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APPROXIMATING FIXED POINTS OF BIANCHINI TYPE MAPPINGS

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Abstract. We introduce and explore two novel types of contractions, namely the (ψ, a, k)-SM-Bianchini and the generalized (ψ, a, k)-SM-Bianchini type contractions. These contractions represent an extension and generalization of existing contraction principles, allowing for a broader and more flexible framework within the study of fixed-point theory. By incorporating the functions ψ, a, and k, we develop a more comprehensive approach that encompasses and extends various classical results. Moreover, to emphasize the practical significance of our theoretical contributions, we present an application of the generalized (ψ, a, k)-SM-Bianchini type contractions to the split feasibility problem.

1. Introduction and preliminaries

Fixed-point theory, especially in relation to contraction mappings, is fundamental across many areas of mathematics due to its broad applications and elegant theoretical foundations. Central to this theory is the concept of contractions, it offers a strong framework for examining the existence and uniqueness of fixed points within metric spaces. Recently, there has been a growing interest in expanding the classical theory of contractions to include more generalized contexts, such as rectangular metric spaces, b-metric spaces, convex metric spaces, etc. The study of a metric space theory began with Fréchet's pioneering work in 1906 [**13**]. Since then, numerous researchers have advanced the field by modifying conditions and refining the metric function. For detailed references, consult [**3**,**4**,**8**,**12**,**14**,**17**,**18**]).

Recently, Berinde has expanded Banach space theory by introducing improved mappings, known as enriched contractions. These are self-mappings T on a normed linear space $(U, \|\cdot\|)$ that meet a symmetric contraction condition: $\|b(u-v)+Tu-Tv\| \leqslant \theta\|u-v\|$, where $u, v \in U$, $b \in [0,+\infty)$ and $\theta \in [0, b+1)$. This new category includes traditional Banach contractions (when the parameter b is zero) and extends to some nonexpansive and Lipschitz mappings. This extension affirms that a fixed point exists in Banach spaces and it can be approximated using the

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Krasnoselskii iteration. Additional generalizations of this concept are discussed in the recent literature [**5**, **15**, **19**] and references therein.

In 2023, Anjum et al. [**1**] introduced new sets and defined a generalized averaged mapping

$$\mho = \{\psi\colon U \to \mathbb{R} : \psi(u) \neq -1, \text{ for all } u \in U\},$$
$$\Omega = \{\lambda\colon U \to \mathbb{R} : \lambda(u) \neq 0, \text{ for all } u \in U\}.$$

An operator $T_\lambda\colon U \to U$ is referred to as a generalized averaged mapping provided it meets the condition $T_\lambda(u) = (1-\lambda(u))u + \lambda(u)Tu$ for all $u \in U$ and for every fixed $\lambda \in \Omega$. This concept was discussed in [**21**]. It is essential to point out that generalized averaged mappings include averaged mappings. The subsequent lemma demonstrates that T_λ and T share the same fixed points.

LEMMA 1.1. *Given $T\colon U \to U$. For any $\lambda \in \Omega$, define $T_\lambda(u) = (1-\lambda(u))u + \lambda(u)Tu$, for all $u \in U$. Then,* $\mathrm{Fix}(T) = \mathrm{Fix}(T_\lambda)$.

This concept was further explored by Bisht and Singh [**7**], who modified the definitions of (ψ, a)-MR-Kannan type contractions and proposed new types of contractions, including (ψ, a, k)-MRB-Kannan and (ψ, α, a, k)-MRB-CRR type contractions. They first introduced two new sets in the following way:

$$\mho^* = \{\psi\colon U \to [0,\infty) \text{ for all } u \in U\},$$
$$\Omega^* = \{\lambda\colon U \to (0,1) \text{ for all } u \in U\}.$$

DEFINITION 1.1. [**7**] A self-mapping T on U is defined as a (ψ, a, k)-MRB-Kannan type contraction, if there are $a \in [0, \frac{1}{2})$, $k \in (0,\infty)$ and $\psi \in \mho^*$ so that

$$\Big\|\frac{u\psi(u) + kTu}{k+\psi(u)} - \frac{v\psi(v) + kTv}{k+\psi(v)}\Big\| \leqslant a\Big(\Big|\frac{k}{k+\psi(u)}\Big|\|u - Tu\| + \Big|\frac{k}{k+\psi(v)}\Big|\|v - Tv\|\Big),$$

for all $u, v \in U$.

