

PROPOSITION 4.1. [**7**] *Let G be a topological group. If G has a normal metrizable closed subgroup H such that G/H has an ω^ω-base, then G has an ω^ω-base.*

THEOREM 4.2. *Let G be a topological group, H a normal closed subgroup of G such that $H \leqslant_T \omega_1$ and $G/H \leqslant_T D$, for a basic order D. Then $G \leqslant_T \omega_1 \times D$.*

Proof. Let $\{W_\alpha : \alpha < \omega_1\}$ be a decreasing sequence of open symmetric neighborhoods of the identity e in G, and let $\{W_\alpha \cap H : \alpha < \omega_1\}$ be an open symmetric neighborhood of the identity e in H.

Let $\mathcal{F}_H^{G/H}$ be a filter neighborhood of the identity H in G/H and $\mathcal{F}_e^G$ a filter neighborhood of the identity e in G. Let D be a basic order such that $G/H \leqslant_T D$, i.e., $\mathcal{F}_H^{G/H} \leqslant_T D$. Let $f : \mathcal{F}_H^{G/H} \to D$ be a Tukey map. Define now map $g : D \to \mathcal{F}_H^{G/H}$ by

$$g(d) = \bigcap\{a \in \mathcal{F}_H^{G/H} : f(a) \leqslant_D d\}.$$

Then g is cofinal monotone (see [**11**, Theorem 5.2]). Let $\{g(d) : d \in D\}$ be an open symmetric neighborhood of the identity H in G/H. Let $q : G \to G/H$ be a quotient map and let us define $U_d = q^{-1}(g(d))$. Let $R_d(\alpha) = W_\alpha \cap U_d$, for $\alpha < \omega_1$ and $d \in D$ and define $\mathcal{R} = \{R_d(\alpha) : d \in D, \alpha < \omega_1\}$. It is obvious that the family $\mathcal{R} = \{R_d(\alpha) : d \in D, \alpha < \omega_1\}$ consists of open symmetric neighborhoods of the identity e in G.

Claim 4.1. *The family* $\mathcal{R}$ *is a base of* G.

Proof. Let $U \in \mathcal{F}_e^G$. Then there is a $V \in \mathcal{F}_e^G$ such that $VV \subseteq U$. Choose $\alpha < \omega_1$ such that $W_\alpha \cap H \subseteq V$. Now consider $W_{\alpha+1} \cap V$. Then $q(W_{\alpha+1} \cap V) \in \mathcal{F}_H^{G/H}$. Then there is $d \in D$ such that $g(d) \subseteq q(W_{\alpha+1} \cap V)$, i.e. $q(U_d) \subseteq q(W_{\alpha+1} \cap V)$. Set $R_d(\alpha+1) = W_{\alpha+1} \cap U_d$. We prove that $R_d(\alpha+1) \subseteq U$. Let $a \in R_d(\alpha+1) = W_{\alpha+1} \cap U_d$. Then $a \in W_{\alpha+1}$ and $a \in U_d \subseteq (W_{\alpha+1} \cap V) \cdot H$. Clearly, $a = bh$ for some $b \in W_{\alpha+1} \cap V$ and $h \in H$. Then $h = b^{-1}a \in (W_{\alpha+1})^{-1} W_{\alpha+1} \subseteq W_\alpha$, i.e. $h \in W_\alpha \cap H$. Finally, $a \in (W_{\alpha+1} \cap V)(W_\alpha \cap H) \subseteq (W_\alpha \cap V)(W_\alpha \cap H) \subseteq VV \subseteq U$. □

Claim 4.2. $\mathcal{R} \leqslant_T \omega_1 \times D$.

Proof. Let $\varphi : \omega_1 \times D \to \mathcal{R}$ be defined by $\varphi(\alpha, d) = R_d(\alpha)$.

