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THE SYNTAX OF POLYTOPAL PROJECTIONS: FROM PERMUTOHEDRA TO ASSOCIAHEDRA

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Abstract. Tonks' projection from the permutohedron to the associahedron and the Loday–Ronco map both send permutations to planar binary trees. We give a syntactic account of these maps in the equational calculus of the free non-symmetric, non-unital operad on one binary generator. The vertex restriction of Tonks' projection is obtained by evaluating the head-insertion encoding on the reversed permutation, while the Loday–Ronco map is obtained by evaluating the decreasing encoding. We also give a local operadic proof that Tonks' vertex map is order-preserving from the weak Bruhat order to the Tamari order.

1. Introduction

The associahedron and the permutohedron are among the central objects of the theory of polytopes, and beyond. In this paper we consider the associahedron $\mathcal{K}^{n+1}$ and the permutohedron $\mathcal{P}^n$, which in our indexing convention both have dimension $n-1$. They belong to the broader family of nestohedra, equivalently hypergraph polytopes (see [**2**]). A classical connection between them is given by Tonks' [**5**] cellular quotient map $\theta\colon \mathcal{P}^n \to \mathcal{K}^{n+1}$. Its restriction to vertices yields a map ϕ from permutations to (planar) binary trees. A closely related map ψ, again from permutations to binary trees, was introduced by Loday–Ronco [**3**] in their study of dendriform algebras: $\psi\colon \mathfrak{S}_n \to \mathbb{Y}_n$

Figure 1 shows the smallest nontrivial case of Tonks' and Loday–Ronco's maps at the level of vertices. We describe both ϕ and ψ in the equational calculus $\mathcal{J}$ of Došen and Petrić [**1**], which axiomatises the free non-symmetric, non-unital operad on one binary generator. The two maps arise from indexed operadic terms: head-insertion for Tonks' map and decreasing insertion for the Loday–Ronco map. The same syntax also gives a local proof of order preservation for Tonks' map.

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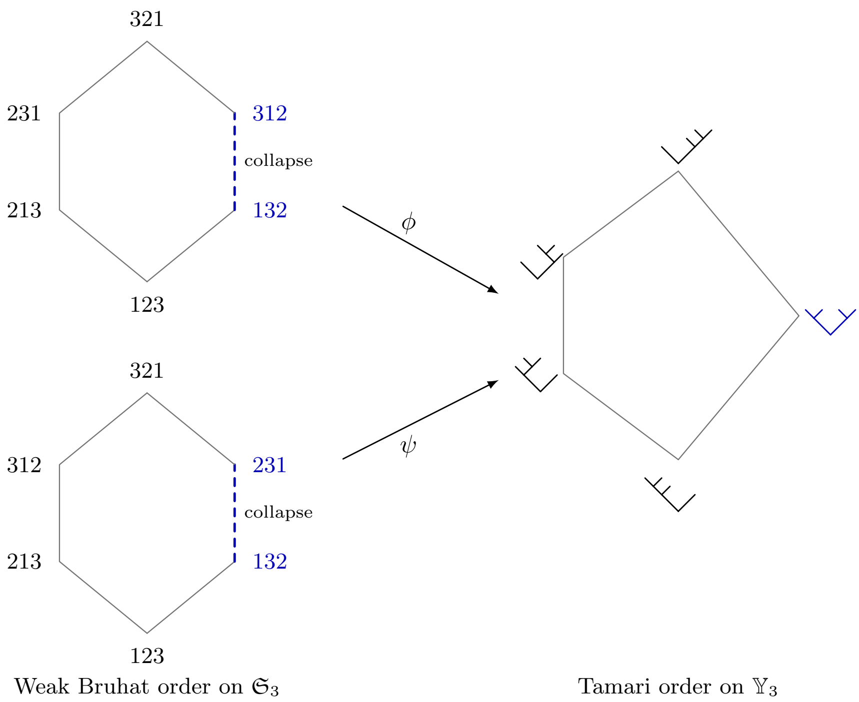


FIGURE 1. The upper hexagon is the right and the lower hexagon is the left weak Bruhat order on $\mathfrak{S}_3$. Tonks' map ϕ collapses the edge 132–312, while the Loday–Ronco map ψ collapses 132–231 to the same binary tree. In both cases, the quotient is the Tamari order on $\mathbb{Y}_3$.

2. Preliminaries

In this section we give a rigorous definition of Tonks' cellular quotient map $\theta\colon \mathcal{P}^n \to \mathcal{K}^{n+1}$, and Loday–Ronco map $\psi\colon \mathfrak{S}_n \to \mathbb{Y}_n$. To that end, we set up some conventions and recall some facts of combinatorics.

Let $[0] = \emptyset$ and $[n] = \{1, \dots, n\}$ for $n \geqslant 1$. Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$, with its elements given in one-line notation; in particular, $\mathfrak{S}_0 = \{\emptyset\}$. For $n > 0$, let $\mathbb{Y}_n$ denote the set of binary trees with n internal nodes, equivalently with $n+1$ leaves, and let $\mathbb{Y}_0 = \{*\}$, where $*$ denotes the trivial tree with a single vertex. Binary trees can be grafted: for $T_1 \in \mathbb{Y}_p$ and $T_2 \in \mathbb{Y}_q$, let $T_1 \vee T_2 \in \mathbb{Y}_{p+q+1}$ be the tree obtained by attaching T_1 and T_2 as the left and right subtrees of a new root, respectively.

Throughout the paper we use the convention that the $(n-1)$-dimensional permutohedron $\mathcal{P}^n$ has vertex set $\mathfrak{S}_n$, while the $(n-1)$-dimensional associahedron

$\mathcal{K}^{n+1}$ has vertex set $\mathbb{Y}_n$. Thus the shift in the notation for $\mathcal{K}^{n+1}$ reflects the indexing of binary trees by their number of leaves.

For a finite word $a = a_1 \cdots a_k$ of distinct integers, its *standardization* $\mathrm{std}(a) \in \mathfrak{S}_k$ is obtained by replacing the smallest letter of a by 1, the second smallest by 2, and so on, while preserving relative order. If $a = \emptyset$, we set $\mathrm{std}(\emptyset) = \emptyset \in \mathfrak{S}_0$. For a word $a = a_1 \cdots a_k$, let $w(a) = a_k \cdots a_1$ denote its *reversal*. For $1 \leqslant t \leqslant k$, write $a_{\leqslant t} = a_1 \cdots a_t$ for a prefix of the word a of length t.

Given a word $a = a_1 \cdots a_k$ of distinct integers and an integer x, let $a^{<x}$ denote the subsequence of entries of a that are strictly less than x, and let $a^{>x}$ denote the subsequence of entries of a that are strictly greater than x. These subsequences are taken in the original left-to-right order and may be empty.

For $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, an *inversion* of π is a pair (i, j) with $1 \leqslant i < j \leqslant n$ and $\pi_i > \pi_j$. We write $\ell(\pi)$ for the number of inversions of π. The (*right weak Bruhat order*) $\leqslant_B$ on $\mathfrak{S}_n$ is the partial order generated by the covering relations $\pi \lessdot_B \pi s_i$, where $s_i = (i, i+1)$ is the adjacent transposition and $\ell(\pi s_i) = \ell(\pi) + 1$. Equivalently,

$$\pi \lessdot_B \pi s_i \quad \text{if and only if} \quad \pi_i < \pi_{i+1},$$

since right multiplication by s_i swaps the entries in positions i and $i+1$. For example, in Figure 1, the weak Bruhat cover $132 \lessdot_B 312$ is collapsed by ϕ to a single tree in $\mathbb{Y}_3$. The (*left*) *weak Bruhat* order is defined analogously using left multiplication: its covers replace π with $s_i\pi$ whenever $\ell(s_i\pi) = \ell(\pi)+1$. In one-line notation this swaps i and $i+1$, and the length increases exactly when i appears to the left of $i+1$. Thus Figure 1 also shows that the left weak order cover from 132 to 231 is collapsed by the Loday–Ronco map. In the remainder of the paper, we will use "weak order" to mean "right weak order".

The set $\mathbb{Y}_n$ has cardinality equal to the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. It is partially ordered by the *Tamari order* $\leqslant_T$, generated by the local rotation $((X, Y), Z) \to (X, (Y, Z))$, where X, Y, and Z are binary trees, and the rotation may occur at any subtree.