DEFINITION 1.2. [**7**] A self-mapping T on U is a (ψ, α, a, k)-MRB-CRR type contraction, if there are $\alpha \in [0,1)$, $a \in [0, \frac{1}{2})$, $\psi \in \mho^*$ and $k \in (0, +\infty)$ so that

$$\Big\|\frac{u\psi(u) + kTu}{k+\psi(u)} - \frac{v\psi(v) + kTv}{k+\psi(v)}\Big\| \leqslant \alpha\|u - v\| + a\Big(\Big|\frac{k}{k+\psi(u)}\Big|\|u - Tu\| + \Big|\frac{k}{k+\psi(v)}\Big|\|v - Tv\|\Big),$$

for all $u, v \in U$.

In a recent work, Bisht [**2**] derived fixed point results for approximating fixed points using the Krasnoselskii iterative method for generalized Bianchini mappings in Banach spaces. In 1966, Browder and Petryshyn [**9**] put forward the idea of asymptotic regularity, and Bianchini [**6**] extended Kannan's fixed-point theorem in 1972 by defining a contraction condition as $d(Tu, Tv) \leqslant a \max d(u, Tu), d(v, Tv)$ for all $u, v \in U$, where $a \in [0, 1)$.

DEFINITION 1.3. [**9**] A self-mapping T is classified as asymptotically regular on U when for each $u \in U$, the distance $d(T^{n+1}u, T^n u) \to 0$ as $n \to +\infty$.

THEOREM 1.1. *Consider* (U,d) *as a complete metric space, and let* $T\colon U \to U$ *be a mapping for which there exists a real number* $0 < h < 1$ *such that for every* $u, v \in U$*, we have* $d(Tu,Tv) \leqslant h \max d(u,Tu), d(v,Tv)$*. Then* T *possesses a unique fixed point in* U*.*

A Banach contraction mapping serves as a fundamental example of an asymptotically regular mapping. However, nonexpansive mappings are not necessarily asymptotically regular in general.

In this study, we propose two new categories of contractions: (ψ, a, k)-SM-Bianchini and generalized (ψ, a, k)-SM-Bianchini type contractions. We provide theoretical results and examples to support our analysis, which generalize several existing results in the literature. Moreover, to emphasize the practical significance of our theoretical contributions, we present an application of the generalized (ψ, a, k)-SM-Bianchini type contractions to the split feasibility problem, a critical issue in optimization and computational mathematics. This application demonstrates the utility of our proposed contractions in solving complex problems beyond the confines of pure theoretical research, thereby underlining their potential impact in various applied mathematical contexts. Through this study, we aim to enrich the existing literature by introducing more general and powerful contraction mappings that can be effectively utilized in both theoretical explorations and practical applications.

2. Main results

First, we introduce the concept of (ψ, a, k)-SM-Bianchini type contractions.

DEFINITION 2.1. A self-mapping T on U is defined as a (ψ, a, k)-SM-Bianchini type contraction, if there are $a \in [0,1)$, $k \in (0,+\infty)$ and $\psi \in \mho^*$ so that

$$\begin{aligned}(2.1)\quad & \Big\| \frac{u\psi(u) + kTu}{k + \psi(u)} - \frac{v\psi(v) + kTv}{k + \psi(v)} \Big\| \\ & \qquad \leqslant a \max \Big\{ \Big| \frac{k}{k+\psi(u)} \Big| \cdot \|u - Tu\|, \Big| \frac{k}{k+\psi(v)} \Big| \cdot \|v - Tv\| \Big\},\end{aligned}$$

for all $u, v \in U$.

Our first main result is stated as follows.

