PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 118 (132) (2025), 97–119 DOI: https://doi.org/10.2298/PIM2532107S

ON TURÁN AND BERNSTEIN-TYPE INTEGRAL MEAN ESTIMATES FOR POLAR DERIVATIVE OF A COMPLEX POLYNOMIAL

Nirmal Kumar Singha and Barchand Chanam

Abstract. If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for $r \geqslant 1$, Aziz [J. Approx. Theory, **55** (1988), 232–239] proved

$$\left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{1/r} \max_{|z|=1} |p'(z)| \geqslant n \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{1/r},$$

whereas Devi et al. [Note Mat., 41 (2021), 19–29] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \leqslant 1$, then for $r > 0$,

$$k^n n \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{1/r} \leqslant \left\{ \int_0^{2\pi} |e^{i\theta} + k^n|^r d\theta \right\}^{1/r} \left\{ n \max_{|z|=1} |p(z)| - \max_{|z|=1} |p'(z)| \right\},$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, where $q(z) = z^n \overline{p(1/\bar{z})}$.

We do not only obtain improved extensions of the above inequalities into polar derivative by involving the leading coefficient and the constant term of the polynomial, but also give integral analogues of inequalities on polar derivative recently proved by Mir and Dar [Filomat, **36**(16) (2022), 5631–5640].

1. Introduction

Empirical findings and inquiries across diverse realms of science and engineering frequently undergo a transformation into mathematical representations and mathematical frameworks. Nearly every realm within mathematics, ranging from algebraic number theory and algebraic geometry to applied analysis, Fourier analysis, numerical analysis, and computer science, possesses its own collection of principles and concepts stemming from the examination of polynomial functions.

Historically, the question relating to polynomials, for example, the solution of polynomial equations and the approximation by polynomials, give rise to some of the most important problems of the day. The well-known Russian mathematician

2020 *Mathematics Subject Classification:* 30A10, 30C10, 30C15.

Key words and phrases: polar derivative, Turán-type inequalities, Bernstein-type inequalities, integral inequalities.

Communicated by Gradimir Milovanović.

Chebyshev (1821–1894) studied some properties of polynomials with the least deviation from a given continuous function and introduced the concept of the best approximation in mathematical analysis. Various interesting inequalities concerning the estimate of the sup-norm of the derivative as an upper bound in terms of the sup-norm of the polynomial itself known as Bernstein-type inequality plays a key role in the literature for proving the inverse theorems in approximation theory (see Borwein and Erdélyi [**6**], Ivanov [**19**], Lorentz [**22**], Telyakovskii [**43**]) and, of course, have their own intrinsic interests. The first result in this area was connected with some investigation of the well-known Russian chemist Mendeleev [**27**]. In fact, Mendeleev's problem was to determine $\max_{-1\leqslant x\leqslant 1}|p'(x)|$, where $p(x)$ is a quadratic polynomial of real variable x with real coefficients and satisfying $-1 \leqslant p(x) \leqslant 1$ for $-1 \leqslant x \leqslant 1$. He himself was able to prove that if $p(x)$ is a quadratic polynomial and $|p(x)| \leqslant 1$ on $[-1, 1]$, then $|p'(x)| \leqslant 4$ on the same interval. Markov [**26**] generalized this result for a polynomial of degree n in the real line. In fact, he proved that if $p(x)$ is an algebraic polynomial of degree at most n with real coefficients, then $\max_{-1\leqslant x\leqslant 1}|p'(x)| \leqslant n^2 \max_{-1\leqslant x\leqslant 1}|p(x)|$.

After about twenty years, Bernstein [**5**] needed the analogue of Markov's theorem for the unit disc in the complex plane instead of the interval $[-1, 1]$ in order to the prove inverse theorem of approximation (see Borwein and Erdélyi [**6**, p. 241]) to estimate how well a polynomial of a certain degree approximates a given continuous function in terms of its derivatives and Lipschitz constants. This leads to the famous well-known result known as Bernstein's inequality which states that if $t \in \tau_n$ (the set of all real trigonometric polynomials of degree at most n), then for $K := [0, 2\pi)$,

$$\max_{\theta\in K}|t^{(m)}(\theta)| \leqslant n^m \max_{\theta\in K}|t(\theta)|. \tag{1.1}$$

The above inequality remains true for all $t \in \tau_n^c$ (the set of all complex trigonometric polynomials of degree at most n), which implies, as a particular case, the following algebraic polynomial version of Bernstein's inequality on the unit disk.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}|p'(z)| \leqslant n \max_{|z|=1}|p(z)|. \tag{1.2}$$

Equality holds in (1.2) *if and only if $p(z)$ has all its zeros at the origin.*

It is really of interest both in theoretical and practical aspects that continuous functions are approximated by polynomials. One such approach of approximation is made through the applications of Bernstein's inequality, particularly the trigonometric version, and in this regard, we have the following interesting result (Theorem 1.2) [**6**, p. 241, Part (a) of E.18] which approximates m times differentiable real-valued function on a half-closed interval $[0, 2\pi)$ by trigonometric polynomials. For the sake of convenience of the readers, we state the above result more precisely.

Let Lip_α, $\alpha \in (0, 1]$, denote the family of all real-valued functions g defined on K satisfying

$$\sup\left\{\frac{|g(x)-g(y)|}{|x-y|^\alpha} : x \neq y \in K\right\} < \infty.$$

If $C(K)$ denotes the set of all continuous functions on K, then for $f \in C(K)$, let

$$E_n(f) := \inf\Big\{\sup_{\theta\in K} |t - f| : t \in \tau_n\Big\}.$$

THEOREM 1.2 (Direct theorem). *Suppose f is m times differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0, 1]$. Then there is a constant C depending only on f so that $E_n(f) \leqslant Cn^{-(m+\alpha)}$, $n = 1, 2, \dots$.*

On the other hand, the converse (inverse) of Theorem 1.2 is essentially of interest and is stated below.

THEOREM 1.3 (Inverse theorem). *Suppose m is a non-negative integer, $\alpha \in (0, 1)$, and $f \in C(K)$. Suppose there is a constant $C > 0$ depending only on f such that $E_n(f) \leqslant Cn^{-(m+\alpha)}$, $n = 1, 2, \dots$. Then f is m times continuously differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$.*

The proof of Theorem 1.3 is done by the application of the well-known result due to Bernstein (inequality (1.1)) given in [**6**].

From the above discussion, it is worth to note that Bernstein and Markov-type inequalities play a significant role in approximation theory. Direct and inverse theorems of approximation and related matters may be found in many books on approximation theory, including Cheney [**8**], Lorentz [**22**], and DeVore and Lorentz [**11**].

Moreover, inequality (1.2) shows how fast a polynomial of degree at most n can change, and is of interest both in mathematics, especially in approximation theory, and in the application areas such as physical systems. Various analogues of these inequalities are known in which the underlying intervals, the sup-norms, and the families of polynomials, are replaced by more general sets, norms, and families of functions, respectively. One such generalization is replacing the sup-norm by a factor involving integral mean.

Let $p(z)$ be a polynomial of degree n over the set of complex numbers and $q(z)$ represent the polynomial $z^n\overline{p(1/\bar{z})}$. For each real number $r > 0$, we define the integral mean of $p(z)$ on the unit circle $|z| = 1$ by $\|p\|_r = \big\{\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\big\}^{1/r}$.

If we take the limit as $r \to \infty$ in the above equality and make use of the well-known fact from analysis [**38, 42**] that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{1/r} = \max_{|z|=1} |p(z)|,$$

we can suitably denote $\|p\|_\infty = \max_{|z|=1} |p(z)|$.

Similarly, we can define

$$\|p\|_0 = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|p(e^{i\theta})| d\theta\right\},$$

and it follows easily that $\lim_{r\to 0^+} \|p\|_r = \|p\|_0$. It would be of further interest that by taking the limit as $r \to 0^+$ the stated results on integral mean inequalities holding for $r > 0$, hold for $r = 0$ as well.

Inequality (1.2) can be obtained by letting $r \to \infty$ in the inequality

$$\|p'\|_r \leqslant n\|p\|_r, \quad r > 0. \tag{1.3}$$

Inequality (1.3) was proved by Zygmund [**46**] for $r \geqslant 1$, and by Arestov [**2**] for $0 < r < 1$.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then inequalities (1.2) and (1.3) can be respectively improved as

$$\|p'\|_\infty \leqslant \frac{n}{2}\|p\|_\infty, \tag{1.4}$$

$$\|p'\|_r \leqslant \frac{n}{\|1+z\|_r}\|p\|_r, \quad r > 0. \tag{1.5}$$

Inequality (1.4) was conjectured by Erdös and later verified by Lax [**21**], whereas inequality (1.5) was proved by de-Bruijn [**9**] for $r \geqslant 1$, and by Rahman and Schmeisser [**36**] for $0 < r < 1$.