2.1. The Loday–Ronco map.

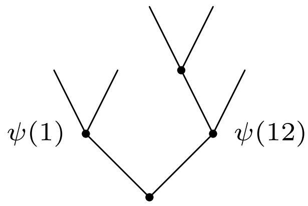
Definition 2.1. We define ψ recursively. For the empty permutation $\emptyset$, let $\psi(\emptyset) = * \in \mathbb{Y}_0$. Let $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ with $n > 0$, and let k be the unique index such that $\pi_k = n$. Define $\pi_L = \pi_1 \cdots \pi_{k-1}$, $\pi_R = \pi_{k+1} \cdots \pi_n$. Then

$$\psi(\pi) = \psi(\mathrm{std}(\pi_L)) \vee \psi(\mathrm{std}(\pi_R)).$$

Example 2.1. Let $\pi = 2413$. The maximum entry is 4, occurring in position 2. Thus, $\pi_L = 2$ and $\pi_R = 13$, so

$$\psi(2413) = \psi(\mathrm{std}(2)) \vee \psi(\mathrm{std}(13)) = \psi(1) \vee \psi(12).$$

Now $\psi(1) = \psi(\emptyset) \vee \psi(\emptyset)$, $\psi(12) = \psi(1) \vee \psi(\emptyset)$. Hence $\psi(2413)$ is the binary tree shown below, obtained by grafting the tree $\psi(1)$ to the left of the root and the tree $\psi(12)$ to the right.



2.2. Tonks' quotient map. Combinatorially, the faces of $\mathcal{P}^n$ are indexed by ordered partitions of $[n]$, while the faces of $\mathcal{K}^{n+1}$ are indexed by planar rooted trees with $n+1$ leaves. At the level of vertices, ordered partitions reduce to permutations and planar rooted trees reduce to planar binary trees. Since only this vertex-level restriction is used below, we pass to the induced map $\phi\colon \mathfrak{S}_n \to \mathbb{Y}_n$.

Definition 2.2. Let $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, and let $(\pi_i, \pi_{i+1}) = (a, b)$ be a pair of entries in adjacent positions. We say that (a, b) is *Tonks-independent* if there exists an index $j > i + 1$ such that $\min(a, b) < \pi_j < \max(a, b)$. Equivalently, (a, b) is Tonks-independent if and only if some entry to the right of the pair has value strictly between a and b.

Remark 2.1. Tonks defines independence more generally for adjacent blocks A_{k-1}, A_k in an ordered partition $(A_1, \dots, A_m)$. In that setting, the condition is that some elements of a block strictly to the right of A_k lie strictly between the minimum and the maximum of $A_{k-1} \cup A_k$. For vertices, where all blocks are singletons, this reduces to Definition 2.2.

Let $\sim$ be the equivalence relation on $\mathfrak{S}_n$ generated by the elementary moves that replace a permutation π by a permutation ρ obtained from π by swapping a Tonks-independent pair. Tonks showed [**5**] that the quotient $\mathfrak{S}_n/\!\sim$ can be canonically identified with $\mathbb{Y}_n$. Under this identification, $\phi\colon \mathfrak{S}_n \to \mathbb{Y}_n$ sends each permutation to the binary tree corresponding to its $\sim$-equivalence class. For example, in Figure 1, the permutations 132 and 312 are equivalent under $\sim$, since the pair $(1, 3)$ in 132 is Tonks-independent.

3. The free non-symmetric operad

To describe Tonks' and Loday–Ronco maps syntactically, we introduce a free non-symmetric, non-unital operad on one binary generator (for general background on operads, see [**4**]).

Definition 3.1. Let $\mathcal{L}$ be the term language generated by a single symbol $\mathbf{2}$ of arity 2, as follows:

- $\mathbf{2}$ is a term of arity $|\mathbf{2}| = 2$;
- if A and B are terms and $1 \leqslant n \leqslant |A|$, then $A \circ_n B$ is a term of arity $|A \circ_n B| = |A| + |B| - 1$.

Each term in $\mathcal{L}$ determines a binary tree whose leaves are ordered from left to right. Under this interpretation, the generator $\mathbf{2}$ corresponds to the 2-corolla, i.e., the unique binary tree with two leaves, and the term $A \circ_n B$ corresponds to the tree obtained by grafting the root of the tree represented by B onto the n-th leaf of the tree represented by A.

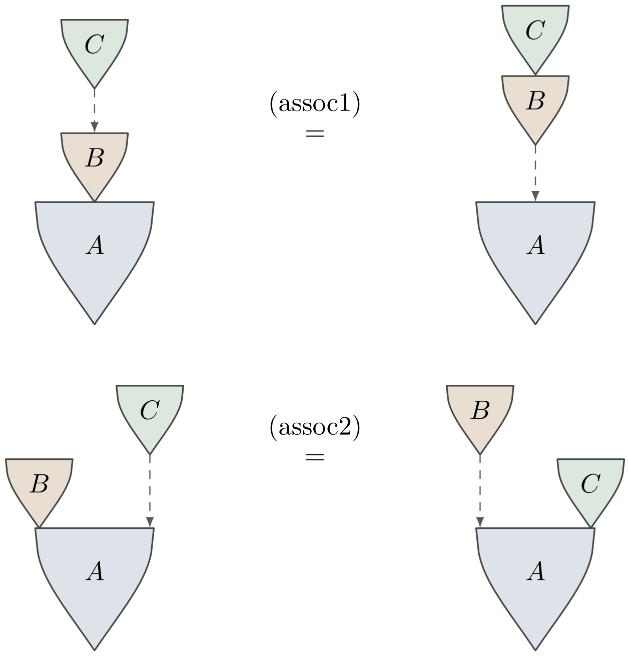
3.1. The equational calculus $\mathcal{J}$. We introduce the equational calculus $\mathcal{J}$ on the term language $\mathcal{L}$. It expresses the laws of partial composition in the free non-symmetric, non-unital operad on one binary generator. Our partial composition $\circ_n$ corresponds to $\triangleleft_n$ of Došen–Petrić [**1**]. The calculus $\mathcal{J}$ used here is their unit-free calculus $\mathcal{J}''$; in particular, we use only (assoc1) and (assoc2) and no unit axioms.

Let $=_{\mathcal{J}}$ be the minimal congruence on $\mathcal{L}$ satisfying the following two axioms:

(assoc1) $\quad (A \circ_n B) \circ_m C = A \circ_n (B \circ_{m-n+1} C) \quad$ if $n \leqslant m < n + |B|$,

(assoc2) $\quad (A \circ_n B) \circ_m C = (A \circ_{m-|B|+1} C) \circ_n B \quad$ if $n + |B| \leqslant m$.

Axiom (assoc1) corresponds to the case in which the grafting of C lands inside the subtree contributed by B, while (assoc2) corresponds to the case in which the graftings of B and C occur at disjoint leaves of A. The following picture illustrates these two situations:



A standard result from [**1**] says that two terms of $\mathcal{L}$ are equal modulo $\mathcal{J}$ if and only if they determine the same binary tree.

3.2. Indexed *l*-factors. To relate the term language $\mathcal{L}$ to permutations, we pass to an indexed version $\mathcal{L}^I$, in which the generator $\mathbf{2}$ is replaced by countably many indexed generators $\mathbf{2}^k$, for $k \in \mathbb{N}^+$. We require that no index occurs more than once in a term. For $\mathbf{A} \in \mathcal{L}^I$, let $\mathrm{ind}(\mathbf{A})$ denote the set of indices occurring in $\mathbf{A}$.

We now define the recursively generated class of l-factors used throughout the paper.

DEFINITION 3.2. A term $\mathbf{A} \in \mathcal{L}^I$ is an *l-factor* if it is obtained recursively as follows:

- $\mathbf{2}^k$ is an l-factor for every $k \in \mathbb{N}^+$;
- if $\mathbf{A}$ is an l-factor, $j \notin \mathrm{ind}(\mathbf{A})$ and $r = \big|\{x \in \mathrm{ind}(\mathbf{A}) : x < j\}\big| + 1$, then $\mathbf{A} \circ_r \mathbf{2}^j$ is an l-factor.

The insertion position r is the rank of j in the increasing order on $\mathrm{ind}(\mathbf{A}) \cup \{j\}$. In particular, $1 \leqslant r \leqslant |\mathbf{A}|$. For an l-factor $\mathbf{A}$, define its *root index* recursively by $\mathrm{root}(\mathbf{2}^i) = i$, $\mathrm{root}(\mathbf{A} \circ_r \mathbf{2}^j) = \mathrm{root}(\mathbf{A})$.

The following normalization lemma is central to what follows.