THEOREM 2.1. *Consider* $(U, \|\cdot\|)$ *as a Banach space and a self-mapping* T *on* U *as an* (ψ, a, k)*-SM-Bianchini type contraction. Then*

(1) $\mathrm{Fix}(T) = \{u^*\}$*;*

(2) *there is some* $\lambda \in \Omega^*$ *so that the sequence* $\{u_n\}$*, which is defined by*

$$(2.2)\qquad u_{n+1} = (1 - \lambda(u_n))u_n + \lambda(u_n)Tu_n,$$

converges to u^* *for any initial approximation* $u_0 \in U$*.*

PROOF. Let $\lambda(u) = \frac{k}{k+\psi(u)}$, for all $u \in U$. For $\psi(u) = 0$, the proof is obvious. Therefore, consider $\psi(u) > 0$ and hence $\lambda \in \Omega^*$. From (2.1), we obtain

$$\Big\|\frac{\lambda(u)}{k}\Big(k\Big(\frac{1}{\lambda(u)}-1\Big)u+kTu\Big)-\frac{\lambda(v)}{k}\Big(k\Big(\frac{1}{\lambda(v)}-1\Big)v+kTv\Big)\Big\| \leqslant a\max\{\lambda(u)\|u-Tu\|,\lambda(v)\|v-Tv\|\},$$

that is,

$$\|T_\lambda u - T_\lambda v\| \leqslant a\max\{\lambda(u)\|u-T_\lambda u\|,\lambda(v)\|v-T_\lambda v\|\}, \tag{2.3}$$

where T_λ is identified as the generalized averaged operator. As $a\in[0,1)$, so by using (2.3), T_λ is a Bianchini contraction. The Picard iteration related to T_λ, that is, $u_{n+1}=T_\lambda(u_n)$ is obtained from the generalized Krasnoselskii iteration process $\{u_n\}_{n=0}^{+\infty}$, defined in (2.2). Considering $u=u_n$ and $v=u_{n-1}$ in (2.3), we obtain

$$\|u_{n+1}-u_n\| \leqslant a\max\{\lambda(u_n)\|u_n-u_{n+1}\|,\lambda(u_{n-1})\|u_{n-1}-u_n\|\}.$$

If $\lambda(u_n)\|u_n-u_{n+1}\| \geqslant \lambda(u_{n-1})\|u_{n-1}-u_n\|$, then clearly we would lead to a contradiction, since $a\in[0,1)$ and $\lambda(u)\in(0,1)$ for all $u\in U$. Therefore,

$$\lambda(u_n)\|u_n-u_{n+1}\| < \lambda(u_{n-1})\|u_{n-1}-u_n\|,$$

that is, $\|u_{n+1}-u_n\| < \|u_{n-1}-u_n\|$ for all $n\in\mathbb{N}$. Some simple calculations show that $\{u_n\}$ is a Cauchy sequence and hence it is convergent in U. Suppose $u^*\in U$, such that $\lim_{n\to+\infty}u_n=u^*$. Also,

$$\begin{aligned}\|u^*-T_\lambda u^*\| &\leqslant \|u^*-u_{n+1}\|+\|u_{n+1}-T_\lambda u^*\|\\ &\leqslant \|u^*-u_{n+1}\|+\|T_\lambda u_n-T_\lambda u^*\|\\ &\leqslant \|u^*-u_{n+1}\|+a\max\{\lambda(u_n)\|u_n-u_{n+1}\|,\lambda(u^*)\|u^*-T_\lambda u^*\|\}.\end{aligned}$$

By taking the limit as $n\to+\infty$ on both sides of the inequality above, we obtain $T_\lambda u^*=u^*$. Next, we will demonstrate that u^* is the only fixed point of T_λ. Suppose $v^*\neq u^*$ is another fixed point of T_λ. Then, by (2.3),

$$\begin{aligned}0<\|u^*-v^*\| &= \|T_\lambda u^*-T_\lambda v^*\|\\ &\leqslant a\max\{\lambda(u^*)\|u^*-T_\lambda u^*\|,\lambda(v^*)\|v^*-T_\lambda v^*\|\}=0.\end{aligned}$$

This implies that T_λ possesses a unique fixed point. By Lemma 1.1, we obtain that T also possesses a unique fixed point u^*. ☐