On the other hand, in 1939 (see [**44**]), Turán obtained a lower bound for the maximum of $|p'(z)|$ on $|z| = 1$, by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then

$$\|p'\|_\infty \geqslant \frac{n}{2}\|p\|_\infty. \tag{1.6}$$

As a generalization Govil [**15**] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then

$$\|p'\|_\infty \geqslant \frac{n}{1+k^n}\|p\|_\infty. \tag{1.7}$$

Whereas, for the class of polynomials not vanishing in $|z| < k$, $k \leqslant 1$, the precise upper bound estimate for the maximum of $|p'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable. For quite some time, it was believed that if $p(z)$ has no zero in $|z| < k$, $k \leqslant 1$, then the inequality that generalizes (1.4) should be $\|p'\|_\infty \leqslant \frac{n}{1+k^n}\|p\|_\infty$, until E. B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

Thus, the approximation does not seem to be known in general, and this problem is still open. However, some special cases in this direction have been considered by many people where some partial extensions of (1.4) are established. In 1980, it was again Govil [**14**], who generalized (1.4) with an extra condition by proving that if $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leqslant 1$, then

$$\|p'\|_\infty \leqslant \frac{n}{1+k^n}\|p\|_\infty, \tag{1.8}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

For the first time in 1984, Malik [**23**] extended inequality (1.6) proved by Turán [**44**] into integral mean version and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $r > 0$ one has $\|1 + z\|_r\|p'\|_\infty \geqslant n\|p\|_r$. The result is sharp and equality holds for $p(z) = (z + 1)^n$.

In 1988, Aziz [**3**] obtained the integral mean extension of inequality (1.7) and proved

THEOREM 1.4. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for $r \geqslant 1$,*

$$\|1 + k^n z\|_r \|p'\|_\infty \geqslant n\|p\|_r. \tag{1.9}$$

The result is sharp and equality holds for $p(z) = \alpha z^n + \beta k^n$, $|\alpha| = |\beta|$.

On the other hand, Devi et al. [**10**] established the integral analogue of inequality (1.8) and proved

THEOREM 1.5. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leqslant 1$, then for $r > 0$,*

$$k^n n\|p\|_r \leqslant \|z + k^n\|_r \{n\|p\|_\infty - \|p'\|_\infty\}, \tag{1.10}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Before proceeding to some other results, let us introduce the concept of the polar derivative involved. For a polynomial $p(z)$ of degree n, we define

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z),$$

the polar derivative of $p(z)$ with respect to the point α (see [**25**] and [**13**, Chap. 6]). The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ in the sense that $\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z)$, uniformly with respect to z for $|z| \leqslant R$, $R > 0$.

Various results of majorization on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et al. [**29**], Marden [**25**], and Rahman and Schmeisser [**35**], where some approaches to obtaining polynomial inequalities are developed on applying the methods and results of the geometric function theory.

In 1998, Aziz and Rather [**4**] established the polar derivative generalization of (1.7) by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every complex number α with $|\alpha| \geqslant k$,

$$\|D_\alpha p\|_\infty \geqslant n\Big(\frac{|\alpha| - k}{1 + k^n}\Big)\|p\|_\infty, \tag{1.11}$$

whereas the corresponding polar derivative analogue of (1.8) was recently given by Mir and Breaz [**30**]. They proved that if $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leqslant 1$, then for every complex number α with $|\alpha| \geqslant 1$,

$$\|D_\alpha p\|_\infty \leqslant n\Big(\frac{|\alpha| + k^n}{1 + k^n}\Big)\|p\|_\infty, \tag{1.12}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Recently, Mir and Dar [**31**] generalized inequalities (1.11) and (1.12) by using some parameters and improved their bounds by incorporating the leading coefficient and the constant term of the polynomial and proved the following results.

THEOREM 1.6. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every complex number α with $|\alpha| \geqslant k$ and a fixed* *complex number* β *with* $|\beta|<1$,

$$\max_{|z|=1}|D_\alpha p(z)+\beta mn|\geqslant n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\Big(1+\frac{L}{n}\Big)\Big(1+\frac{M}{2}\Big)\Big(\max_{|z|=1}|p(z)|+|\beta|m\Big), \tag{1.13}$$

where

$$L=\frac{k^n|a_n|-|a_0|-|\beta|m}{k^n|a_n|+|a_0|+|\beta|m},\quad M=\frac{(k^n|a_n|-|a_0|-|\beta|m)(k-1)}{k^n|a_n|+|a_0|k+|\beta|mk}$$

and $m=\min_{|z|=k}|p(z)|$.

Theorem 1.7. *If* $p(z)=\sum_{j=0}^n a_jz^j$ *is a polynomial of degree* n *having no zero in* $|z|<k$, $k\leqslant 1$, *then for every complex number* α *with* $|\alpha|\geqslant 1$ *and a fixed complex number* β *with* $|\beta|<1$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)|\leqslant \frac{n}{1+k^n}&\Big\{|\alpha|(1+k^n)-k^n(|\alpha|-1)\Big(1+\frac{X}{n}\Big)\Big(1+\frac{Y}{2}\Big)\Big\}\\ &\times\max_{|z|=1}|p(z)|-\frac{n(|\alpha|-1)}{1+k^n}\Big(1+\frac{X}{n}\Big)\Big(1+\frac{Y}{2}\Big)|\beta|m,\end{aligned} \tag{1.14}$$

where

$$X=\frac{|a_0|-|a_n|k^n-|\beta|m}{|a_0|+|a_n|k^n+|\beta|m},\quad Y=\frac{(|a_0|-|a_n|k^n-|\beta|m)(1-k)}{k|a_0|+|a_n|k^n+|\beta|m}$$

and $m=\min_{|z|=k}|p(z)|$, *provided* $|p'(z)|$ *and* $|q'(z)|$ *attain their maxima at the same point on* $|z|=1$.

For about 20 years, there has been no generalization and extension of inequality (1.11) until due to Aziz and Rather [**4**] one appeared in 1998 into integral analogue. In an attempt to obtain the integral versions of Turán-type inequalities of the class of polynomials with zeros lying in $|z|\leqslant k$, $k\geqslant 1$, it was only in 2017 that Rather and Bhat [**37**] gave the extension of inequality (1.11) in integral mean setting by applying the Gauss–Lucas theorem and a well-known property of subordination [**18**]. In fact, they proved

Theorem 1.8. *If* $p(z)=\sum_{j=0}^n a_jz^j$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k$, $k\geqslant 1$, *then for every complex number* α *with* $|\alpha|\geqslant k$ *and for each* $r>0$,

$$n(|\alpha|-k)\|p\|_r\leqslant C_r\|D_\alpha p\|_\infty, \tag{1.15}$$

where $C_r=\big\{\int_0^{2\pi}|1+k^ne^{i\theta}|^rd\theta\big\}^{1/r}$.

It is worth mentioning that the polynomial inequalities in sup-norm on the unit circle in the complex plane are a special case of polynomial inequalities in an integral setting. For example, if we let $r\to\infty$ in (1.15), and noting that $C_r\to 1+k^n$, we get inequality (1.11) due to Aziz and Rather [**4**] in polar derivative. Moreover, if we divide both sides of inequality (1.11) by $|\alpha|$ and let $|\alpha|\to\infty$, it reduces to the ordinary derivative inequality (1.7) proved by Govil [**15**]. Thus the direction of extending inequalities concerning ordinary or polar derivatives into integral versions, has better higher orders or meanings in the sense of the above discussion.

But in the current paper, we have proved the integral version of inequality (1.13) of Theorem 1.6, which further provides the improved integral extension of inequality (1.11) in a simpler approach than done by Rather and Bhat [**37**], entirely based on some existing inequalities on polynomials.

On the other hand, for the last more than 40 years, it has been of interest to obtain the integral setting of inequality (1.8) due to Govil [**14**]; Devi et al. [**10**] gave the integral extension of inequality (1.8). In seeking improvement and extension into polar derivative of inequality (1.10) due to Devi et al. [**10**], we also prove an improved integral extension of it in polar derivative which is the integral analogue of inequality (1.14) of Theorem 1.7 and it further provides an improved and generalized integral version of inequality (1.12) due to Mir and Breaz [**30**].

The improvement and generalization of the inequalities concerning complex polynomials is a widely studied topic, and for more information in this direction, we refer to the recently published papers [**1**, **28**, **39**, **40**], etc.

The present paper is organized as follows. In Section 2, we present the main results in integral norm along with remarks and corollaries. In Section 3, we present some auxiliary results necessary in proving the main results. Then the proofs of our main results are given in Section 4. Finally, Section 5 contains the conclusion.

2. Main results

In this paper, as mentioned above, we are able to prove the following integral extensions of both Theorems 1.6 and 1.7 of Mir and Dar [**31**], which further give improved and generalized integral versions in polar derivative of Theorems 1.4 and 1.5 respectively.