LEMMA 3.1. *Let $\mathbf{A}$ be an l-factor, $i = \mathrm{root}(\mathbf{A})$, and let*

$$L = \{x \in \mathrm{ind}(\mathbf{A}) : x < i\} \quad \textit{and} \quad R = \{x \in \mathrm{ind}(\mathbf{A}) : x > i\}.$$

Let $\mathbf{A}_L$ and $\mathbf{A}_R$ be l-factors with $\mathrm{ind}(\mathbf{A}_L) = L$ and $\mathrm{ind}(\mathbf{A}_R) = R$. Then $\mathbf{A}$ is equivalent modulo $\mathcal{I}$ to a term in one of the following forms:

- *if $L, R \neq \emptyset$, then $\mathbf{A} =_{\mathcal{I}} (\mathbf{2}^i \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L$;*
- *if $L = \emptyset$, then $\mathbf{A} =_{\mathcal{I}} \mathbf{2}^i \circ_2 \mathbf{A}_R$;*
- *if $R = \emptyset$, then $\mathbf{A} =_{\mathcal{I}} \mathbf{2}^i \circ_1 \mathbf{A}_L$;*
- *if $\mathrm{ind}(\mathbf{A}) = \{i\}$, then $\mathbf{A} = \mathbf{2}^i$.*

PROOF. We argue by induction on $N(\mathbf{A}) := \big|\mathrm{ind}(\mathbf{A})\big|$. If $N(\mathbf{A}) = 1$, then $\mathbf{A} = \mathbf{2}^i$, and the claim is immediate. Assume now that $N(\mathbf{A}) > 1$. By Definition 3.2, we may write $\mathbf{A} = \mathbf{B} \circ_r \mathbf{2}^x$, where $\mathbf{B}$ is an l-factor, $x \notin \mathrm{ind}(\mathbf{B})$, and $r = \big|\{y \in \mathrm{ind}(\mathbf{B}) : y < x\}\big| + 1$. Moreover, $\mathrm{root}(\mathbf{A}) = \mathrm{root}(\mathbf{B}) = i$. Since $i \in \mathrm{ind}(\mathbf{B})$ and $x \notin \mathrm{ind}(\mathbf{B})$, we have $x \neq i$.

By the inductive hypothesis, one of the following holds for $\mathbf{B}$:

(1) There exist l-factors $\mathbf{B}_L, \mathbf{B}_R$ such that

$$\mathrm{ind}(\mathbf{B}_L) = \{y \in \mathrm{ind}(\mathbf{B}) : y < i\}, \quad \mathrm{ind}(\mathbf{B}_R) = \{y \in \mathrm{ind}(\mathbf{B}) : y > i\},$$
$$\mathbf{B} =_{\mathcal{I}} (\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L.$$

(2) There exists an l-factor $\mathbf{B}_R$ such that

$$\mathrm{ind}(\mathbf{B}_R) = \{y \in \mathrm{ind}(\mathbf{B}) : y > i\}, \quad \mathbf{B} =_{\mathcal{I}} \mathbf{2}^i \circ_2 \mathbf{B}_R.$$

(3) There exists an l-factor $\mathbf{B}_L$ such that

$$\mathrm{ind}(\mathbf{B}_L) = \{y \in \mathrm{ind}(\mathbf{B}) : y < i\}, \quad \mathbf{B} =_{\mathcal{I}} \mathbf{2}^i \circ_1 \mathbf{B}_L.$$

(4) $\mathbf{B} = \mathbf{2}^i$.

We distinguish two cases.

Case 1: $x < i$. Then every index greater than i is also greater than x. If

$$\mathbf{B} =_{\mathcal{I}} (\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L,$$

then

$$\{y \in \mathrm{ind}(\mathbf{B}) : y < x\} = \{y \in \mathrm{ind}(\mathbf{B}_L) : y < x\},$$

so the insertion index r is exactly the one prescribed for inserting x into $\mathbf{B}_L$. In particular, $1 \leqslant r \leqslant |\mathbf{B}_L|$. Applying (assoc1) gives

$$((\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L) \circ_r \mathbf{2}^x = (\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_1 (\mathbf{B}_L \circ_r \mathbf{2}^x).$$

Set $\mathbf{A}_L := \mathbf{B}_L \circ_r \mathbf{2}^x$, $\mathbf{A}_R := \mathbf{B}_R$. Then $\mathbf{A}_L$ is an l-factor, $\mathbf{A}_R$ is an l-factor, and

$$\operatorname{ind}(\mathbf{A}_L) = \{y \in \operatorname{ind}(\mathbf{A}) : y < i\}, \quad \operatorname{ind}(\mathbf{A}_R) = \{y \in \operatorname{ind}(\mathbf{A}) : y > i\}.$$

Thus $\mathbf{A} =_{\mathcal{J}} (\mathbf{2}^i \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L$. If either $\mathbf{B} =_{\mathcal{J}} \mathbf{2}^i \circ_2 \mathbf{B}_R$ or $\mathbf{B} =_{\mathcal{J}} \mathbf{2}^i \circ_1 \mathbf{B}_L$, the cases are proven similarly, while the case where $\mathbf{B} = \mathbf{2}^i$ is trivial.

Case 2: $x > i$. Then every index smaller than i is also smaller than x. If

$$\mathbf{B} =_{\mathcal{J}} (\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L,$$

let $s = \big|\{y \in \operatorname{ind}(\mathbf{B}_R) : y < x\}\big| + 1$. Since every index in $\mathbf{B}_L$, as well as the root index i, is smaller than x, we have $r = |\mathbf{B}_L| + s$ and $1 + |\mathbf{B}_L| \leqslant r$. Hence (assoc2) gives

$$((\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L) \circ_r \mathbf{2}^x = \big((\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_{s+1} \mathbf{2}^x\big) \circ_1 \mathbf{B}_L.$$

Now $2 \leqslant s+1 < 2 + |\mathbf{B}_R|$, so (assoc1) yields $(\mathbf{2}^i \circ_2 \mathbf{B}_R) \circ_{s+1} \mathbf{2}^x = \mathbf{2}^i \circ_2 (\mathbf{B}_R \circ_s \mathbf{2}^x)$. Set $\mathbf{A}_L := \mathbf{B}_L$ and $\mathbf{A}_R := \mathbf{B}_R \circ_s \mathbf{2}^x$. Then $\mathbf{A}_L$ and $\mathbf{A}_R$ are l-factors, and

$$\operatorname{ind}(\mathbf{A}_L) = \{y \in \operatorname{ind}(\mathbf{A}) : y < i\}, \quad \operatorname{ind}(\mathbf{A}_R) = \{y \in \operatorname{ind}(\mathbf{A}) : y > i\}.$$

Thus $\mathbf{A} =_{\mathcal{J}} (\mathbf{2}^i \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L$. Again, if either $\mathbf{B} =_{\mathcal{J}} \mathbf{2}^i \circ_2 \mathbf{B}_R$ or $\mathbf{B} =_{\mathcal{J}} \mathbf{2}^i \circ_1 \mathbf{B}_L$, the proofs are analogous, with the last case $\mathbf{B} = \mathbf{2}^i$ being trivial. □

3.3. Soundness of the axioms under evaluation. Let $\mathbb{Y} = \bigcup_{n \geqslant 0} \mathbb{Y}_n$. We define an evaluation map $\varepsilon \colon \mathcal{L}^I \to \mathbb{Y}$ by forgetting the indices on the generators and interpreting each $\mathbf{2}^k$ as the 2-corolla, and each partial composition $\circ_i$ as grafting at the i-th leaf. Thus $\varepsilon(\mathbf{A})$ is the binary tree determined by the term $\mathbf{A}$. We use the same symbol $=_{\mathcal{J}}$ for the congruence on $\mathcal{L}^I$ generated by the same axiom schemes (assoc1) and (assoc2).

Proposition 3.1. *If $t =_{\mathcal{J}} t'$, then $\varepsilon(t) = \varepsilon(t')$. Equivalently, each axiom scheme* (assoc1) *and* (assoc2) *preserves the evaluated binary tree.*

Proof. By definition of ε, the term $(\mathbf{A} \circ_n \mathbf{B}) \circ_m \mathbf{C}$ is evaluated by first grafting the tree $\varepsilon(\mathbf{B})$ onto the n-th leaf of $\varepsilon(\mathbf{A})$, and then grafting $\varepsilon(\mathbf{C})$ onto the appropriate leaf of the resulting tree. In axiom (assoc1), the second grafting lands inside the subtree contributed by $\mathbf{B}$. Thus the two sides describe the same iterated grafting and evaluate to the same binary tree. In axiom (assoc2), the graftings of $\mathbf{B}$ and $\mathbf{C}$ occur at disjoint leaves of $\varepsilon(\mathbf{A})$. Therefore the order in which these two graftings are performed does not affect the resulting binary tree, and the two sides again have the same evaluation.

Since $=_{\mathcal{J}}$ is the congruence generated by these axioms, and since ε is compatible with partial composition, ε is constant on $\mathcal{J}$-equivalence classes. □

Došen and Petrić [**1**] show for the unit-free calculus $\mathcal{J}''$ that two terms are equal in the calculus *if and only if* they determine the same binary tree. In the present paper, only the soundness direction of Proposition 3.1 is needed.