EXAMPLE 2.1. Consider $U=[0,1]$ along with the Euclidean norm $\|\cdot\|$. Let $T\colon U\to U$ be defined as $Tu=u(2-3u^2)/8$, for all $u\in[0,1]$. Also take $\psi(u)=u^2, k=2$ and $\lambda(u)=\frac{2}{2+u^2}$. So, the (ψ,a,k)-SM-Bianchini type contraction condition given by

$$\Big\|\frac{u\psi(u)+kTu}{k+\psi(u)}-\frac{v\psi(v)+kTv}{k+\psi(v)}\Big\| \leqslant a\max\Big\{\Big|\frac{k}{k+\psi(u)}\Big|\cdot\|u-Tu\|,\Big|\frac{k}{k+\psi(v)}\Big|\cdot\|v-Tv\|\Big\},$$

becomes

$$\Big\|\frac{u}{4}-\frac{v}{4}\Big\| \leqslant a\max\Big\{\Big|\frac{2}{2+u^2}\Big|\Big\|u-\frac{u}{4}\Big\|,\Big|\frac{2}{2+v^2}\Big|\Big\|v-\frac{v}{4}\Big\|\Big\}.$$

That is,

$$\frac{1}{4}\|u-v\| \leqslant a \max\Big\{\Big|\frac{2}{2+u^2}\Big|\Big\|u-\frac{u}{4}\Big\|, \Big|\frac{2}{2+v^2}\Big|\Big\|v-\frac{v}{4}\Big\|\Big\}$$
$$\frac{1}{4}\|u-v\| \leqslant \frac{3a}{4} \max\Big\{\Big|\frac{2}{2+u^2}\Big|\|u\|, \Big|\frac{2}{2+v^2}\Big|\|v\|\Big\}.$$

This is satisfied for all $u, v \in U$ and for any $a \in [\frac{1}{2}, 1)$. Thus, T is a (ψ, a, k)-SM-Bianchini type contraction. From Theorem 2.1, T possesses a unique fixed point, which is 0.

The next result corresponds to Theorem 2.1 of [**7**].

Corollary 2.1. *Consider $(U, \|\cdot\|)$ to be a Banach space and $T\colon U \to U$ as a (ψ, a, k)-MRB-Kannan type contraction. Then*

(1) $\mathrm{Fix}(T) = \{u^*\}$;
(2) *there is some $\lambda \in \Omega^*$ so that the sequence u_n, which corresponds to the generalized Krasnoselskii iteration related to T, which is defined in the following way $u_{n+1} = (1-\lambda(u_n))u_n + \lambda(u_n)Tu_n, n \geqslant 0$, converges to u^* for any initial approximation $u_0 \in U$.*

Proof. We can easily see that every (ψ, a, k)-SM-Bianchini type contraction is a (ψ, a, k)-MRB-Kannan type contraction, so applying Theorem 2.1, we achieve the intended outcome. □

The next corollary is Theorem 2.0.3 of [**1**], which can be deduced from Corollary 2.1 by taking $k = 1$.

Corollary 2.2. *Consider $(U, \|\cdot\|)$ as a Banach space and T as a self-mapping on U satisfying the (ψ, a)-MR-Kannan type contraction condition, in other words, it is an operator satisfying*

$$\Big\|\frac{u\psi(u)+Tu}{1+\psi(u)} - \frac{v\psi(v)+Tv}{1+\psi(v)}\Big\| \leqslant a\Big(\Big|\frac{1}{1+\psi(u)}\Big|\|u-Tu\| + \Big|\frac{1}{1+\psi(v)}\Big|\|v-Tv\|\Big),$$

for every $u, v \in U$, where $\psi \in \mho^*$ and $0 \leqslant a < \frac{1}{2}$. Then, T admits a unique fixed point.

Now, we consider a generalized (ψ, a, k)-SM-Bianchini type contraction.