- φ is monotone. Indeed, let $(\alpha, d), (\beta, e) \in \omega_1 \times D$ and $d \leqslant_D e$ and $\alpha \leqslant \beta$. Since g is monotone, $g(e) \subseteq g(d)$. This imply that $U_e = q^{-1}(g(e)) \subseteq q^{-1}(g(d)) = U_d$. Since $\{W_\alpha : \alpha < \omega_1\}$ is a descreasing sequence, then $W_\beta \subseteq W_\alpha$ for $\alpha \leqslant \beta$. Thus,
$$\varphi(\beta, e) = W_\beta \cap U_e \subseteq W_\alpha \cap U_d = \varphi(\alpha, d).$$
- φ is cofinal. Let $A \subseteq \omega_1 \times D$ be a cofinal subset. We show that $\varphi[A]$ is cofinal subset of $\mathcal{R}$. Let $N \in \mathcal{R}$. Then there are $d \in D$ and $\alpha < \omega_1$ such that $N = R_d(\alpha)$, i.e. $N = R_d(\alpha) = \varphi(\alpha, d)$. Since A is cofinal, there is $a \in A$ such that $(\alpha, d) \leqslant a$. Since φ is monotone, then
$$\varphi(a) \subseteq \varphi(\alpha, d) = R_d(\alpha) = N.$$ □

Now by lemma 2.1, $\mathcal{R} \leqslant_T \omega_1 \times D$. This finishes the proof of the theorem. □

Corollary 4.1. (CH) *Let* G *be a topological group,* H *a normal closed subgroup of* G *such that* $H \leqslant_T \omega_1$ *and* $G/H \leqslant_T \omega^\omega$*. Then* $G \leqslant_T \omega^\omega$*.*

Proof. Under CH, $\omega_1 \leqslant_T \omega^\omega$. Then by Theorem 4.2
$$G \leqslant \omega_1 \times \omega^\omega \leqslant_T \omega^\omega \times \omega^\omega \equiv_T \omega^\omega.$$ □

Now we give some applications of three-space property applied on finite exact sequences.

An *exact sequence* is a sequence of morphisms between objects (for example, groups, rings, modules, and, more generally, objects of Abelian categories) such that the image of one morphism equals the kernel of the next.

In the context of group theory, a sequence

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} G_n$$

of groups and group homomorphisms is said to be *exact at* G_i if $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$. The sequence is called *exact* if it is exact at each G_i for all $1 \leqslant i < n$, i.e., if the image of each homomorphism is equal to the kernel of the next.

It is helpful to consider relatively simple cases where the sequence is of group homomorphisms, is finite, and begins or ends with the *trivial group* denoted 0.

Lemma 4.1. [**11**] *Let G be a topological group and H a subgroup of G. Then $H \leqslant_T G$.*

Lemma 4.2. [**11**] *Let G be a topological group and $N \lhd G$ a closed normal subgroup of G. Then $G/N \leqslant_T G$.*

Lemma 4.3. [**11**] *Let G and H be topological groups, and let $\varphi : G \to H$ be an open and continuous homomorphism. Then $H \leqslant_T G$.*

Lemma 4.4. *Let G_1, G_2 and G_3 be topological groups and let $G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3$ be finite exact sequence such that:*

(1) *$G_1 \leqslant_T \omega$ and $G_3 \leqslant_T \omega$, i.e., G_1 and G_3 are metrizable,* (2) *f_1 is open.*

Then $G_2 \leqslant_T \omega$, i.e. G_2 is metrizable.

Proof. Let e_1, e_2 and e_3 be identities of groups G_1, G_2 and G_3, respectively.

By definition of finite exact sequence we know that $\mathrm{Im}(f_1) = \mathrm{Ker}(f_2)$. Since $\mathrm{Ker}(f_2) \leqslant G_2$ and f_1 is open, then by Lemma 4.3, $\mathrm{Ker}(f_2) \leqslant_T G_1 \leqslant_T \omega$. Note that $\mathrm{Ker}(f_2) = f_2^{-1}[\{e_3\}]$. Since G_3 is T_1, then $\{e_3\}$ is a closed subset of G_3, and since f_2 is continuous, then $f_2^{-1}[\{e_3\}]$ is closed. Thus, $\mathrm{Ker}(f_2)$ is a closed normal subgroup of G_2. Also, $G_2/\mathrm{Ker}(f_2) \cong \mathrm{Im}(f_2)$ which is subgroup of G_3. By Lemma 4.1 $\mathrm{Im}(f_2) \leqslant_T G_3 \leqslant_T \omega$, i.e., $G_2/\mathrm{Ker}(f_2) \leqslant_T \omega$. Since metrizability is three-space property, then $G_2 \leqslant_T \omega$, i.e. G_2 is metrizable. □

Lemma 4.5. *Let G_1, G_2 and G_3 be metrizable topological groups and let*

$$0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3$$

be a finite exact sequence. Then G_2 is metrizable.