We begin by proving the result for the class of polynomials having all its zeros in $|z| \leqslant k$, $k \geqslant 1$ which is the integral mean analogue of inequality (1.13) of Theorem 1.6. More precisely, we prove

THEOREM 2.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every complex number α, β with $|\alpha| \geqslant k$, $|\beta| < 1$, for each γ, $0 \leqslant \gamma < 2\pi$ and $r > 0$,*

$$\|D_\alpha\{p(e^{i\theta}) + \beta m\}\|_r \geqslant \frac{|\alpha| - k}{2} A E_r \|p(e^{i\theta}) + \beta m\|_r, \tag{2.1}$$

where

$$A = \Big\{n + \frac{k^n|a_n| - |a_0| - |\beta|m}{k^n|a_n| + |a_0| + |\beta|m}\Big\}, \quad E_r = \frac{\Big\{\int_0^{2\pi} \Big|\frac{|a_n|k^{n+1} + |a_0| + |\beta|m}{|a_n|k^n + k|a_0| + |\beta|mk} + e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi} \Big|1 + e^{i\gamma}k^n\Big|^r d\gamma\Big\}^{1/r}}$$

and $m = \min_{|z|=k} |p(z)|$.

REMARK 2.1. Letting $r \to \infty$ on both sides of (2.1), we obtain

$$\max_{|z|=1} |D_\alpha\{p(z) + \beta m\}| \geqslant \frac{|\alpha| - k}{2} A \frac{\Big\{\frac{|a_n|k^{n+1} + |a_0| + |\beta|m}{|a_n|k^n + k|a_0| + |\beta|mk} + 1\Big\}}{(1 + k^n)} \max_{|z|=1} |p(z) + \beta m|. \tag{2.2}$$

Suppose z_0 on $|z|=1$ be such that $\max_{|z|=1}|p(z)| = |p(z_0)|$. Then, in particular,

$$\max_{|z|=1}|p(z)+\beta m| \geqslant |p(z_0)+\beta m|. \tag{2.3}$$

Now to appropriate choice of the argument of β, we can get

$$|p(z_0)+\beta m| = |p(z_0)| + |\beta| m. \tag{2.4}$$

Using (2.4) in (2.3), we have

$$\max_{|z|=1}|p(z)+\beta m| \geqslant |p(z_0)| + |\beta| m. \tag{2.5}$$

On combining (2.2) and (2.5), we have

$$\max_{|z|=1}|D_\alpha\{p(z)+\beta m\}| \geqslant \frac{|\alpha|-k}{2} A \frac{\left\{\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}+1\right\}}{(1+k^n)}\Big(\max_{|z|=1}|p(z)| + |\beta|m\Big),$$

which on simplification gives inequality (1.13) of Theorem 1.6.

If we divide both sides of (2.1) of Theorem 2.1 by $|\alpha|$ and let $|\alpha| \to \infty$, we get an interesting result, which gives the integral mean analogue of an inequality in ordinary derivative due to Mir and Dar [**31**, Corollary 1] which further improves inequality (1.9).

Corollary 2.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every complex number β with $|\beta| < 1$, for each γ, $0 \leqslant \gamma < 2\pi$ and $r > 0$,*

$$\|p'\|_r \geqslant \frac{AE_r}{2}\|p(e^{i\theta})+\beta m\|_r, \tag{2.6}$$

where A, E_r and m are as defined in Theorem 2.1.

Remark 2.2. Further, taking the limit as $r \to \infty$ on both sides of (2.6) and following similar arguments of Remark 2.1 for routing inequality (2.5), namely, $\max_{|z|=1}|p(z)+\beta m| \geqslant \max_{|z|=1}|p(z)| + |\beta| m$ for some suitable β, we get the result which is due to Mir and Dar [**31**, Corollary 1] and it sharpens inequality (1.7).

Putting $\beta = 0$ in (2.1) of Theorem 2.1, we further get the following interesting result, which is an integral extension of a result due to Mir and Breaz [**30**, Theorem 3].

Corollary 2.2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every complex number α with $|\alpha| \geqslant k$, for each γ, $0 \leqslant \gamma < 2\pi$ and $r > 0$,*

$$\|D_\alpha p\|_r \geqslant \frac{|\alpha|-k}{2}\Big\{n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\Big\} \frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}}\|p\|_r. \tag{2.7}$$

Remark 2.3. Taking the limit as $r \to \infty$ on both sides of (2.7), we get the result due to Mir and Breaz [**30**, Theorem 3] and it sharpens inequality (1.11).

If we divide both sides of (2.7) of Corollary 2.2 by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following interesting result, which gives the integral mean analogue of a result due to Mir and Breaz [**30**, Remark 7] which also improves inequality (1.9).

Corollary 2.3. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for each γ, $0 \leqslant \gamma < 2\pi$ and $r > 0$,*

$$\|p'\|_r \geqslant \frac{1}{2}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\} \frac{\Big\{\int_0^{2\pi} \Big|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|} + e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi} |1 + e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}} \|p\|_r. \tag{2.8}$$

Remark 2.4. Further, taking the limit as $r \to \infty$ on both sides of (2.8), we get the result due to Mir and Breaz [**30**, Remark 7] and it sharpens inequality (1.7).

Next, we prove the following result, which deals with a subclass of polynomials having no zero in $|z| < k$, $k \leqslant 1$ which is the integral mean analogue of inequality (1.14) of Theorem 1.7. More precisely, we prove

Theorem 2.2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k$, $k \leqslant 1$, then for every complex number α, β with $|\alpha| \geqslant 1$, $|\beta| < 1$, for each γ, $0 \leqslant \gamma < 2\pi$ and $r > 0$,*

$$\frac{k^n(|\alpha| - 1)}{2} A' E'_r \Big\|p(e^{i\theta}) + \frac{\bar{\beta} m e^{in\theta}}{k^n}\Big\|_r \leqslant n|\alpha| \|p\|_\infty - \|D_\alpha p\|_\infty, \tag{2.9}$$

where

$$A' = \Big\{n + \frac{|a_0| - |a_n|k^n - |\beta|m}{|a_0| + |a_n|k^n + |\beta|m}\Big\}, \quad E'_r = \frac{\Big\{\int_0^{2\pi} \Big|\frac{|a_0|+|a_n|k^{n+1}+|\beta|mk}{|a_0|k+k^n|a_n|+|\beta|m} + e^{i\gamma}|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi} |k^n + e^{i\gamma}|^r d\gamma\Big\}^{1/r}}$$

and $m = \min_{|z|=k} |p(z)|$, provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Remark 2.5. Letting $r \to \infty$ on both sides of (2.9), we obtain

$$\begin{aligned} \frac{k^n(|\alpha| - 1)}{2} A' \frac{\Big\{\frac{|a_0|+|a_n|k^{n+1}+|\beta|mk}{|a_0|k+k^n|a_n|+|\beta|m} + 1\Big\}}{1 + k^n} & \max_{|z|=1} \Big|p(z) + \frac{\bar{\beta} m z^n}{k^n}\Big| \\ & \leqslant n|\alpha| \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)|. \end{aligned} \tag{2.10}$$

Suppose z_0 on $|z| = 1$ be such that $\max_{|z|=1}|p(z)| = |p(z_0)|$. Then, in particular,

$$\Big|p(z_0) + \frac{\bar{\beta} m z_0^n}{k^n}\Big| \leqslant \max_{|z|=1} \Big|p(z) + \frac{\bar{\beta} m z^n}{k^n}\Big|. \tag{2.11}$$

Now to appropriate the choice of the argument of β, we can get

$$\Big|p(z_0) + \frac{\bar{\beta} m z_0^n}{k^n}\Big| = |p(z_0)| + |\beta| \frac{m}{k^n}. \tag{2.12}$$

Using (2.12) in (2.11), we have

$$|p(z_0)| + |\beta|\frac{m}{k^n} \leqslant \max_{|z|=1}\Big|p(z) + \frac{\bar{\beta}mz^n}{k^n}\Big|. \tag{2.13}$$

On combining (2.10) and (2.13), we have

$$\frac{k^n(|\alpha|-1)}{2}A'\frac{\left\{\frac{|a_0|+|a_n|k^{n+1}+|\beta|mk}{|a_0|k+k^n|a_n|+|\beta|m}+1\right\}}{1+k^n}\Big(\max_{|z|=1}|p(z)| + |\beta|\frac{m}{k^n}\Big)$$
$$\leqslant n|\alpha|\max_{|z|=1}|p(z)| - \max_{|z|=1}|D_\alpha p(z)|,$$

which on simplification gives inequality (1.14) of Theorem 1.7.

If we divide both sides of (2.9) of Theorem 2.2 by $|\alpha|$ and let $|\alpha| \to \infty$, we get an interesting result, which gives the integral mean analogue of an inequality in ordinary derivative due to Mir and Dar [**31**, Corollary 2] which further improves inequality (1.10).