Definition 2.2. A self-mapping T on U is defined as a generalized (ψ, a, k)-SM-Bianchini type contraction, if there are $a \in [0, \infty)$, $k \in (0, \infty)$ and $\psi \in \mho^*$ so that

$$\Big\|\frac{u\psi(u)+kTu}{k+\psi(u)} - \frac{v\psi(v)+kTv}{k+\psi(v)}\Big\| \leqslant a \max\Big\{\Big|\frac{k}{k+\psi(u)}\Big|\cdot\|u-Tu\|, \Big|\frac{k}{k+\psi(v)}\Big|\cdot\|v-Tv\|\Big\}, \tag{2.4}$$

for every $u, v \in U$.

We present our second main result.

THEOREM 2.2. *Consider* $(U, \|\cdot\|)$ *as a Banach space and a self-mapping* T *on* U *as a generalized* (ψ, a, k)*-SM-Bianchini type contraction. Let* T_λ *be an asymptotically regular mapping. Then*

(1) $\mathrm{Fix}(T) = \{u^*\}$;

(2) *there is some* $\lambda \in \Omega^*$ *so that the sequence* $\{u_n\}$*, which is defined by*

$$u_{n+1} = (1 - \lambda(u_n))u_n + \lambda(u_n)Tu_n, \tag{2.5}$$

converges to u^* *for any initial approximation* $u_0 \in U$.

PROOF. Case I: If $\psi(u) = 0$, then the proof is obvious.

Case II: Consider $\psi(u) > 0$ and $\lambda(u) = \frac{k}{k+\psi(u)}$, for all $u \in U$. Using (2.1), we obtain

$$\Big\| \frac{\lambda(u)}{k}\Big(k\Big(\frac{1}{\lambda(u)} - 1\Big)u + kTu\Big) - \frac{\lambda(v)}{k}\Big(k\Big(\frac{1}{\lambda(v)} - 1\Big)v + kTv\Big)\Big\| \\ \leqslant a \max\{\lambda(u)\|u - Tu\|, \lambda(v)\|v - Tv\|\},$$

that is, $\|T_\lambda u - T_\lambda v\| \leqslant a \max\{\lambda(u)\|u - T_\lambda u\|, \lambda(v)\|v - T_\lambda v\|\}$, where T_λ is identified as the generalized averaged operator. As $a \in [0, 1)$, so by using (2.3), T_λ is a Bianchini contraction. The Picard iteration related to T_λ, that is, $u_{n+1} = T_\lambda(u_n)$ is obtained from the generalized Krasnoselskii iteration process $\{u_n\}_{n=0}^{+\infty}$, which is defined in (2.5).

The proof of $\{u_n\}$ to be a Cauchy sequence is on the similar lines of Theorem 2.3 of Bisht [**2**]. As U is a Banach space, we easily obtain $\lim_{n\to+\infty} u_n = u^*$. Using the asymptotically regular condition of T_λ, we instantly get $u^* = T_\lambda u^*$. Thus, using $\mathrm{Fix}(T) = \mathrm{Fix}(T_\lambda)$, we obtain $Tz^* = z^*$. It is straightforward to demonstrate that the mapping has a unique fixed point. □

The next corollary was given in [**7**] as Theorem 3.4.

COROLLARY 2.3. *Consider* $(U, \|\cdot\|)$ *as a Banach space and a self-mapping* T *on* U *as a generalized* (ψ, a, k)*-MRB-Kannan type contraction. Let* T_λ *be an asymptotically regular mapping, then*

(1) $\mathrm{Fix}(T) = \{u^*\}$;

(2) *there is some* $\lambda \in \Omega^*$ *so that the generalized Krasnoselskii iteration related to* T*, in other words, the sequence* $\{u_n\}$*, which is defined in the following way* $u_{n+1} = (1 - \lambda(u_n))u_n + \lambda(u_n)Tu_n$, $n \geqslant 0$*, converges to* u^* *for any initial approximation* $u_0 \in U$.

PROOF. We can easily see that every generalized (ψ, a, k)-SM-Bianchini type contraction is a generalized (ψ, a, k)-MRB-Kannan type contraction, so applying Theorem 2.2, we achieve the intended outcome. □

Now, we introduce a less restrictive concept instead of $a \in [0, +\infty)$. Let $k \in [0, +\infty)$, and define Q as the set of mappings $\phi\colon [0, +\infty) \to [0, +\infty)$ that satisfy,

(a) for each $t > 0$, $\phi(t) < t(k + 1)$;

(b) $t_n \to t \geqslant 0$ implies $\limsup_{n\to+\infty} \phi(t_n) \leqslant \phi(t)$, meaning that ϕ is upper semicontinuous.