Proof. Let e be the only element of trivial group 0 and let e_1, e_2 and e_3 be indentites of groups G_1, G_2 and G_3, respectively. Then $\mathrm{Im}(f_0) = \{e_1\} = \mathrm{Ker}(f_1)$. This imply that f_1 is one-to-one map, thus $G_1 \cong \mathrm{Im}(f_1)$. Since $G_1 \leqslant_T \omega$, then $\mathrm{Ker}(f_2) = \mathrm{Im}(f_1) \leqslant_T \omega$. The rest of the proof is the same as in the previous lemma. □

5. Fréchet paratopological groups with a D-sn-network

Let X be a space. For $P \subseteq X$, the set P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P. P is a *sequentially open* subset of X if P is a sequential neighborhood of x in X for each $x \in P$. X is said to be a *sequential space* [**6**] if each sequentially open subset is open in X.

Definition 5.1. Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathcal{P}_x$, then $W \subseteq U \cap V$ for some $W \in \mathcal{P}_x$; (b) the family $\mathcal{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with U open in X, then $P \subseteq U$ for some $P \in \mathcal{P}_x$.

(1) The family $\mathcal{P}$ is called a *sn-network* (*sequential-neighborhood network*) [**12**] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. X is called *snf-countable* [**12**] if X has an *sn*-network $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.
(2) The family $\mathcal{P}$ is called a *weak base* [**2**] for X if for every $A \subseteq X$, the set A is open in X whenever for each $x \in A$ there exists $P \in \mathcal{P}_x$ such that $P \subseteq A$. The space X is *weakly first-countable* if $\mathcal{P}_x$ is countable for each $x \in X$.

Definition 5.2. A topological space X is said to have a *D-sn-network*, for a basic order D, if the space X has a *sn*-network $\bigcup_{x \in X} \mathcal{P}_x$ such that $\mathcal{P}_x = \{U_d(x) : d \in D\}$ for each $x \in X$ and $U_a(x) \subseteq U_d(x)$ for each $x \in X$ and $d \leqslant_D a$ in D.

Definition 5.3. A topological space X is called *Fréchet space* if for every $A \subseteq X$ and $x \in \overline{A}$ there is a sequence $\langle x_n : n < \omega \rangle \subseteq A$ such that $\lim_{n \to \omega} x_n = x$.

Note that if G is a topological group, then it is Fréchet if for every $A \subseteq G$ with $e \in \overline{A}$ there is a sequence $\langle x_n : n < \omega \rangle \subseteq A$ such that $\lim_{n \to \omega} x_n = e$.

Definition 5.4. A topological space X is said to have a *D-weak base* if the space X has a weak base $\bigcup_{x \in X} \mathcal{U}_x$ such that $\mathcal{U}_x = \{U_d(x) : d \in D\}$ for each $x \in X$ and $U_a(x) \subseteq U_d(x)$ for each $x \in X$ and $d \leqslant_D a$ in D.

The following three lemmas prove analogously to Lemma 3.3, Lemma 3.4 and Lemma 3.5 from [**10**].

Lemma 5.1. *Suppose that* $\mathcal{U} = \{U_d(x) : x \in X, d \in D\}$ *and* $\mathcal{V} = \{V_d(x) : x \in X, d \in D\}$ *are two D-weak bases for a Fréchet Hausdorff space* X*. Then, for each* $x \in X$ *and each* $d \in D$*, there is* $a \in D$ *such that* $V_a(x) \subseteq U_d(x)$*.*

Lemma 5.2. *Let* G *be a semitopological group and* D *a basic order. Suppose that* $\{V_d(x) : x \in G, d \in D\}$ *is a D-weak base for* G*. For each* $x \in G$ *and* $d \in D$*, put* $W_d(x) = x \cdot V_d(e)$*. Then* $\{W_d(x) : x \in G, d \in D\}$ *is a D-weak base for* G*.*