4. The maps as syntactic projections

We now identify Tonks' map and the Loday–Ronco map with the two syntactic encodings defined below.

4.1. Encoding permutations. We define two encodings of permutations in the indexed language $\mathcal{L}^I$. The first is the head-insertion encoding, and the second is the decreasing encoding.

Definition 4.1. Let $a = a_1 \cdots a_n$ be a word of distinct positive integers. We define $h(a) \in \mathcal{L}^I$ recursively. If $n = 1$, then $h(a_1) = \mathbf{2}^{a_1}$. If $n > 1$, let $a' = a_1 \cdots a_{n-1}$. Then $h(a) = h(a') \circ_r \mathbf{2}^{a_n}$, where

$$r = 1 + \big|\{\, a_j \text{ occurring in } a' : a_j < a_n \,\}\big|.$$

In particular, if $\pi \in \mathfrak{S}_n$, then $h(\pi)$ is obtained by applying this construction to π viewed as a word on $[n]$. The recursive definition of h immediately gives the following fact.

Lemma 4.1. *For every word $a = a_1 \cdots a_n$ of distinct positive integers, the term $h(a)$ is an l-factor. Moreover,* $\mathrm{ind}(h(a)) = \{a_1, \ldots, a_n\}$, $\mathrm{root}(h(a)) = a_1$.

We now describe the second encoding.

Definition 4.2. Let $a = a_1 \cdots a_r$ be a nonempty word of distinct positive integers, and let $\kappa(a) = \kappa_1 \cdots \kappa_r$ be the decreasing rearrangement of the letters of a, so that $\kappa_1 > \kappa_2 > \cdots > \kappa_r$. For x occurring in a, let

$$u_a(x) = \big|\{\, a_j \text{ to the left of } x \text{ in } a : a_j > x \,\}\big|.$$

Define $f(a) \in \mathcal{L}^I$ as follows. If $r = 1$, set $f(a) = \mathbf{2}^{\kappa_1}$. If $r > 1$, set

$$f(a) = (\cdots((\mathbf{2}^{\kappa_1} \circ_{m_{\kappa_2}} \mathbf{2}^{\kappa_2}) \circ_{m_{\kappa_3}} \mathbf{2}^{\kappa_3}) \cdots) \circ_{m_{\kappa_r}} \mathbf{2}^{\kappa_r},$$

where, for $q = 2, \ldots, r$, we have $m_{\kappa_q} = u_a(\kappa_q) + 1$. This is well defined: when $\mathbf{2}^{\kappa_q}$ is inserted, the term already contains the generators whose indices are entries of a greater than κ_q, namely $\kappa_1, \ldots, \kappa_{q-1}$. Hence the current arity is q, while

$$0 \leqslant u_a(\kappa_q) \leqslant q - 1.$$

Therefore $1 \leqslant m_{\kappa_q} \leqslant q$, so the insertion index is valid.

In particular, if $\pi \in \mathfrak{S}_n$, then $\kappa(\pi) = n(n-1) \cdots 1$, and the preceding definition gives

$$f(\pi) = (\cdots((\mathbf{2}^n \circ_{m_{n-1}} \mathbf{2}^{n-1}) \circ_{m_{n-2}} \mathbf{2}^{n-2}) \cdots) \circ_{m_1} \mathbf{2}^1,$$

where $m_x = u_\pi(x) + 1$ $(1 \leqslant x < n)$.

Example 4.1. We illustrate the two encodings on the permutation $\pi = 312 \in \mathfrak{S}_3$. For the head-insertion encoding h, we first compute

$$h(312) = h(31) \circ_2 \mathbf{2}^2 = (\mathbf{2}^3 \circ_1 \mathbf{2}^1) \circ_2 \mathbf{2}^2.$$

To compute the decreasing encoding f, we process the values in the order $3, 2, 1$. Since $u_\pi(2) = 1$ and $u_\pi(1) = 1$ we have $m_2 = u_\pi(2) + 1 = 2$, $m_1 = u_\pi(1) + 1 = 2$. Hence $f(312) = (\mathbf{2}^3 \circ_2 \mathbf{2}^2) \circ_2 \mathbf{2}^1$.

4.2. Tonks' projection via *h*. We now define a recursive insertion map on permutations and show that it agrees with Tonks' vertex map ϕ. Let $t \in \mathbb{Y}_1$ be the 2-corolla. For $T \in \mathbb{Y}_p$ and $1 \leqslant i \leqslant p+1$, let $T \triangleleft_i t \in \mathbb{Y}_{p+1}$ denote the binary tree obtained by grafting the 2-corolla t at the i-th leaf of T.

DEFINITION 4.3. For each $n \geqslant 0$, define $\widehat{\phi}\colon \mathfrak{S}_n \to \mathbb{Y}_n$ recursively. Set $\widehat{\phi}(\emptyset) = *$, where $* \in \mathbb{Y}_0$ is the unique tree with a single vertex. $\widehat{\phi}(\pi) = t$, if $\pi \in \mathfrak{S}_1$, and $\widehat{\phi}(\pi) = \widehat{\phi}(\mathrm{std}(\pi_2 \cdots \pi_n)) \triangleleft_{\pi_1} t$ for $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ with $n > 1$.

This is well defined, since $\mathrm{std}(\pi_2 \cdots \pi_n) \in \mathfrak{S}_{n-1}$, the tree $\widehat{\phi}(\mathrm{std}(\pi_2 \cdots \pi_n))$ lies in $\mathbb{Y}_{n-1}$, and therefore has exactly n leaves, while $\pi_1 \in [n]$.

At the level of vertices, Tonks' construction may be described (as in [**5**]) by the following evaluation procedure for a product $x_1 x_2 \cdots x_{n+1}$. An ordered partition of $[n]$ records how the n binary compositions are grouped into stages. On vertices, where the partition consists of singletons, this procedure is determined by a permutation $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, and the compositions are carried out successively in the order $\pi_1, \ldots, \pi_n$. In particular, the first step composes the variables x_{π_1} and x_{π_1+1}. On the tree side, this corresponds to grafting a 2-corolla at the π_1-st leaf. After this first composition, the remaining $n-1$ steps act on a product of length n, so their relative order is encoded by the standardized tail $\mathrm{std}(\pi_2 \cdots \pi_n)$. We record this in the following proposition.

PROPOSITION 4.1. *For every permutation* $\pi \in \mathfrak{S}_n$*,* $\widehat{\phi}(\pi) = \phi(\pi)$*.*

Although $\widehat{\phi}$ is introduced only as an auxiliary map, it plays an important role in what follows. It leads to an alternative recursive description of Tonks' map, which we denote by φ. This formulation is better suited to the operadic arguments developed later, and in particular it will allow us to use the normalization lemma for l-factors proved earlier in the paper in the proof of our main result on Tonks' map.

DEFINITION 4.4. For each $n \geqslant 0$, define $\varphi\colon \mathfrak{S}_n \to \mathbb{Y}_n$ recursively by $\varphi(\emptyset) = *$, where $* \in \mathbb{Y}_0$ is the unique tree with a single vertex. For $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ $(n > 0)$, set $\varphi(\pi) = \varphi(\mathrm{std}(\pi^{<\pi_n})) \vee \varphi(\mathrm{std}(\pi^{>\pi_n}))$, where $\vee$ denotes grafting at a new root. If one of the two subsequences is empty, the corresponding tree is $* \in \mathbb{Y}_0$.

THEOREM 4.1. *For every permutation* $\pi \in \mathfrak{S}_n$*,* $\widehat{\phi}(\pi) = \varphi(\pi)$*.*

PROOF. We argue by induction on n. If $n = 0$, then both $\widehat{\phi}(\emptyset)$ and $\varphi(\emptyset)$ are equal to the unique tree $* \in \mathbb{Y}_0$. If $n = 1$, then both maps send the unique permutation in $\mathfrak{S}_1$ to the unique tree $t \in \mathbb{Y}_1$. Assume now that $n > 1$, and let $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$. Set $\rho = \mathrm{std}(\pi_2 \cdots \pi_n) \in \mathfrak{S}_{n-1}$. By Definition 4.3, $\widehat{\phi}(\pi) = \widehat{\phi}(\rho) \triangleleft_{\pi_1} t$. By the inductive hypothesis, $\widehat{\phi}(\rho) = \varphi(\rho)$. Thus $\widehat{\phi}(\pi) = \varphi(\rho) \triangleleft_{\pi_1} t$. Since $\rho \in \mathfrak{S}_{n-1}$, the recursive definition of φ yields $\varphi(\rho) = \varphi(\mathrm{std}(\rho^{<\rho_{n-1}})) \vee \varphi(\mathrm{std}(\rho^{>\rho_{n-1}}))$. We distinguish two cases.