Corollary 2.4. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k, k \leqslant 1$, then for every complex number β with $|\beta| < 1$, for each γ, $0 \leqslant \gamma < 2\pi$ and $r > 0$,*

$$\frac{k^n A' E'_r}{2}\Big\|p(e^{i\theta}) + \frac{\bar{\beta}me^{in\theta}}{k^n}\Big\|_r \leqslant n\|p\|_\infty - \|p'\|_\infty, \tag{2.14}$$

where A', E'_r and m are as defined in Theorem 2.2, provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Remark 2.6. Further, taking the limit as $r \to \infty$ on both sides of (2.14) and following similar arguments of Remark 2.5 for routing inequality (2.13), namely, $\max_{|z|=1}|p(z)| + |\beta|\frac{m}{k^n} \leqslant \max_{|z|=1}\big|p(z) + \frac{\bar{\beta}mz^n}{k^n}\big|$ for some suitable β, we get the result which is due to Mir and Dar [**31**, Corollary 2] and it sharpens inequality (1.8).

3. Lemmas

For the proofs of the theorems, we require the following definition and lemmas.

Definition 3.1. For $\gamma = (\gamma_0, \dots, \gamma_n) \in \mathbb{C}^{n+1}$ and $p(z) = \sum_{j=0}^{n} a_j z^j$, we define $\Lambda_\gamma p(z) = \sum_{j=0}^{n} \gamma_j a_j z^j$.

The operator Λ_γ is said to be *admissible* if it preserves one of the following properties:

(1) $p(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leqslant 1\}$,
(2) $p(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geqslant 1\}$.

The following lemma is due to Arestov [**2**].

Lemma 3.1. *Let $\phi(x) = \psi(\log x)$, where ψ is a convex non-decreasing function on $\mathbb{R}$. Then for all polynomials $p(z) = \sum_{j=0}^{n} a_j z^j$ of degree n and each admissble operator Λ_γ,*

$$\int_0^{2\pi}\phi(|\Lambda_\gamma p(e^{i\theta})|)d\theta \leqslant \int_0^{2\pi}\phi(c(\gamma,n)|p(e^{i\theta})|)d\theta,$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

In particular, the lemma applies with $\phi\colon x \mapsto x^r$ for every $r \in (0, +\infty)$ and with $\phi\colon x \mapsto \log x$ as well. Therefore we have

$$\|\Lambda_\gamma p\|_r \leqslant c(\gamma, n)\|p\|_r, \quad (0 \leqslant r < +\infty). \tag{3.1}$$

LEMMA 3.2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *having no zero in* $|z| < 1$*, then for any* $k \geqslant 1$*, for each* γ*,* $\gamma \in \mathbb{R}$ *and* $r > 0$*,*

$$\int_0^{2\pi} |k^n p(e^{i\theta}/k) + e^{i\gamma} p(ke^{i\theta})|^r d\theta \leqslant |1 + e^{i\gamma} k^n|^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta. \tag{3.2}$$

PROOF. For $k \geqslant 1$ and $\gamma \in \mathbb{R}$, the polynomial $\sum_{j=0}^{n} \binom{n}{j}(k^{n-j} + e^{i\gamma} k^j) z^j$ has all its zeros on the unit circle (see [**33**, Problem 26, p. 108]). Hence, if $p(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish for $|z| < 1$, then by Szegö's convolution theorem [**41**] the same is true for

$$\begin{aligned}\Lambda p(z) :&= (k^n + e^{i\gamma})a_0 + (k^{n-1} + e^{i\gamma} k)a_1 z + \cdots + (1 + e^{i\gamma} k^n) a_n z^n \\ &= k^n p(z/k) + e^{i\gamma} p(kz).\end{aligned}$$

Therefore, Λ is an admissible operator. Applying (3.1), we obtain (3.2). □

The next lemma is a generalization of the well-known Schwarz lemma and is due to Osserman [**32**].

LEMMA 3.3. *Let* $f(z)$ *be analytic in* $|z| < 1$ *such that* $|f(z)| < 1$ *for* $|z| < 1$ *and* $f(0) = 0$*. Then* $|f(z)| \leqslant |z| \frac{|z| + |f'(0)|}{1 + |f'(0)||z|}$ *for* $|z| < 1$.

LEMMA 3.4. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zero in* $|z| < 1$*, then for any* $k \geqslant 1$*, for each* γ*,* $0 \leqslant \gamma < 2\pi$ *and* $r > 0$*,*

$$\int_0^{2\pi}\int_0^{2\pi} |q(ke^{i\theta}) + e^{i\gamma} p(ke^{i\theta})|^r d\gamma\, d\theta \leqslant \int_0^{2\pi} |1 + e^{i\gamma} k^n|^r d\gamma \int_0^{2\pi} |p(e^{i\theta})|^r d\theta.$$

PROOF. Since $q(z) = z^n \overline{p(1/\bar{z})}$, so $q(kz) = (kz)^n \overline{p(1/k\bar{z})}$. In particular, for each θ, $0 \leqslant \theta < 2\pi$,

$$|q(ke^{i\theta})| = \left|k^n e^{in\theta} \overline{p\Big(\frac{1}{ke^{-i\theta}}\Big)}\right| = k^n |p(e^{i\theta}/k)|. \tag{3.3}$$

Now for points $e^{i\theta}$, for which $p(ke^{i\theta}) \neq 0$, we obtain by using (3.3), for each $r > 0$,

$$\begin{aligned}\int_0^{2\pi} |q(ke^{i\theta}) + e^{i\gamma} p(ke^{i\theta})|^r d\gamma &= |p(ke^{i\theta})|^r \int_0^{2\pi} \left|\frac{q(ke^{i\theta})}{p(ke^{i\theta})} + e^{i\gamma}\right|^r d\gamma \\ &= |p(ke^{i\theta})|^r \int_0^{2\pi} \left|\left|\frac{q(ke^{i\theta})}{p(ke^{i\theta})}\right| + e^{i\gamma}\right|^r d\gamma \\ &= \int_0^{2\pi} ||q(ke^{i\theta})| + e^{i\gamma}|p(ke^{i\theta})||^r d\gamma \\ &= \int_0^{2\pi} |k^n|p(e^{i\theta}/k)| + e^{i\gamma}|p(ke^{i\theta})||^r d\gamma.\end{aligned} \tag{3.4}$$

Since (3.4) is trivially true for points $e^{i\theta}$ for which $p(ke^{i\theta})=0$, it follows that

$$\int_0^{2\pi} |q(ke^{i\theta})+e^{i\gamma}p(ke^{i\theta})|^r d\gamma = \int_0^{2\pi} |k^n|p(e^{i\theta}/k)|+e^{i\gamma}|p(ke^{i\theta})||^r d\gamma. \tag{3.5}$$

Integrating (3.5) both sides with respect to θ from 0 to 2π, and using Lemma 3.2, we get

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} |q(ke^{i\theta})+e^{i\gamma}p(ke^{i\theta})|^r d\gamma\, d\theta &= \int_0^{2\pi}\int_0^{2\pi} |k^n|p(e^{i\theta}/k)|+e^{i\gamma}|p(ke^{i\theta})||^r d\gamma\, d\theta\\ &= \int_0^{2\pi}\left\{\int_0^{2\pi} |k^n p(e^{i\theta}/k)+e^{i\gamma}p(ke^{i\theta})|^r d\gamma\right\}d\theta\\ &= \int_0^{2\pi}\left\{\int_0^{2\pi} |k^n p(e^{i\theta}/k)+e^{i\gamma}p(ke^{i\theta})|^r d\theta\right\}d\gamma\\ &\leqslant \int_0^{2\pi}\left\{|1+e^{i\gamma}k^n|^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}d\gamma\end{aligned}$$

or

$$\int_0^{2\pi}\int_0^{2\pi} |q(ke^{i\theta})+e^{i\gamma}p(ke^{i\theta})|^r d\gamma\, d\theta \leqslant \int_0^{2\pi} |1+e^{i\gamma}k^n|^r d\gamma \int_0^{2\pi} |p(e^{i\theta})|^r d\theta. \qquad \square$$

Lemma 3.5. *If $p(z)=\sum_{j=0}^n a_j z^j$ is a polynomial of degree $n\geqslant 1$ having no zero in $|z|<1$, then for any $k\geqslant 1$, for each γ, $0\leqslant\gamma<2\pi$ and $r>0$,*

$$\left\{\int_0^{2\pi} |p(ke^{i\theta})|^r d\theta\right\}^{1/r} \leqslant \frac{\left\{\int_0^{2\pi} |1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi} \left|\frac{|a_0|k+|a_n|}{|a_0|+k|a_n|}+e^{i\gamma}\right|^r d\gamma\right\}^{1/r}} \left\{\int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta\right\}^{1/r}. \tag{3.6}$$