Lemma 5.3. *Let* G *be a paratopological group and* D *a basic order. Suppose that* $\{V_d(x) : x \in G, d \in D\}$ *is a D-weak base for* G*. For each* $x \in G$ *and* $d \in D$*, put* $W_d(x) = x \cdot V_d(e) \cdot V_d(e)$*. Then* $\{W_d(x) : x \in G, d \in D\}$ *is a D-weak base for* G*.*

Theorem 5.1. *A Frećhet Hausdorff paratopological group* G *has a D-weak base if and only if it has a D-base, for a basic order* D*.*

PROOF. It is obvious that "if" part is true. We show the "only if" part. Let $\{V_d(x) : x \in G, d \in D\}$ be a D-weak base for G. By Lemma 5.2, we can assume that $V_d(x) = x \cdot V_d(e)$ for each $x \in G$ and $d \in D$. For each $d \in D$, let us define

$$U_d = \{y \in V_d(e) : y \cdot V_a(e) \subseteq V_d(e) \text{ for some } a \in D\}.$$

It is obvious that $e \in U_d \subseteq V_d(e)$. It suffices to show that U_d is open in G. Let $y \in U_d$ be arbitrary chosen. Then $y \cdot V_a(e) \subseteq V_d(e)$ for some $a \in D$. By Lemma 5.1 and 5.3 there is $b \in D$ such that $y \cdot V_b(e) \cdot V_b(e) \subseteq y \cdot V_a(e)$. Hence

$$(y \cdot V_b(e)) \cdot V_b(e) \subseteq V_d(e).$$

This implies that $V_b(y) = y \cdot V_b(e) \subseteq U_d$. Thus, U_d is open in G for each $d \in D$.

Now $\{W_d(x) : x \in G, d \in D\}$ is a D-base, where $W_d(x) = x \cdot U_d$. □

LEMMA 5.4. [**3**] *Let P be a poset.*

(1) *The uniformity $\mathcal{U}_X$ of a uniform space X has a P-base iff $P \succeq \mathcal{U}_X$ iff $\mathcal{U}_X \leqslant_T P$.*
(2) *A topological space X has a neighborhood P-base at a point $x \in X$ iff $P \succeq \mathcal{T}_x(X)$ iff $\mathcal{T}_x(X) \leqslant_T P$.*

COROLLARY 5.1. [**15**] *A Fréchet topological group G has a basic point if and only if it is metrizable.*

COROLLARY 5.2. *Every Fréchet topological group G with a D-weak base, for a basic order D, is metrizable.*

PROOF. Since every topological group is paratopological group, and since G has a D-weak base, then by Theorem 5.1, G has a D-base. Now by Lemma 5.4(2), filter neighborhood of the identity e of group G is Tukey reducible to D, i.e. e is basic point. By Corollary 5.1, G is metrizable. □

DEFINITION 5.5. A paratopological group G is said to have *the property* $(**)$ if there exists a non-trivial sequence $\langle x_n : n < \omega\rangle$ in G such that both $\langle x_n : n < \omega\rangle$ and $\left\langle x_n^{-1} : n < \omega\right\rangle$ converge to the identity of G.

LEMMA 5.5. *Every Fréchet paratopological group G having the property $(**)$ satisfies that (strong α_4-property) for any subset $\langle x_{m,n} : m, n < \omega\rangle \subseteq G$ satisfying $\lim_{n\to\infty} x_{m,n} = x \in G$ for each $m < \omega$, it is possible to choose strictly increasing sequences $\langle i_k : k < \omega\rangle$ and $\langle j_k : k < \omega\rangle$ of natural numbers such that $\lim_{k\to\infty} x_{i_k,j_k} = x$.*

THEOREM 5.2. *Every Fréchet Hausdorff paratopological group G having the property $(**)$ with a D-sn-network, for a basic order D, is first-countable.*

PROOF. Let $\{U_d : d \in D\}$ be a D-*sn*-network for a basic order D. Since D is a separable metric space, there is a countable base $\{B_n : n < \omega\}$ of D. For each $d \in D$ and $n < \omega$ with $d \in B_n$, define $D_n(d) = \bigcap\{U_a : a \in B_n\}$. For each $d \in D$, we list an increasing sequence of natural numbers $\left\langle n_i^d : i < \omega\right\rangle$ such that $d \in B_{n_i^d}$ for each $i < \omega$ and $\{B_{n_i^d} : i < \omega\}$ is a decreasing base of D.