Case 1: $\pi_1 < \pi_n$. Then $\rho_{n-1} = \pi_n - 1$. Let

$$L = \varphi(\mathrm{std}(\rho^{<\rho_{n-1}})), \quad R = \varphi(\mathrm{std}(\rho^{>\rho_{n-1}})).$$

Thus $\varphi(\rho) = L \vee R$. The entries of ρ smaller than ρ_{n-1} are exactly the standardized images of the entries of π that are smaller than π_n, except for the first entry π_1 itself. Hence, if $\alpha = \mathrm{std}(\pi^{<\pi_n})$, then $\alpha_1 = \pi_1$ and $\mathrm{std}(\alpha_2 \cdots \alpha_{\pi_n - 1}) = \mathrm{std}(\rho^{<\rho_{n-1}})$. By Definition 4.3 and the inductive hypothesis, $L \triangleleft_{\pi_1} t = \widehat{\phi}(\alpha) = \varphi(\alpha)$. Moreover, the entries of ρ greater than ρ_{n-1} are exactly the standardized images of the entries of π greater than π_n, so $R = \varphi(\mathrm{std}(\pi^{>\pi_n}))$. Since $L \in \mathbb{Y}_{\pi_n - 2}$, the tree L has exactly $\pi_n - 1$ leaves. As $\pi_1 < \pi_n$, the insertion at the global leaf π_1 of $L \vee R$ therefore occurs inside the left subtree. Therefore $(L \vee R) \triangleleft_{\pi_1} t = (L \triangleleft_{\pi_1} t) \vee R$. Hence

$$\widehat{\phi}(\pi) = \varphi(\mathrm{std}(\pi^{<\pi_n})) \vee \varphi(\mathrm{std}(\pi^{>\pi_n})) = \varphi(\pi).$$

Case 2: $\pi_1 > \pi_n$. Then $\rho_{n-1} = \pi_n$. Let $L = \varphi(\mathrm{std}(\rho^{<\rho_{n-1}}))$, $R = \varphi(\mathrm{std}(\rho^{>\rho_{n-1}}))$. In this case $L = \varphi(\mathrm{std}(\pi^{<\pi_n}))$. Let $\beta = \mathrm{std}(\pi^{>\pi_n})$. Then the first entry of β is $\pi_1 - \pi_n$, and the standardized tail of β is $\mathrm{std}(\rho^{>\rho_{n-1}})$. Hence, by Definition 4.3 and the inductive hypothesis, $R \triangleleft_{\pi_1 - \pi_n} t = \widehat{\phi}(\beta) = \varphi(\beta)$. Since L has exactly π_n leaves, insertion at the global leaf π_1 of $L \vee R$ occurs in the right subtree, at leaf $\pi_1 - \pi_n$. Therefore $(L \vee R) \triangleleft_{\pi_1} t = L \vee (R \triangleleft_{\pi_1 - \pi_n} t)$. It follows that

$$\widehat{\phi}(\pi) = \varphi(\mathrm{std}(\pi^{<\pi_n})) \vee \varphi(\mathrm{std}(\pi^{>\pi_n})) = \varphi(\pi). \qquad \square$$

We now specialize Lemma 3.1 to the head-insertion encoding. The general lemma gives a root decomposition for every l-factor. For terms of the form $h(a)$, the additional point is that the indices of left and right terms in the decomposition inherit the order of the corresponding subwords of a.

LEMMA 4.2. *Let $a = a_1 \cdots a_n$ be a nonempty word of distinct positive integers, let $i = a_1$, and let $a^{<i}$ and $a^{>i}$ denote the subwords of a consisting of the entries less than i and greater than i, respectively. Then $h(a)$ is equivalent modulo $\mathcal{I}$ to a term in one of the following forms:*

- *if $a^{<i}, a^{>i} \neq \emptyset$, then $h(a) =_{\mathcal{I}} (\mathbf{2}^i \circ_2 h(a^{>i})) \circ_1 h(a^{<i})$;*
- *if $a^{<i} = \emptyset$, then $h(a) =_{\mathcal{I}} \mathbf{2}^i \circ_2 h(a^{>i})$;*
- *if $a^{>i} = \emptyset$, then $h(a) =_{\mathcal{I}} \mathbf{2}^i \circ_1 h(a^{<i})$;*
- *if $a = i$, then $h(a) = \mathbf{2}^i$.*

PROOF. By Lemma 4.1, $h(a)$ is an l-factor and $i = \mathrm{root}(h(a)) = a_1$. Lemma 3.1 gives a root decomposition of $h(a)$ as $h(a) =_{\mathcal{I}} (\mathbf{2}^i \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L$, with the evident one-sided variants, where

$$\mathrm{ind}(\mathbf{A}_L) = \{x \in \mathrm{ind}(h(a)) : x < i\}, \quad \mathrm{ind}(\mathbf{A}_R) = \{x \in \mathrm{ind}(h(a)) : x > i\}.$$

It remains only to check that, for the particular term $h(a)$, the indices appearing in $\mathbf{A}_L$ and $\mathbf{A}_R$ occur in the order inherited from the word a. We argue by induction on the length of prefixes of a. For $1 \leqslant t \leqslant n$, recall that we have defined $a_{\leqslant t} = a_1 \cdots a_t$.

We claim that $h(a_{\leqslant t})$ satisfies the decomposition stated in the lemma, with $a^{<i}_{\leqslant t}$ and $a^{>i}_{\leqslant t}$ in place of $a^{<i}$ and $a^{>i}$, respectively. The case $t = 1$ is immediate. Assume the claim for $a_{\leqslant t}$, and write $a_{\leqslant t+1} = a_{\leqslant t} x$.

If $x < i$, then $a^{<i}_{\leqslant t+1} = a^{<i}_{\leqslant t} x$, and $a^{>i}_{\leqslant t+1} = a^{>i}_{\leqslant t}$. The insertion position of x in $h(a_{\leqslant t+1})$ is $1 + \big|\{y \in a_{\leqslant t} : y < x\}\big|$. Since every entry of $a^{>i}_{\leqslant t}$ is greater than x, this is exactly the insertion position of x in $h(a^{<i}_{\leqslant t})$. Hence the right-hand term in the root decomposition is unchanged, while the left-hand term becomes $h(a^{<i}_{\leqslant t}x) = h(a^{<i}_{\leqslant t+1})$.

The case $x > i$ is analogous: the left-hand term is unchanged, and the insertion position of x in the right-hand term is precisely the position prescribed in the construction of $h(a^{>i}_{\leqslant t}x) = h(a^{>i}_{\leqslant t+1})$.

Taking $t = n$, the induction gives $h(a) =_{\mathcal{I}} (\mathbf{2}^i \circ_2 h(a^{>i})) \circ_1 h(a^{<i})$ when both subwords are nonempty, and the corresponding one-sided congruences when one of them is empty. These are precisely the four cases stated in the lemma. □

We now show that the grafting map φ is realized by the syntactic encoding h applied to the reversed permutation. For the rest of the paper, for every nonempty word a of distinct positive integers, write $g(a) = \varepsilon(h(w(a)))$.

Theorem 4.2. *For every $n \geqslant 1$ and every permutation $\pi \in \mathfrak{S}_n$, $\varphi(\pi) = g(\pi)$.*

Proof. We argue by induction on n. If $n = 1$, then $\pi = 1$, $w(\pi) = 1$, and $h(w(\pi)) = \mathbf{2}^1$. Hence $g(\pi)$ is the 2-corolla, which is exactly $\varphi(1)$.

Assume now that the statement holds for all permutations of size $< n$, and let $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$. Set $a = w(\pi) = \pi_n\pi_{n-1}\cdots\pi_1$. The first letter of a is π_n. Moreover, the subword of a consisting of letters smaller than π_n is $w(\pi^{<\pi_n})$, and the subword consisting of letters greater than π_n is $w(\pi^{>\pi_n})$.

Suppose first that both $\pi^{<\pi_n}$ and $\pi^{>\pi_n}$ are nonempty. By Lemma 4.2,

$$h(w(\pi)) =_{\mathcal{I}} \left(\mathbf{2}^{\pi_n} \circ_2 h(w(\pi^{>\pi_n}))\right) \circ_1 h(w(\pi^{<\pi_n})).$$

Applying ε and using Proposition 3.1, we obtain $g(\pi) = g(\pi^{<\pi_n}) \vee g(\pi^{>\pi_n})$. Standardization preserves the relative order of a word. Since the insertion positions in h depend only on this relative order, $h(w(a))$ and $h(w(\mathrm{std}(a)))$ have the same term shape and differ only in their indices; after applying ε, these indices are forgotten. Therefore $g(\pi^{<\pi_n}) = g(\mathrm{std}(\pi^{<\pi_n}))$ and $g(\pi^{>\pi_n}) = g(\mathrm{std}(\pi^{>\pi_n}))$. By the inductive hypothesis,

$$g(\mathrm{std}(\pi^{<\pi_n})) = \varphi(\mathrm{std}(\pi^{<\pi_n})), \quad g(\mathrm{std}(\pi^{>\pi_n})) = \varphi(\mathrm{std}(\pi^{>\pi_n})).$$

Hence $g(\pi) = \varphi(\mathrm{std}(\pi^{<\pi_n})) \vee \varphi(\mathrm{std}(\pi^{>\pi_n})) = \varphi(\pi)$.