Proof. Since $p(z)$ has no zero in the disk $|z|<1$, its conjugate reciprocal polynomial $q(z)=z^n\overline{p(1/\bar{z})}$ has all its zeros in $|z|\leqslant 1$. Then $\frac{zq(z)}{p(z)}$ satisfies the hypothesis of Lemma 3.3, and hence we have for $|z|<1$,

$$|zq(z)| \leqslant |z|\frac{|z||a_0|+|a_n|}{|a_0|+|a_n||z|}|p(z)|,$$

which gives

$$|q(z)| \leqslant \frac{|z||a_0|+|a_n|}{|a_0|+|a_n||z|}|p(z)|. \tag{3.7}$$

Replacing z by $1/z$ in inequality (3.7), we have for $|z|>1$,

$$|p(z)| \leqslant \frac{|a_0|+|z||a_n|}{|a_0||z|+|a_n|}|q(z)|. \tag{3.8}$$

Note that inequality (3.8) is true for all z on $|z|=1$ also, and therefore for any $k\geqslant 1$ and $0\leqslant\theta\leqslant 2\pi$,

$$|p(ke^{i\theta})| \leqslant \frac{|a_0|+k|a_n|}{|a_0|k+|a_n|}|q(ke^{i\theta})|$$

or

$$\frac{|a_0|k+|a_n|}{|a_0|+k|a_n|} \leqslant \Big|\frac{q(ke^{i\theta})}{p(ke^{i\theta})}\Big|, \quad \text{where} \quad \frac{|a_0|k+|a_n|}{|a_0|+k|a_n|} \geqslant 1. \tag{3.9}$$

It can be easily verified that for every real number γ and $L \geqslant l \geqslant 1$, we have $|L+e^{i\gamma}| \geqslant |l+e^{i\gamma}|$.

This implies for each $r>0$,

$$\int_0^{2\pi} |L+e^{i\gamma}|^r d\gamma \geqslant \int_0^{2\pi} |l+e^{i\gamma}|^r d\gamma. \tag{3.10}$$

For points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, for which $p(ke^{i\theta}) \neq 0$, we take $L = \big|\frac{q(ke^{i\theta})}{p(ke^{i\theta})}\big|$ and $l = \frac{|a_0|k+|a_n|}{|a_0|+k|a_n|}$, then by (3.9), $L \geqslant l \geqslant 1$, and from (3.10), we get for each $r>0$,

$$\begin{aligned} \int_0^{2\pi} |q(ke^{i\theta})+e^{i\gamma}p(ke^{i\theta})|^r d\gamma &= |p(ke^{i\theta})|^r \int_0^{2\pi} \Big|\frac{q(ke^{i\theta})}{p(ke^{i\theta})}+e^{i\gamma}\Big|^r d\gamma \\ &= |p(ke^{i\theta})|^r \int_0^{2\pi} \Big|\Big|\frac{q(ke^{i\theta})}{p(ke^{i\theta})}\Big|+e^{i\gamma}\Big|^r d\gamma \\ &\geqslant |p(ke^{i\theta})|^r \int_0^{2\pi} \Big|\frac{|a_0|k+|a_n|}{|a_0|+k|a_n|}+e^{i\gamma}\Big|^r d\gamma. \end{aligned} \tag{3.11}$$

Since (3.11) is trivially true for points $e^{i\theta}$ for which $p(ke^{i\theta})=0$, it follows that

$$\int_0^{2\pi} |q(ke^{i\theta})+e^{i\gamma}p(ke^{i\theta})|^r d\gamma \geqslant |p(ke^{i\theta})|^r \int_0^{2\pi} \Big|\frac{|a_0|k+|a_n|}{|a_0|+k|a_n|}+e^{i\gamma}\Big|^r d\gamma. \tag{3.12}$$

Integrating (3.12) both sides with respect to θ from 0 to 2π, and using in Lemma 3.4, we get

$$\int_0^{2\pi} |p(ke^{i\theta})|^r d\theta \int_0^{2\pi} \Big|\frac{|a_0|k+|a_n|}{|a_0|+k|a_n|}+e^{i\gamma}\Big|^r d\gamma \leqslant \int_0^{2\pi} |1+e^{i\gamma}k^n|^r d\gamma \int_0^{2\pi} |p(e^{i\theta})|^r d\theta,$$

which immediately leads to (3.6). □

LEMMA 3.6. *If $p(z)=\sum_{j=0}^n a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for each γ, $0 \leqslant \gamma < 2\pi$ and $r>0$,*

$$\Big\{\int_0^{2\pi} |p(ke^{i\theta})|^r d\theta\Big\}^{1/r} \geqslant k^n \frac{\Big\{\int_0^{2\pi} \Big|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi} |1+e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}} \Big\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\Big\}^{1/r}. \tag{3.13}$$

PROOF. Since $p(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, the polynomial $g(z) = p(kz)$ has all its zeros in the unit disc $|z| \leqslant 1$. Let $h(z) = z^n g(1/z)$. Then $h(z)$ is a polynomial of degree atmost n having no zero in $|z|<1$. Therefore applying Lemma 3.5 to $h(z)$, we have for $k \geqslant 1$,

$$\Big\{\int_0^{2\pi} |h(ke^{i\theta})|^r d\theta\Big\}^{1/r} \leqslant \frac{\Big\{\int_0^{2\pi} |1+e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi} \Big|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}} \Big\{\int_0^{2\pi} |h(e^{i\theta})|^r d\theta\Big\}^{1/r}.$$

Since $|h(e^{i\theta})| = |g(e^{i\theta})|$ on $|z| = 1$,

$$\left\{\int_0^{2\pi} |g(e^{i\theta})|^r d\theta\right\}^{1/r} \geqslant \frac{\left\{\int_0^{2\pi} \left|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|} + e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi} |1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}} \left\{\int_0^{2\pi} |h(ke^{i\theta})|^r d\theta\right\}^{1/r}. \tag{3.14}$$

But $h(e^{i\theta}) = e^{in\theta}g(1/e^{i\theta}) = e^{in\theta}p(k/e^{i\theta})$ and so,

$$\left\{\int_0^{2\pi} |h(ke^{i\theta})|^r d\theta\right\}^{1/r} = k^n \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{1/r}. \tag{3.15}$$

Using (3.15) in (3.14), we get

$$\left\{\int_0^{2\pi} |g(e^{i\theta})|^r d\theta\right\}^{1/r} \geqslant k^n \frac{\left\{\int_0^{2\pi} \left|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|} + e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi} |1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}} \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{1/r}. \tag{3.16}$$

Using the fact

$$\left\{\int_0^{2\pi} |g(e^{i\theta})|^r d\theta\right\}^{1/r} = \left\{\int_0^{2\pi} |p(ke^{i\theta})|^r d\theta\right\}^{1/r}$$

in inequality (3.16), we get the desired inequality (3.13). □

NOTE: It is interesting to note that Lemmas 3.5 and 3.6 gives the integral analogues of two results due to Kumar [**20**, Lemmas 2.3 and 2.4].

LEMMA 3.7 (Malik [**24**]). *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for $|z| = 1$, $|q'(z)| \leqslant k|p'(z)|$.*

LEMMA 3.8 (Dubinin [**12**]). *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $|z| = 1$,*

$$|p'(z)| \geqslant \frac{1}{2}\left\{n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right\}|p(z)|.$$

LEMMA 3.9. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n, then for any $k \geqslant 1$ and $r > 0$,*

$$\left\{\int_0^{2\pi} |p(ke^{i\theta})|^r d\theta\right\}^{1/r} \leqslant k^n \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{1/r}. \tag{3.17}$$

As far as Lemma 3.9 is concerned, it is difficult to trace its origin. It was deduced from a well-known result of Hardy [**17**], according to which for every function $f(z)$ analytic in $|z| < t_0$, and for every $r > 0$, $\left\{\int_0^{2\pi} |f(e^{i\theta})|^r d\theta\right\}^{1/r}$ is a non-decreasing function of t for $0 < t < t_0$. If $p(z)$ is a polynomial of degree n, then $f(z) = z^n\overline{p(1/\bar{z})}$ is again a polynomial, that is, an entire function and by Hardy's result for $r > 0$,

$$\left\{\int_0^{2\pi} \left|f(te^{i\theta})\right|^r d\theta\right\}^{1/r} \leqslant \left\{\int_0^{2\pi} |f(e^{i\theta})|^r d\theta\right\}^{1/r},$$

for $t = \frac{1}{k} < 1$. This is equivalent to (3.17).

Lemma 3.10. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k > 0$, *then for every complex number* β *with* $|\beta| < 1$ *and* $m = \min_{|z|=k} |p(z)|$, *we have* $k^n|a_n| \geqslant |\beta|m + |a_0|$.

Proof. By hypothesis, $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k > 0$. Then, the polynomial $P(z) = e^{-i \arg a_0} p(z)$ has the same zeros as $p(z)$. Here,

$$\begin{aligned} P(z) &= e^{-i \arg a_0}\big\{|a_0|e^{i \arg a_0} + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n\big\} \\ &= |a_0| + e^{-i \arg a_0}\big\{a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n\big\}. \end{aligned}$$

In the case $m = \min_{|z|=k} |p(z)| \neq 0$, consider the polynomial $R(z) = P(z) + |\beta|m$, where β is a complex number with $|\beta| < 1$.