CLAIM 5.1. *For each $d \in D$, there is $i < \omega$ such that $D_{n_i^d}(d)$ is a sequential neighborhood of the identity e.*

PROOF. Suppose that the claim fails. Then there is some $d \in D$ such that for each $i < \omega$, $D_{n_i^d}(d)$ is not sequential neighborhood at e. Then for each $i < \omega$, there is a sequence $\langle x_{n,i} : n < \omega \rangle$ which converge to e but is not eventually in $D_{n_i^d}(d)$. With out loss of generality we may and will assume that $x_{n,i} \notin D_{n_i^d}(d)$ for each $i < \omega$. By Lemma 5.5 there are strictly increasing sequences $\langle n_k : k < \omega \rangle$ and $\langle i_k : k < \omega \rangle$ such that $\lim_{k\to\infty} x_{n_k,i_k} = e$. Since $x_{n_k,i_k} \notin D_{n_{i_k}^d}(d)$, for each $k < \omega$ there is $d_k \in B_{n_{i_k}^d}$ such that $x_{n_k,i_k} \notin U_{d_k}$. Also, by the choice of d_k, for $k < \omega$, we have that $\lim_{k\to\infty} d_k = d$.

Since D is a basic order, there is a bounded subsequence

$$\langle d_{k_l} : l < \omega \rangle \subseteq \langle d_k : k < \omega \rangle .$$

Let $d' \in D$ be a boundary for $\langle d_{k_l} : l < \omega \rangle$. Since D is directed set, we may assume that $d \leqslant_D d'$. By Definition 5.2 $U_{d'} \subseteq U_{d_{k_l}}$ for each $l < \omega$. Going to the subsequence $\left\langle x_{n_{k_m}} : m < \omega \right\rangle$ of $\langle x_{n_k} : k < \omega \rangle$ such that $x_{n_{k_m},i_{k_m}} \notin U_{d_{k_m}}$ for each $m < \omega$, we have $x_{n_{k_m},i_{k_m}} \notin U_{d'}$ for each $m < \omega$. Contradiction with assumption that $U_{d'}$ is sequential neighborhood of the identity e. □

For every $d \in D$ let i_d denote the least natural number such that $D^d_{n_{i_d}}(d)$ is a sequential neighborhood of e. We can do this by Claim 5.1. Now we show that $D^d_{n_{i_d}}(d)$ is a neighborhood of e. It suffices to show that $e \in \mathrm{Int}(D^d_{n_{i_d}}(d)) \subseteq D^d_{n_{i_d}}(d)$.

Assume that $e \in G \smallsetminus \mathrm{Int}(D^d_{n_{i_d}}(d))$. Since $G \smallsetminus \mathrm{Int}(D^d_{n_{i_d}}(d))$ is closed, then $G \smallsetminus \mathrm{Int}(D^d_{n_{i_d}}(d)) = \overline{G \smallsetminus \mathrm{Int}(D^d_{n_{i_d}}(d))}$. Since G is Fréchet, there is a sequence $\langle x_n : n < \omega \rangle \subseteq G \smallsetminus \mathrm{Int}(D^d_{n_{i_d}}(d))$ which converge to e. Contradiction with the fact that $D^d_{n_{i_d}}(d)$ is a sequential neighborhood of e.

Now, since $\{D^d_{n_{i_d}}(d) : d \in D\}$ is countable, then $\{\mathrm{Int}(D^d_{n_{i_d}}(d)) : d \in D\}$ is countable neighborhood base of the identity e. □

6. cs*-character and basic orders

The main result of this section is Theorem 6.1. In the proof of that Theorem we will follow the proof of [**14**, Theorem 3.3].

DEFINITION 6.1. [**8**] A family $\mathcal{B}$ of subsets of a topological space X is called a *cs***-network at a point* $x \in X$ if for each sequence $\langle x_n : n < \omega \rangle$ in X converging to x and for each neighborhood O_x of x there is a set $N \in \mathcal{B}$ such that $x \in N \supseteq O_x$ and the set $\{n < \omega : x_n \in N\}$ is infinite.

The *cs***-character* of a topological group G is the least cardinality of cs*-network at the identity e of G.