If $\pi^{<\pi_n} = \emptyset$, then Lemma 4.2 gives $h(w(\pi)) =_{\mathcal{I}} \mathbf{2}^{\pi_n} \circ_2 h(w(\pi^{>\pi_n}))$. After applying ε, this says $g(\pi) = * \vee g(\pi^{>\pi_n})$. By standardization and the inductive hypothesis, $g(\pi^{>\pi_n}) = g(\mathrm{std}(\pi^{>\pi_n})) = \varphi(\mathrm{std}(\pi^{>\pi_n}))$. Thus

$$g(\pi) = * \vee \varphi(\mathrm{std}(\pi^{>\pi_n})) = \varphi(\pi).$$

The case $\pi^{>\pi_n} = \emptyset$ is analogous. Then $h(w(\pi)) =_{\mathcal{I}} \mathbf{2}^{\pi_n} \circ_1 h(w(\pi^{<\pi_n}))$, and hence $g(\pi) = \varphi(\mathrm{std}(\pi^{<\pi_n})) \vee * = \varphi(\pi)$. □

Corollary 4.1. *For every $n \geqslant 1$ and every permutation $\pi \in \mathfrak{S}_n$, $\phi(\pi) = g(\pi)$.*

Proof. By Proposition 4.1 and Theorem 4.1, we have $\phi(\pi) = \varphi(\pi)$. The conclusion now follows from Theorem 4.2. □

4.3. The Loday–Ronco map via f. We now relate the Loday–Ronco map ψ to the decreasing encoding f. Recall that ψ is defined recursively by splitting a permutation π at its maximal entry n. Thus, if viewed as a word on $[n]$ we have that $\pi = \pi_L\, n\, \pi_R$, and then $\psi(\pi) = \psi(\mathrm{std}(\pi_L)) \vee \psi(\mathrm{std}(\pi_R))$. To identify ψ with the evaluation of f, we first prove a normalization lemma for the decreasing encoding, analogous to Lemma 3.1.

LEMMA 4.3. *Let $a = a_1 \cdots a_r$ be a nonempty word of distinct positive integers, and let $\kappa(a) = \kappa_1 \cdots \kappa_r$ be the decreasing rearrangement of the letters of a and write $a = a_L\, \kappa_1\, a_R$, where a_L and a_R are the subwords to the left and to the right of κ_1, respectively. Let $\mathbf{B}_L =_{\mathcal{J}} f(a_L)$ and $\mathbf{B}_R =_{\mathcal{J}} f(a_R)$. Then, $f(a)$ is equivalent modulo $\mathcal{J}$ to a term in one of the following forms:*

- *if $a_L, a_R \neq \emptyset$, then $f(a) =_{\mathcal{J}} (\mathbf{2}^{\kappa_1} \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L$;*
- *if $a_L = \emptyset$, then $f(a) =_{\mathcal{J}} \mathbf{2}^{\kappa_1} \circ_2 \mathbf{B}_R$;*
- *if $a_R = \emptyset$, then $f(a) =_{\mathcal{J}} \mathbf{2}^{\kappa_1} \circ_1 \mathbf{B}_L$;*
- *if $a = \kappa_1$, then $f(a) = \mathbf{2}^{\kappa_1}$.*

PROOF. We argue by induction on $r = \mathrm{len}(a)$. If $r = 1$, then $a = \kappa_1$ and $f(a) = \mathbf{2}^{\kappa_1}$, so the claim is immediate. Assume now that $r > 1$. Let a' be the word obtained from a by deleting its smallest letter κ_r. Then $f(a) = f(a') \circ_{u_a(\kappa_r)+1} \mathbf{2}^{\kappa_r}$. The maximum of a' is still κ_1. Write $a' = a'_L\, \kappa_1\, a'_R$. Since each axiom scheme preserves arity, congruent terms have the same arity.

We distinguish two cases.

Case 1: κ_r occurs in a_L. Then $a'_R = a_R$, while a'_L is obtained from a_L by deleting κ_r. Moreover, $u_a(\kappa_r) = u_{a_L}(\kappa_r)$, because every letter of a that is greater than κ_r and lies to its left is already contained in a_L.

If both a'_L and a'_R are nonempty, then by the inductive hypothesis there exist terms $\mathbf{A}_L, \mathbf{A}_R \in \mathcal{L}^I$ such that $\mathbf{A}_L =_{\mathcal{J}} f(a'_L)$, $\mathbf{A}_R =_{\mathcal{J}} f(a_R)$, and

$$f(a') =_{\mathcal{J}} (\mathbf{2}^{\kappa_1} \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L.$$

Since $\mathbf{A}_L =_{\mathcal{J}} f(a'_L)$, we have $|\mathbf{A}_L| = |f(a'_L)| = \mathrm{len}(a'_L) + 1 = \mathrm{len}(a_L)$. Hence $1 \leqslant u_{a_L}(\kappa_r) + 1 \leqslant |\mathbf{A}_L|$, so (assoc1) applies:

$$((\mathbf{2}^{\kappa_1} \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L) \circ_{u_a(\kappa_r)+1} \mathbf{2}^{\kappa_r} = (\mathbf{2}^{\kappa_1} \circ_2 \mathbf{A}_R) \circ_1 \left(\mathbf{A}_L \circ_{u_{a_L}(\kappa_r)+1} \mathbf{2}^{\kappa_r}\right).$$

Set $\mathbf{B}_L := \mathbf{A}_L \circ_{u_{a_L}(\kappa_r)+1} \mathbf{2}^{\kappa_r}$, $\mathbf{B}_R := \mathbf{A}_R$. Since $\mathbf{A}_L =_{\mathcal{J}} f(a'_L)$, it follows from the definition of $f(a_L)$ that $\mathbf{B}_L =_{\mathcal{J}} f(a_L)$. Therefore $f(a) =_{\mathcal{J}} (\mathbf{2}^{\kappa_1} \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L$.

If either $a'_L = \emptyset$ and $a'_R \neq \emptyset$, or $a'_R = \emptyset$ and $a'_L \neq \emptyset$, we proceed analogously, and if $a' = \kappa_1$ this case is trivial.

Case 2: κ_r occurs in a_R. Then $a'_L = a_L$, while a'_R is obtained from a_R by deleting κ_r. Put $s = u_{a_R}(\kappa_r) + 1$. Since every letter of a_L, as well as κ_1 itself, lies to the left of κ_r and is greater than κ_r, we have $u_a(\kappa_r) = \mathrm{len}(a_L) + u_{a_R}(\kappa_r) + 1$. Again, we prove only the main case, since the other cases are similar.

If both a'_L and a'_R are nonempty, then by the inductive hypothesis there exist terms $\mathbf{A}_L, \mathbf{A}_R \in \mathcal{L}^I$ such that $\mathbf{A}_L =_{\mathcal{J}} f(a_L)$, $\mathbf{A}_R =_{\mathcal{J}} f(a'_R)$, and

$$f(a') =_{\mathcal{J}} (\mathbf{2}^{\kappa_1} \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L.$$

Since $\mathbf{A}_L =_\jmath f(a_L)$, we have $|\mathbf{A}_L| = |f(a_L)| = \mathrm{len}(a_L) + 1$. Therefore $u_a(\kappa_r) + 1 = |\mathbf{A}_L| + s$. Applying (assoc2) gives

$$((\mathbf{2}^{\kappa_1} \circ_2 \mathbf{A}_R) \circ_1 \mathbf{A}_L) \circ_{u_a(\kappa_r)+1} \mathbf{2}^{\kappa_r} = \big((\mathbf{2}^{\kappa_1} \circ_2 \mathbf{A}_R) \circ_{s+1} \mathbf{2}^{\kappa_r}\big) \circ_1 \mathbf{A}_L.$$

Now $\mathbf{A}_R =_\jmath f(a'_R)$, so $|\mathbf{A}_R| = |f(a'_R)| = \mathrm{len}(a'_R) + 1 = \mathrm{len}(a_R)$. Hence $2 \leqslant s + 1 < 2 + |\mathbf{A}_R|$, and (assoc1) yields

$$(\mathbf{2}^{\kappa_1} \circ_2 \mathbf{A}_R) \circ_{s+1} \mathbf{2}^{\kappa_r} = \mathbf{2}^{\kappa_1} \circ_2 \big(\mathbf{A}_R \circ_s \mathbf{2}^{\kappa_r}\big).$$