Now, on $|z| = k$ $|\beta|m < m \leqslant |P(z)|$.

Then by Rouche's theorem [**7**], it follows that $R(z)$ has all its zeros in $|z| < k$ and in case $m = 0$, $R(z) = P(z)$. Thus in any case $R(z)$ has all its zeros in $|z| \leqslant k$. Now, applying Vieta's formula [**45**] to $R(z)$, we get $\frac{|a_0|+|\beta|m}{|a_n|} \leqslant k^n$, i.e., $k^n|a_n| \geqslant |\beta|m + |a_0|$. □

The next lemma is a simple deduction from the Maximum Modulus Principle which is due to G. Pólya and G. Szegö [**34**] (or see [**29**]).

Lemma 3.11. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n, *then for any* $k \geqslant 1$,

$$\max_{|z|=k} |p(z)| \leqslant k^n \max_{|z|=1} |p(z)|.$$

Lemma 3.12 (Govil and Rahman [**16**]). *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n, *then on* $|z| = 1$, $|p'(z)| + |q'(z)| \leqslant n \max_{|z|=1} |p(z)|$.

Lastly, we require the following result.

Lemma 3.13. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then for every complex number* β *with* $|\beta| < 1$, *for each* γ, $0 \leqslant \gamma < 2\pi$ *and* $r > 0$, $\|p'\|_\infty \geqslant \frac{AE_r}{2}\|p(e^{i\theta}) + \beta m\|_r$, *where* A, E_r *and* m *are as defined in Theorem* 2.1.

Proof. By hypothesis, $p(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$. Consider the polynomial $R(z) = p(z) + \beta m$, where β is a complex number with $|\beta| < 1$ and $m = \min_{|z|=k} |p(z)|$. Then by Rouche's theorem [**7**], it follows that $R(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$. And so the polynomial $P(z) = R(kz)$ has all its zeros in $|z| \leqslant 1$.

Applying Lemma 3.8 to $P(z)$, we have for $|z| = 1$,

$$\max_{|z|=1} |P'(z)| \geqslant \frac{1}{2}\Big\{n + \frac{k^n|a_n| - |a_0 + \beta m|}{k^n|a_n| + |a_0 + \beta m|}\Big\}|P(z)|,$$

which is equivalent to

$$k \max_{|z|=k} |p'(z)| \geqslant \frac{1}{2}\Big\{n + \frac{k^n|a_n| - |a_0 + \beta m|}{k^n|a_n| + |a_0 + \beta m|}\Big\}|p(kz) + \beta m|.$$

Therefore for any $r > 0$, we have

$$k^r\Big\{\max_{|z|=k}|p'(z)|\Big\}^r \geqslant \frac{1}{2^r}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}^r|p(ke^{i\theta})+\beta m|^r, \quad 0\leqslant\theta<2\pi,$$

and hence

$$k\max_{|z|=k}|p'(z)| \geqslant \frac{1}{2}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}\Big\{\frac{1}{2\pi}\int_0^{2\pi}|p(ke^{i\theta})+\beta m|^r d\theta\Big\}^{1/r}. \tag{3.18}$$

Applying Lemma 3.6 to $R(z)$, we get

$$\Big\{\int_0^{2\pi}|R(ke^{i\theta})|^r d\theta\Big\}^{1/r} \geqslant k^n\frac{\Big\{\int_0^{2\pi}\Big|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}} \times\Big\{\int_0^{2\pi}|R(e^{i\theta})|^r d\theta\Big\}^{1/r}$$

or

$$\Big\{\int_0^{2\pi}|p(ke^{i\theta})+\beta m|^r d\theta\Big\}^{1/r} \geqslant k^n\frac{\Big\{\int_0^{2\pi}\Big|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}} \times\Big\{\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\Big\}^{1/r}. \tag{3.19}$$

Using (3.19) in (3.18), we get

$$\begin{aligned} k\max_{|z|=k}|p'(z)| \geqslant{}& \frac{k^n}{2}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\} \\ &\times\frac{\Big\{\int_0^{2\pi}\Big|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}} \\ &\times\Big\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\Big\}^{1/r}. \end{aligned} \tag{3.20}$$

Since $p'(z)$ is a polynomial of degree at most $n-1$, applying Lemma 3.11 to $p'(z)$ for $k\geqslant 1$, we have

$$\max_{|z|=k}|p'(z)| \leqslant k^{n-1}\max_{|z|=1}|p'(z)|. \tag{3.21}$$

Using (3.21) in (3.20), we get

$$\begin{aligned}\max_{|z|=1}|p'(z)| \geqslant \frac{1}{2}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}\\ \times\frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}}\\ \times\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\right\}^{1/r}.\end{aligned}\tag{3.22}$$

Further, proceeding the similar arguments in the end parts of the proof of Theorem 2.1 for routing inequalities (4.11), (4.12), and (4.16), and using these in inequality (3.22), we get

$$\begin{aligned}\max_{|z|=1}|p'(z)| \geqslant \frac{1}{2}\Big\{n+\frac{k^n|a_n|-|a_0|-|\beta|m}{k^n|a_n|+|a_0|+|\beta|m}\Big\}\\ \times\frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}+e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}}\\ \times\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\right\}^{1/r}.\end{aligned}$$ □

NOTE: It is worth to see that Lemma 3.13 provides the integral setting of a result due to Mir and Dar [**31**, Corollary 1].

4. Proofs of the main results

PROOF OF THEOREM 2.1. By hypothesis, $p(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$. In the case $m = \min_{|z|=k}|p(z)| \neq 0$, consider the polynomial $R(z) = p(z) + \beta m$, where β is a complex number with $|\beta| < 1$.

Now, on $|z| = k$, $|\beta m| < m \leqslant |p(z)|$. Then by Rouche's theorem [**7**], it follows that $R(z)$ has all its zeros in $|z| < k$ and in the case $m = 0$, $R(z) = p(z)$. Thus in any case $R(z)$ has all its zeros in $|z| \leqslant k$. Then the polynomial $P(z) = R(kz)$ has all its zeros in $|z| \leqslant 1$.

It is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|,\tag{4.1}$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$. Applying Lemma 3.7 to $P(z)$, we have for $|z| = 1$,

$$|Q'(z)| \leqslant |P'(z)|.\tag{4.2}$$

Using (4.1) and (4.2), we have for $\left|\frac{\alpha}{k}\right| \geqslant 1$ and $|z| = 1$,

$$\begin{aligned}(4.3)\qquad |D_{\frac{\alpha}{k}}P(z)| &= \left|nP(z)+\Big(\frac{\alpha}{k}-z\Big)P'(z)\right| \\ &\geqslant \left|\frac{\alpha}{k}\right||P'(z)|-|nP(z)-zP'(z)| \\ &= \left|\frac{\alpha}{k}\right||P'(z)|-|Q'(z)| \\ &\geqslant \Big(\left|\frac{\alpha}{k}\right|-1\Big)|P'(z)|.\end{aligned}$$

Applying Lemma 3.8 to $P(z)$, we have for $|z|=1$,

$$(4.4)\qquad |P'(z)| \geqslant \frac{1}{2}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}|P(z)|.$$

Combining (4.4) and (4.3), we get

$$\left|D_{\frac{\alpha}{k}}P(z)\right| \geqslant \frac{|\alpha|-k}{2k}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}|P(z)|.$$

Replacing $P(z)$ by $R(kz)$ in the above inequality, we obtain

$$(4.5)\qquad \left|nR(kz)+\Big(\frac{\alpha}{k}-z\Big)kR'(kz)\right| \geqslant \frac{|\alpha|-k}{2k}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}|R(kz)|.$$

Inequality (4.5) is equivalent to

$$|nR(kz)+(\alpha-kz)R'(kz)| \geqslant \frac{|\alpha|-k}{2k}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}|R(kz)|.$$

Therefore for any $r>0$, we have

$$|D_\alpha R(ke^{i\theta})|^r \geqslant \Big(\frac{|\alpha|-k}{2k}\Big)^r\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\}^r|R(ke^{i\theta})|^r, \quad 0\leqslant\theta<2\pi,$$

and hence

$$\begin{aligned}(4.6)\qquad \Big\{\int_0^{2\pi}|D_\alpha R(ke^{i\theta})|^r d\theta\Big\}^{1/r} &\geqslant \frac{|\alpha|-k}{2k}\Big\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\Big\} \\ &\quad\times\Big\{\int_0^{2\pi}|R(ke^{i\theta})|^r d\theta\Big\}^{1/r}.\end{aligned}$$