THEOREM 6.1. *Let D be a basic order. If a topological space X has a D-weak base, then it has countable cs***-character.*

PROOF. By Definition 5.4, there is a D-weak base $\bigcup_{x\in X}\mathcal{U}_x$ such that $\mathcal{U}_x = \{U_d(x) : d \in D\}$ for each $x \in X$. First we need the following claim.

CLAIM 6.1. *For every $x \in X, d \in D$ and a sequence $\langle x_n : n < \omega\rangle$ which converge to x, $\langle x_n : n < \omega\rangle$ is eventually in $U_d(x)$.*

PROOF. Assume, for the contrary, that the claim is false. Then there are $x \in X$, $d \in D$ and a sequence $\langle x_n : n < \omega\rangle$ which converge to x, such that $\{n < \omega : x_n \notin U_d(x)\}$ is infinite. Then there is a subsequence $\langle x_{n_k} : k < \omega\rangle \subseteq \langle x_n : n < \omega\rangle$ such that $\langle x_{n_k} : k < \omega\rangle \cap U_d(x) = \emptyset$ and $\langle x_{n_k} : k < \omega\rangle$ converge to x. Then for each $y \notin \{x\} \cup \langle x_{n_k} : k < \omega\rangle$ there is $a \in D$ such that $y \in U_a(y)$ and $U_a(y) \cap \langle x_{n_k} : k < \omega\rangle = \emptyset$. Since $\bigcup_{x\in X}\mathcal{U}_x$ is a weak-base for X, and

$$U_a(y) \subseteq X \smallsetminus \langle x_{n_k} : k < \omega\rangle,$$

for some $a \in D$, then $X \smallsetminus \langle x_{n_k} : k < \omega\rangle$ is open. Contradiction with the fact that $\langle x_{n_k} : k < \omega\rangle$ converge to x. □

Since D is separable metric space, there is a countable base $\{B_n : n < \omega\}$ for D. We may and will assume that $B_{n+1} \subseteq B_n$ for $n < \omega$. For each $x \in X$, $d \in D$ and $n < \omega$ with $d \in B_n$, define $D^x_n(d) = \bigcap\{U_a(x) : a \in B_n\}$. Note that for each $b, c \in B_n$, $D^x_n(b) = D^x_n(c)$.

For each $x \in X$, set $P_x = \{D^x_n(d) : d \in D,\ n < \omega\}$. It is obvious that P_x is countable for each $x \in X$.

CLAIM 6.2. *For each $x \in X$, P_x is a $cs^\star$-network at x.*

PROOF. Let $\langle x_n : n < \omega\rangle \subseteq X$ be a sequence which converge to x and let U be a neighborhood of x. Then there is $d \in D$ such that $x \in U_d(x) \subseteq U$. It suffices to show there is some $n < \omega$ such that $\{k < \omega : x_k \in D^x_n(d)\}$ is infinite.

Assume, for the contrary, that $\{k < \omega : x_k \in D^x_n(d)\}$ is finite for each $n < \omega$. Then by induction we can get an increasing sequence $\langle n_l : l < \omega\rangle$ of natural numbers such that $x_{n_l} \notin D^x_{n_l}(d)$ and $d \in B_{n_l}$, for $l < \omega$. Then there is some $d_l \in B_{n_l}$ such that $x_{n_l} \notin U_{d_l}(x)$. Since $\{B_{n_l} : l < \omega\}$ is a decreasing base for D and $d \in B_{n_l}$ for each $l < \omega$, then $\langle d_l : l < \omega\rangle$ converges to d. Sice D is basic order, there is a bounded subsequence $\left\langle d_{l_j} : j < \omega\right\rangle$. Let $d' \in D$ be an upper boundary of $\left\langle d_{l_j} : j < \omega\right\rangle$. Since D is directed set we may assume that $d \leqslant_D d'$ for each $d \in D$. By Definition 5.4, $U_{d'}(x) \subseteq U_{d_{l_j}}(x)$ for each $j < \omega$. Thus, $\langle x_{n_l} : l < \omega\rangle \cap U_{d'}(x) = \emptyset$. Contradiction with fact that $\langle x_{n_l} : l < \omega\rangle$ converges to x. □

This finishes the proof of the theorem. □

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