Set $\mathbf{B}_L := \mathbf{A}_L$, $\mathbf{B}_R := \mathbf{A}_R \circ_s \mathbf{2}^{\kappa_r}$. Since $\mathbf{A}_R =_\jmath f(a'_R)$, it follows from the definition of $f(a_R)$ that $\mathbf{B}_R =_\jmath f(a_R)$. Therefore $f(a) =_\jmath (\mathbf{2}^{\kappa_1} \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L$. □

Theorem 4.3. *For every $n \geqslant 1$ and every permutation $\pi \in \mathfrak{S}_n$, $\psi(\pi) = \varepsilon(f(\pi))$.*

Proof. We argue by induction on n. If $n = 1$, then $f(1) = \mathbf{2}^1$, and $\varepsilon(f(1))$ is the 2-corolla. This is exactly $\psi(1)$. Assume now that the statement holds for all permutations of size $< n$, and let $\pi = \pi_L\, n\, \pi_R \in \mathfrak{S}_n$. Apply Lemma 4.3 to the word $a = \pi$. Since $\kappa_1 = \max(\pi) = n$, the root generator is $\mathbf{2}^n$, while the left and right branches are determined by the subwords π_L and π_R.

If both π_L and π_R are nonempty, Lemma 4.3 gives $f(\pi) =_\jmath (\mathbf{2}^n \circ_2 \mathbf{B}_R) \circ_1 \mathbf{B}_L$, where $\mathbf{B}_L =_\jmath f(\pi_L)$, $\mathbf{B}_R =_\jmath f(\pi_R)$. Applying ε and using Proposition 3.1, we obtain $\varepsilon(f(\pi)) = \varepsilon(f(\pi_L)) \vee \varepsilon(f(\pi_R))$. Standardization preserves the relative order of a word. Since the insertion positions in the decreasing encoding f depend only on this relative order, the terms $f(a)$ and $f(\mathrm{std}(a))$ have the same shape and differ only in their indices for every nonempty word a of distinct positive integers. Applying ε forgets those indices, and hence $\varepsilon(f(a)) = \varepsilon(f(\mathrm{std}(a)))$. Applying this to $a = \pi_L$ and $a = \pi_R$, we get $\varepsilon(f(\pi_L)) = \varepsilon(f(\mathrm{std}(\pi_L)))$, $\varepsilon(f(\pi_R)) = \varepsilon(f(\mathrm{std}(\pi_R)))$. By the inductive hypothesis, $\varepsilon(f(\mathrm{std}(\pi_L))) = \psi(\mathrm{std}(\pi_L))$, $\varepsilon(f(\mathrm{std}(\pi_R))) = \psi(\mathrm{std}(\pi_R))$. Therefore $\varepsilon(f(\pi)) = \psi(\mathrm{std}(\pi_L)) \vee \psi(\mathrm{std}(\pi_R)) = \psi(\pi)$.

If $\pi_L = \emptyset$, then Lemma 4.3 gives $f(\pi) =_\jmath \mathbf{2}^n \circ_2 \mathbf{B}_R$, $\mathbf{B}_R =_\jmath f(\pi_R)$. Hence

$$\varepsilon(f(\pi)) = \psi(\emptyset) \vee \varepsilon(f(\pi_R)).$$

Since π_R is nonempty in this case, standardization and the inductive hypothesis give $\varepsilon(f(\pi_R)) = \varepsilon(f(\mathrm{std}(\pi_R))) = \psi(\mathrm{std}(\pi_R))$. Thus

$$\varepsilon(f(\pi)) = \psi(\emptyset) \vee \psi(\mathrm{std}(\pi_R)) = \psi(\pi).$$

The case $\pi_R = \emptyset$ is analogous. In that case $f(\pi) =_\jmath \mathbf{2}^n \circ_1 \mathbf{B}_L$, $\mathbf{B}_L =_\jmath f(\pi_L)$, and hence $\varepsilon(f(\pi)) = \psi(\mathrm{std}(\pi_L)) \vee \psi(\emptyset) = \psi(\pi)$. □

Corollary 4.2. *For every $n \geqslant 0$ and every permutation $\pi \in \mathfrak{S}_n$, $\phi(\pi) = \psi(\pi^{-1})$.*

Proof. We argue by induction on n. If $n = 0$, then $\pi = \emptyset$, and both $\phi(\emptyset)$ and $\psi(\emptyset^{-1}) = \psi(\emptyset)$ are equal to $* \in \mathbb{Y}_0$. If $n = 1$, the claim is immediate. Assume now that the statement holds for all permutations of size $< n$, and let $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$. Set $\sigma = \pi^{-1} \in \mathfrak{S}_n$. Since $\sigma_{\pi_n} = n$, the recursive definition of ψ gives

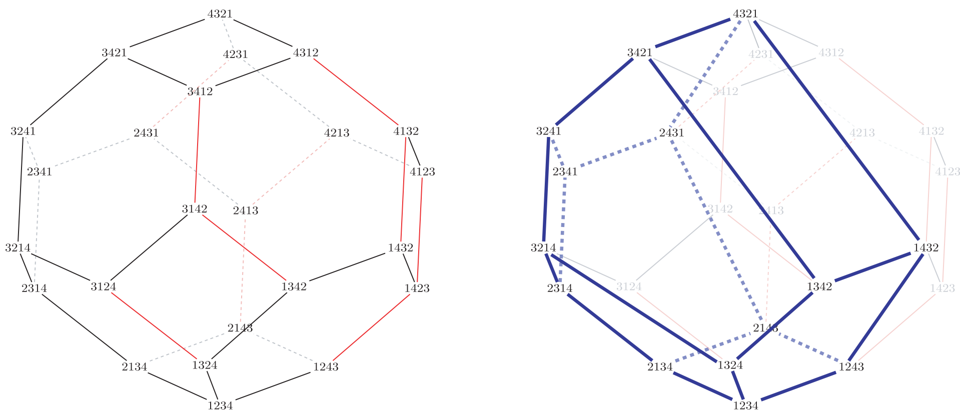
$$\psi(\sigma) = \psi(\mathrm{std}(\sigma_L)) \vee \psi(\mathrm{std}(\sigma_R)),$$

where $\sigma_L = \sigma_1 \cdots \sigma_{\pi_n - 1}$, $\sigma_R = \sigma_{\pi_n + 1} \cdots \sigma_n$. Now let $\alpha = \pi^{<\pi_n}$, $\beta = \pi^{>\pi_n}$. The word σ_L consists of the positions, in π, of the letters of α, listed in increasing order of their values. Standardizing σ_L replaces these original positions by their relative positions among the entries of α. This is exactly the inverse of the standardized word $\operatorname{std}(\alpha)$. Therefore $\operatorname{std}(\sigma_L) = \big(\operatorname{std}(\alpha)\big)^{-1}$. The same argument applied to the entries greater than π_n gives $\operatorname{std}(\sigma_R) = \big(\operatorname{std}(\beta)\big)^{-1}$. By the inductive hypothesis, $\psi(\operatorname{std}(\sigma_L)) = \phi\big(\operatorname{std}(\sigma_L)^{-1}\big) = \phi(\operatorname{std}(\alpha))$, and likewise $\psi(\operatorname{std}(\sigma_R)) = \phi(\operatorname{std}(\beta))$. Hence $\psi(\pi^{-1}) = \phi(\operatorname{std}(\pi^{<\pi_n})) \vee \phi(\operatorname{std}(\pi^{>\pi_n}))$. By the recursive definition of φ and the equality $\phi = \varphi$, we also have $\phi(\pi) = \phi(\operatorname{std}(\pi^{<\pi_n})) \vee \phi(\operatorname{std}(\pi^{>\pi_n}))$. Therefore $\psi(\pi^{-1}) = \phi(\pi)$, as required. □

5. Order preservation

We show that Tonks' map $\phi\colon \mathfrak{S}_n \to \mathbb{Y}_n$ is order-preserving from the weak Bruhat order to the Tamari order. Since the weak Bruhat order is the reflexive transitive closure of its covering relation, it is enough to consider a cover $\pi = \alpha\, u\, v\, \beta$, $\rho = \alpha\, v\, u\, \beta$, $u < v$, so that $\pi \lessdot_B \rho$. Set $\tau = w(\beta)$, $\gamma = w(\alpha)$. Then $w(\pi) = \tau\, v\, u\, \gamma$, $w(\rho) = \tau\, u\, v\, \gamma$. The local difference between $w(\pi)$ and $w(\rho)$ lies in the adjacent block vu versus uv, after the common prefix τ. Since the insertion indices in h depend only on the letters already processed, the prefix τ determines the local comparison.