Applying Lemma 3.6 to $R(z)$, we get

$$\begin{aligned}(4.7)\qquad \Big\{\int_0^{2\pi}|R(ke^{i\theta})|^r d\theta\Big\}^{1/r} &\geqslant k^n\frac{\Big\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}} \\ &\quad\times\Big\{\int_0^{2\pi}|R(e^{i\theta})|^r d\theta\Big\}^{1/r}.\end{aligned}$$

Using (4.7) in (4.6), we get

$$\left\{\int_0^{2\pi}|D_\alpha R(ke^{i\theta})|^r d\theta\right\}^{1/r} \geqslant k^n\frac{|\alpha|-k}{2k}\left\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\right\} \times \frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}}\left\{\int_0^{2\pi}|R(e^{i\theta})|^r d\theta\right\}^{1/r}. \tag{4.8}$$

Since $D_\alpha R(z)$ is a polynomial of degree at most $n-1$, applying Lemma 3.9 to $D_\alpha R(z)$ for $k \geqslant 1$, we have

$$\left\{\int_0^{2\pi}|D_\alpha R(ke^{i\theta})|^r d\theta\right\}^{1/r} \leqslant k^{n-1}\left\{\int_0^{2\pi}|D_\alpha R(e^{i\theta})|^r d\theta\right\}^{1/r}. \tag{4.9}$$

Using (4.9) in (4.8), we get

$$k^{n-1}\left\{\int_0^{2\pi}|D_\alpha R(e^{i\theta})|^r d\theta\right\}^{1/r} \geqslant k^n\frac{|\alpha|-k}{2k}\left\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\right\} \times \frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}}\left\{\int_0^{2\pi}|R(e^{i\theta})|^r d\theta\right\}^{1/r},$$

which is equivalent to

$$\left\{\int_0^{2\pi}|D_\alpha\{p(e^{i\theta})+\beta m\}|^r d\theta\right\}^{1/r} \geqslant \frac{|\alpha|-k}{2}\left\{n+\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|}\right\} \times \frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\right\}^{1/r}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{1/r}}\left\{\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\right\}^{1/r}. \tag{4.10}$$

For every $\beta \in \mathbb{C}$, we have $|a_0+\beta m| \leqslant |a_0|+|\beta|m$, and since the functions

$$x \mapsto \frac{k^n|a_n|-x}{k^n|a_n|+x} \quad \text{and} \quad x \mapsto \frac{|a_n|k^{n+1}+x}{|a_n|k^n+kx},$$

are both non-increasing for $x \geqslant 0$ and for every $k \geqslant 1$, it follows that

$$\frac{k^n|a_n|-|a_0+\beta m|}{k^n|a_n|+|a_0+\beta m|} \geqslant \frac{k^n|a_n|-|a_0|-|\beta|m}{k^n|a_n|+|a_0|+|\beta|m}, \tag{4.11}$$

where

$$\frac{k^n|a_n|-|a_0|-|\beta|m}{k^n|a_n|+|a_0|+|\beta|m} \geqslant 0, \tag{4.12}$$

and

$$\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|} \geqslant \frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}, \tag{4.13}$$

where

$$\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}\geqslant 1. \tag{4.14}$$

(4.12) and (4.14) follow readily in view of Lemma 3.10.

It can be easily verified that for every real number γ and $L\geqslant l\geqslant 1$,

$$|L+e^{i\gamma}|\geqslant|l+e^{i\gamma}|.$$

This implies for each $r>0$,

$$\int_0^{2\pi}|L+e^{i\gamma}|^r d\gamma\geqslant\int_0^{2\pi}|l+e^{i\gamma}|^r d\gamma. \tag{4.15}$$

Now, we take $L=\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}$ and $l=\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}$, then by (4.13) and (4.14), $L\geqslant l\geqslant 1$, and from (4.15), we get for each $r>0$,

$$\int_0^{2\pi}\Big|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\Big|^r d\gamma\geqslant\int_0^{2\pi}\Big|\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}+e^{i\gamma}\Big|^r d\gamma. \tag{4.16}$$

From (4.11), (4.12), (4.16), and (4.10), we get

$$\begin{aligned}\Big\{\int_0^{2\pi}|D_\alpha\{p(e^{i\theta})+\beta m\}|^r d\theta\Big\}^{1/r}&\geqslant\frac{|\alpha|-k}{2}\Big\{n+\frac{k^n|a_n|-|a_0|-|\beta|m}{k^n|a_n|+|a_0|+|\beta|m}\Big\}\\&\times\frac{\Big\{\int_0^{2\pi}\Big|\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\Big\}^{1/r}}\Big\{\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\Big\}^{1/r}. \quad\square\end{aligned}$$

Proof of Theorem 2.2. Since $p(z)$ has no zero in $|z|<k$, $k\leqslant 1$, the polynomial $q(z)$ of degree n has all its zeros in $|z|\leqslant\frac{1}{k}$, $\frac{1}{k}\geqslant 1$. Applying Lemma 3.13 to $q(z)$, we get for $|\beta|<1$,

$$\max_{|z|=1}|q'(z)|\geqslant\frac{A'E''_r}{2}\Big\{\frac{1}{2\pi}\int_0^{2\pi}|q(e^{i\theta})+\beta m'|^r d\theta\Big\}^{1/r}, \tag{4.17}$$

where

$$m'=\min_{|z|=\frac{1}{k}}|q(z)|=\min_{|z|=\frac{1}{k}}\big|z^n\overline{p(1/\bar z)}\big|=\frac{1}{k^n}\min_{|z|=k}|p(z)|=m/k^n,$$

$$A'=\Big\{n+\frac{\frac{1}{k^n}|a_0|-|a_n|-|\beta|\frac{m}{k^n}}{\frac{1}{k^n}|a_0|+|a_n|+|\beta|\frac{m}{k^n}}\Big\}=\Big\{n+\frac{|a_0|-|a_n|k^n-|\beta|m}{|a_0|+|a_n|k^n+|\beta|m}\Big\}$$

and

$$\begin{aligned}E''_r&=\frac{\Big\{\int_0^{2\pi}\Big|\frac{|a_0|\frac{1}{k^{n+1}}+|a_n|+|\beta|\frac{m}{k^n}}{|a_0|\frac{1}{k^n}+\frac{1}{k}|a_n|+|\beta|\frac{m}{k^{n+1}}}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi}\Big|1+e^{i\gamma}\frac{1}{k^n}\Big|^r d\gamma\Big\}^{1/r}}\\&=k^n\frac{\Big\{\int_0^{2\pi}\Big|\frac{|a_0|+|a_n|k^{n+1}+|\beta|mk}{|a_0|k+k^n|a_n|+|\beta|m}+e^{i\gamma}\Big|^r d\gamma\Big\}^{1/r}}{\Big\{\int_0^{2\pi}|k^n+e^{i\gamma}|^r d\gamma\Big\}^{1/r}}=k^nE'_r.\end{aligned}$$

We have

$$|q(z)+\beta m'| = \Big|z^n\overline{p(1/\bar{z})}+\beta\frac{m}{k^n}\Big| = |z|^n\Big|\overline{p(1/\bar{z})}+\frac{\beta m}{k^n z^n}\Big| = |z|^n\Big|p(1/\bar{z})+\frac{\bar{\beta} m}{k^n \bar{z}^n}\Big|.$$

In particular, for each θ, $0 \leqslant \theta < 2\pi$,

$$|q(e^{i\theta})+\beta m'| = \Big|p(e^{i\theta})+\frac{\bar{\beta} m e^{in\theta}}{k^n}\Big|. \tag{4.18}$$

Using (4.18) in (4.17), we get

$$\max_{|z|=1}|q'(z)| \geqslant \frac{k^n A' E_r'}{2}\Big\{\frac{1}{2\pi}\int_0^{2\pi}\Big|p(e^{i\theta})+\frac{\bar{\beta} m e^{in\theta}}{k^n}\Big|^r d\theta\Big\}^{1/r}. \tag{4.19}$$

By Lemma 3.12, we have for $|z|=1$,

$$|p'(z)|+|q'(z)| \leqslant n\max_{|z|=1}|p(z)|. \tag{4.20}$$

Since $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z|=1$, let z_0 on $|z|=1$ be such that $\max_{|z|=1}|q'(z)| = |q'(z_0)|$, then $\max_{|z|=1}|p'(z)| = |p'(z_0)|$.