The case $n = 4$ is displayed in Figure 2. The red intervals in the weak Bruhat order are precisely those collapsed by ϕ, and the resulting quotient is the 1-skeleton of the associahedron $\mathcal{K}^5$, with vertices indexed by the elements of $\mathbb{Y}_4$ and ordered by the Tamari order.



(A) Weak Bruhat order on $\mathfrak{S}_4$.

(B) Induced Tamari order on the quotient $\mathfrak{S}_4/\sim$.

FIGURE 2. Weak Bruhat order on $\mathfrak{S}_4$ and the corresponding 1-skeleton of the associahedron $\mathcal{K}^5$.

LEMMA 5.1. *Let τ be a word of distinct integers, and let $u < v$ be integers not occurring in τ. Let $n = k(v; \tau)$ and $m = k(u; \tau v)$ where*

$$k(a; \sigma) = 1 + \big|\{x \text{ occurring in } \sigma : x < a\}\big|$$

is the insertion index used in the definition of h. Then

$$n - m = \big|\{x \text{ occurring in } \tau : u < x < v\}\big|.$$

LEMMA 5.2. *Let $\pi = \alpha\, u\, v\, \beta$, $\rho = \alpha\, v\, u\, \beta$, $u < v$, and set $\tau = w(\beta)$. If (u, v) is Tonks-independent in π, then $h(w(\pi)) =_\jmath h(w(\rho))$. Consequently, $\phi(\pi) = \phi(\rho)$.*

PROOF. Write $w(\pi) = \tau\, v\, u\, \gamma$, $w(\rho) = \tau\, u\, v\, \gamma$, where $\gamma = w(\alpha)$. Since (u, v) is Tonks-independent in π, there is an entry of β strictly between u and v. Hence β is nonempty, and therefore $\tau = w(\beta)$ is nonempty. Let $\mathbf{A} = h(\tau)$.

Set $n = k(v; \tau)$, $m = k(u; \tau v)$. Since $u < v$, we also have

$$k(u; \tau) = k(u; \tau v) = m.$$

Since $u < v$, inserting u before v increases the insertion index of v by 1, so

$$k(v; \tau u) = k(v; \tau) + 1 = n + 1.$$

Therefore $h(\tau v u) = (\mathbf{A} \circ_n \mathbf{2}^v) \circ_m \mathbf{2}^u$, $h(\tau u v) = (\mathbf{A} \circ_m \mathbf{2}^u) \circ_{n+1} \mathbf{2}^v$. Since (u, v) is Tonks-independent in π, there exists $x \in \beta$ with $u < x < v$. Equivalently, there exists $x \in \tau$ with $u < x < v$. By Lemma 5.1, we have $m < n$. Hence $m + 2 \leqslant n + 1$, because $|\mathbf{2}^u| = 2$. We may therefore apply (assoc2) to the term $h(\tau u v)$, obtaining

$$(\mathbf{A} \circ_m \mathbf{2}^u) \circ_{n+1} \mathbf{2}^v = (\mathbf{A} \circ_n \mathbf{2}^v) \circ_m \mathbf{2}^u.$$

Thus $h(\tau u v) =_\jmath h(\tau v u)$. The remaining letters, namely the letters of γ, are then inserted in the same order on both sides. At each step the two partial words contain the same set of letters, so the next insertion has the same index in both terms. Since $=_\jmath$ is a congruence, it follows that $h(w(\rho)) =_\jmath h(w(\pi))$. Applying ε and Corollary 4.1, we conclude $\phi(\rho) = g(\rho) = g(\pi) = \phi(\pi)$. □

LEMMA 5.3. *Let $\pi = \alpha\, u\, v\, \beta$, $\rho = \alpha\, v\, u\, \beta$, $u < v$, and set $\tau = w(\beta)$. If (u, v) is not Tonks-independent in π, then $\phi(\pi) <_T \phi(\rho)$.*

PROOF. Write $w(\pi) = \tau\, v\, u\, \gamma$, $w(\rho) = \tau\, u\, v\, \gamma$, where $\gamma = w(\alpha)$.

We first treat the case $\tau \neq \emptyset$. Let $\mathbf{A} = h(\tau)$ and set $n = k(v; \tau)$, $m = k(u; \tau v)$. Since (u, v) is not Tonks-independent in π, there is no $x \in \beta$ with $u < x < v$, hence no $x \in \tau$ with $u < x < v$. By Lemma 5.1, we have $m = n$.

Since $u < v$, we also have $k(u; \tau) = k(u; \tau v) = m = n$. Therefore

$$h(\tau v u) = (\mathbf{A} \circ_n \mathbf{2}^v) \circ_n \mathbf{2}^u.$$

By (assoc1), $(\mathbf{A} \circ_n \mathbf{2}^v) \circ_n \mathbf{2}^u = \mathbf{A} \circ_n (\mathbf{2}^v \circ_1 \mathbf{2}^u)$. Moreover, since $u < v$,

$$k(v; \tau u) = k(v; \tau) + 1 = n + 1.$$

Hence $h(\tau u v) = (\mathbf{A} \circ_n \mathbf{2}^u) \circ_{n+1} \mathbf{2}^v$. Applying (assoc1) again, we obtain

$$(\mathbf{A} \circ_n \mathbf{2}^u) \circ_{n+1} \mathbf{2}^v = \mathbf{A} \circ_n (\mathbf{2}^u \circ_2 \mathbf{2}^v).$$

Now $\varepsilon(\mathbf{2}^v \circ_1 \mathbf{2}^u)$ and $\varepsilon(\mathbf{2}^u \circ_2 \mathbf{2}^v)$ differ by exactly one right rotation, so

$$\varepsilon(h(\tau vu)) <_T \varepsilon(h(\tau uv)).$$

If $\tau = \emptyset$, then $h(vu) = \mathbf{2}^v \circ_1 \mathbf{2}^u$, $h(uv) = \mathbf{2}^u \circ_2 \mathbf{2}^v$, and therefore again

$$\varepsilon(h(vu)) <_T \varepsilon(h(uv)).$$

It remains to pass from the prefixes τvu and τuv to the words $w(\pi)$ and $w(\rho)$. Let the letters of γ be inserted one after another. At each step the two partial words have the same set of letters, and the insertion index for the next letter depends only on this set. Hence the next insertion has the same index on both sides.

Tamari order is closed under substitution into a fixed leaf context: if $S <_T T$, then replacing the same leaf of any binary tree by S and by T again gives a strict Tamari inequality. The rotation witnessing $S <_T T$ occurs inside the substituted subtree, while the surrounding context is unchanged. Applying this observation successively to the common insertions of the letters of γ, we obtain $g(\pi) <_T g(\rho)$. Using Corollary 4.1, we conclude $\phi(\pi) <_T \phi(\rho)$. □

THEOREM 5.1. *The map $\phi\colon \mathfrak{S}_n \to \mathbb{Y}_n$ is order-preserving from the weak Bruhat order on $\mathfrak{S}_n$ to the Tamari order on $\mathbb{Y}_n$.*

PROOF. It is enough to consider a cover $\pi \lessdot_B \rho$, so that $\pi = \alpha\, u\, v\, \beta$, $\rho = \alpha\, v\, u\, \beta$, $u < v$. If (u, v) is Tonks-independent, then Lemma 5.2 gives $\phi(\pi) = \phi(\rho)$. If (u, v) is not Tonks-independent, then Lemma 5.3 gives $\phi(\pi) <_T \phi(\rho)$. In either case, $\phi(\pi) \leqslant_T \phi(\rho)$. Since the weak Bruhat order is the reflexive transitive closure of its covering relation, it follows that ϕ is order-preserving. □

6. Conclusion

We have given a syntactic account of the two maps from permutations to binary trees considered in this paper. Tonks' map ϕ is obtained by evaluating the head-insertion encoding h on the reversed permutation, while the Loday–Ronco map ψ is obtained by evaluating the decreasing encoding f on the permutation itself. Both constructions therefore reside in the same equational calculus $\mathcal{I}$ for the free non-symmetric, non-unital operad on one binary generator.

The comparison also explains the relation between the two recursive descriptions. For Tonks' map, the relevant decomposition is governed by the first letter of the reversed word, equivalently by the last letter of the original permutation. For the Loday–Ronco map, the decreasing encoding is normalized by splitting at the maximal entry.

In particular, the order-preservation property of Tonks' map admits a local operadic explanation. In the independent case, the identification of adjacent swaps is governed by instances of (assoc2). In the dependent case, the strict Tamari increase is exposed by the corresponding local rewriting analysis, which yields the relevant rotation on the tree side.

It remains to ask whether similar normal-form methods apply to other nestohedra where canonical projections and quotient constructions also occur.

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