Now, in particular, (4.20) gives $|p'(z_0)|+|q'(z_0)| \leqslant n\max_{|z|=1}|p(z)|$, which implies

$$\max_{|z|=1}|q'(z)| \leqslant n\max_{|z|=1}|p(z)| - \max_{|z|=1}|p'(z)|. \tag{4.21}$$

Using (4.21) in (4.19), we get

$$n\max_{|z|=1}|p(z)| \geqslant \max_{|z|=1}|p'(z)| + \frac{k^n A' E_r'}{2}\Big\{\frac{1}{2\pi}\int_0^{2\pi}\Big|p(e^{i\theta})+\frac{\bar{\beta} m e^{in\theta}}{k^n}\Big|^r d\theta\Big\}^{1/r}. \tag{4.22}$$

Also, by using Lemma 3.12, we have for $|\alpha| \geqslant 1$ and $|z|=1$,

$$\begin{aligned}
|D_\alpha p(z)| &= |np(z)+(\alpha - z)p'(z)| \\
&\leqslant |np(z)-zp'(z)|+|\alpha||p'(z)| \\
&= |q'(z)|+|\alpha||p'(z)| \quad (\because |q'(z)| = |np(z)-zp'(z)| \text{ for } |z|=1) \\
&= |q'(z)|+|p'(z)|-|p'(z)|+|\alpha||p'(z)| \\
&\leqslant n\max_{|z|=1}|p(z)|+(|\alpha|-1)|p'(z)| \\
&\leqslant n\max_{|z|=1}|p(z)|+(|\alpha|-1)\max_{|z|=1}|p'(z)|.
\end{aligned} \tag{4.23}$$

Combining (4.23) and (4.22), we get

$$\begin{aligned}
\max_{|z|=1}|D_\alpha p(z)| \leqslant n\max_{|z|=1}|p(z)| + (|\alpha|-1)\Big[&n\max_{|z|=1}|p(z)| \\
&- \frac{k^n A' E_r'}{2}\Big\{\frac{1}{2\pi}\int_0^{2\pi}\Big|p(e^{i\theta})+\frac{\bar{\beta} m e^{in\theta}}{k^n}\Big|^r d\theta\Big\}^{1/r}\Big],
\end{aligned}$$

which is equivalent to

$$\frac{k^n(|\alpha|-1)}{2}A' E_r'\Big\{\frac{1}{2\pi}\int_0^{2\pi}\Big|p(e^{i\theta})+\frac{\bar{\beta} m e^{in\theta}}{k^n}\Big|^r d\theta\Big\}^{1/r} \leqslant n|\alpha|\max_{|z|=1}|p(z)| - \max_{|z|=1}|D_\alpha p(z)|. \quad \square$$

5. Conclusion

Studying the extremal problems of functions of a complex variable and generalizing the classical polynomial inequalities are topical in geometric function theory. In the past few years, a series of papers related both to Bernstein and Turán-type inequalities have been published and significant advances in terms of extension, improvement, as well as generalization, have been achieved in different directions. One such generalization is replacing the sup-norm by a factor involving the integral mean. These types of inequalities are of interest both in mathematics and in the application areas such as physical systems.

Acknowledgment. The authors are very grateful to the referee for the valuable suggestions in upgrading the paper to the present form.

References

1. R. R. Akopyan, P. Kumar, G. V. Milovanović, *On the inequalities of Zygmund and de Bruijn*, Anal. Math. **50**(4) (2024), 967–986.
2. V. V. Arestov, *On inequalities for trigonometric polynomials and their derivative*, Izv. Akad. Nauk SSSR, Ser. Mat. **45**(1) (1981), 3–22.
3. A. Aziz, *Integral mean estimates for polynomials with restricted zeros*, J. Approx. Theory **55** (1988), 232–239.
4. A. Aziz, N. A. Rather, *A refinement of a theorem of Paul Turán concerning polynomial*, J. Math. Inequal. Appl. **1** (1998), 231–238.
5. S. Bernstein, *Sur l' ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mem. Acad. R. Belg. **4** (1912), 1–103.
6. P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1995.
7. J. W. Brown, R. V. Churchill, *Complex Variables and Applications*, McGraw-Hill, New York, NY, 1948.
8. E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, NY, 1966.
9. N. G. De-Bruijn, *Inequalities concerning polynomials in the complex domain*, Nederl. Akad. Wet., Proc., Ser. A **50** (1947), 1265–1272; Indag. Math. **9** (1947), 591–598.
10. K. B. Devi, K. Krishnadas, B. Chanam, *L^r inequalities for the derivative of a polynomial*, Note Mat. **41**(2) (2021), 19–29.
11. R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.
12. V. N. Dubinin, *Applications of the Schwarz lemma to inequalities for entire functions with constraints on zeros*, J. Math. Sci. **143** (2007), 3069–3076.
13. R. B. Gardner, N. K. Govil, G. V. Milovanović, *Extremal Problems and Inequalities of Markov–Bernstein Type for Algebraic Polynomials*, Math. Anal. Appl., Elsevier/ Academic Press, London, 2022.
14. N. K. Govil, *On a theorem of S. Bernstein*, Proc. Nat. Acad. Sci. **50** (1980), 50–52.
15. ______, *On the derivative of a polynomial*, Proc. Am. Math. Soc. **41** (1973), 543–546.
16. N. K. Govil, Q. I. Rahman, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Am. Math. Soc. **137** (1969), 501–517.
17. G. H. Hardy, *The mean value of the modulus of an analytic function*, Proc. Lond. Math. Soc. **14** (1915), 269–277.
18. E. Hille, *Analytic Function Theory*, Vol. 2, Ginn. and Company, New York, Toronto, 1962.
19. V. I. Ivanov, *Some extremal properties of polynomials and inverse inequalities in approximation theory*, Trudy Mat. Inst. Steklov **145** (1979), 79–110.
20. P. Kumar, *On the inequalities concerning polynomials*, Complex Anal. Oper. Theory **14**(65) (2020), 65.

21. P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Am. Math. Soc. **50** (1944), 509–513.
22. G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston, New York, 1966.
23. M. A. Malik, *An integral mean estimates for polynomials*, Proc. Am. Math. Soc. **91**(2) (1984), 281–284.
24. ———, *On the derivative of a polynomial*, J. Lond. Math. Soc. **1**(2) (1969), 57–60.
25. M. Marden, *Geometry of Polynomials*, Math. Surv. **3**, Am. Math, Soc., Providence, 1966.
26. A. A. Markov, *On a problem of D. I. Mendeleev*, Zap. Imp. Akad. Nauk, St. Petersburg **62** (1889), 1–24.
27. D. Mendeleev, *Investigations of Aqueous Solutions Based on Specific Gravity*, St. Petersberg, 1887. (in Russian).
28. G. V. Milovanović, A. Mir, *On the Erdös–Lax and Turán inequalities concerning polynomials*, Math. Inequal. Appl. **25** (2022), 407–419.
29. G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, *Topics in Polynomials, Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.
30. A. Mir, D. Breaz, *Bernstein and Turán-type inequalities for a polynomial with constraints on its zeros*, RACSAM **115** (2021), (Art. 124) 1–12.
31. A. Mir, I. H. Dar, *Inequalities for the polar derivative of a complex polynomial*, Filomat **36**(16) (2022), 5631–5640.
32. R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Am. Math. Soc. **128** (2000), 3513–3517.
33. G. Pólya, G. Szegö, *Problems and Theorems in Analysis*, Vol. 1, Springer-Verlag, New York/ Heidelberg/ Berlin, 1972.
34. ———, *Problems and Theorems in Analysis*, Vol. 2, Springer-Verlag, New York/ Heidelberg/ Berlin, 1976.
35. Q. I. Rahman, G. Schmeisser, *Analytic Theory of Polynomials*, Oxford University Press, Oxford, 2002.
36. ———, L^p *inequalities for polynomials*, J. Approx. Theory **53** (1988), 26–32.
37. N. A. Rather, F. A. Bhat, *Inequalities for the polar derivative of a polynomial*, Appl. Math. E-Notes **17** (2017), 231–241.
38. W. Rudin, *Real and Complex Analysis*, Tata McGraw-Hill, 1977.
39. T. B. Singh, K. B. Devi, B. Chanam, *Turán-type integral mean inequalities of the polar derivative of apolynomial*, Palest. J. Math. **12**(3) (2023), 85–93.
40. R. Soraisam, M. S. Singh, B. Chanam, *Turán-type* L^r*-inequalities for polar derivative of a polynomial*, Nonlinear Funct. Anal. Appl. **28**(3) (2023), 731–751.
41. G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Z., **13** (1922), 28–55.
42. A. E. Taylor, *Introduction to Functional Analysis*, John Wiley and Sons, New York, 1958.
43. S. A. Telyakovskii, *Research in the theory of approximation of functions at the mathematical institute of the academy of sciences*, Trudi Mat. Inst. Steklov **182** (1988); English trans. in: Proc. Steklov Inst. Math. (1990), 141–197.
44. P. Turán, *Über die ableitung von polynomen*, Compos. Math. **7** (1939), 89–95.
45. E. B. Vinberg, *A Course in Algebra*, Grad. Stud. Math. **56**, Am. Math. Soc., 2003.
46. A. Zygmund, *A remark on conjugate series*, Proc. Lond. Math. Soc. **34** (1932), 392–400.

Department of Mathematics
National Institute of Technology Manipur
Langol
India
nirmalsingha99@gmail.com
barchand_2004@yahoo.co.in

(Received 28 11 2023)
(Revised 20 04 2024)