Moreover, a new class of function $\mathcal{F}$ is defined as $F\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ so that:
(c) for all $t \geqslant 0$, $F(tu, tv) = tF(u, v)$ and $F(0, 0) = 0$,
(d) F is continuous at the point $(0, 0)$.

Our third main result is given as follows.

Theorem 2.3. *Consider a Banach space* $(U, \|\cdot\|)$ *and a mapping* $T\colon U \to U$ *for which one can find* $\phi \in Q$, $\psi \in \mho^*$, $0 \leqslant k < +\infty$, *and* $F \in \mathcal{F}$ *so that*

$$\Big\|\frac{u\psi(u) + kTu}{k + \psi(u)} - \frac{v\psi(v) + kTv}{k + \psi(v)}\Big\| \leqslant \phi(\|u - v\|) + F\Big(\Big|\frac{k}{k + \psi(u)}\Big|\|u - Tu\|, \Big|\frac{k}{k + \psi(v)}\Big|\cdot\|v - Tv\|\Big),$$

for each $u, v \in U$ *and* $T_{\frac{k}{k+\psi(u)}}$ *is an asymptotically regular mapping. Then* T *has a unique fixed point* $u^* \in U$ *if it is continuous and as* $n \to +\infty$, *the sequence given by* $u_{n+1} = T_{\frac{k}{k+\psi(u)}} u_n, n = 0, 1, 2, \dots$, *for each* $u_0 \in U$, *converges to* u^*.

Proof. We will consider the following two cases:
Case I: If $\psi(u) = 0$, then the proof is easy.
Case II: Let $\psi(u) > 0$ and let us denote $\lambda(u) = \frac{k}{k+\psi(u)} \in (0, 1)$. Analogically as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}\|T_\lambda u - T_\lambda v\| &\leqslant \phi(\|u - v\|) + F\Big(\Big|\frac{k}{k + \psi(u)}\Big|\cdot\|u - Tu\|, \Big|\frac{k}{k + \psi(v)}\Big|\cdot\|v - Tv\|\Big)\\ &\leqslant \phi(\|u - v\|) + F(\lambda(u)\|u - T_\lambda u\|, \lambda(v)\|v - T_\lambda v\|). \qquad (2.6)\end{aligned}$$

Consider the initial point $u_0 \in U$, and define the sequence $\{u_n\}$ by $u_{n+1} = T_\lambda u_n$ for $n = 0, 1, 2, \dots$. If there exists some $n_0 \in \mathbb{N}$ such that $u_{n_0+1} = u_{n_0}$, it follows that $(1 - \lambda)u_{n_0} + \lambda TU_{n_0} = u_{n_0}$, which implies that $Tu_{n_0} = u_{n_0}$, meaning that u_{n_0} is a fixed point of T. So $u_{n+1} \neq u_n$ holds for every $n \geqslant 0$, and assuming the sequence $\{u_n\}$ is not a Cauchy sequence, there exist a positive number ϵ and sequences of integers $\{n_i\}$ and $\{m_i\}$ in $\mathbb{N}$ with $m_i > n_i \geqslant i$ so that $\|u_{n_i} - u_{m_i}\| \geqslant \epsilon$ for $i = 1, 2, \dots$. We can choose m_i to be as small as possible $\|u_{n_i} - u_{m_i-1}\| < \epsilon$. Taking into account the triangle inequality, we have

$$\epsilon \leqslant \|u_{n_i} - u_{m_i}\| \leqslant \|u_{n_i} - u_{m_i-1}\| + \|u_{m_i-1} - u_{m_i}\| \leqslant \epsilon + \|u_{m_i-1} - u_{m_i}\|,$$

for every $i \in \mathbb{N}$. Using the asymptotic regularity condition, it follows that

$$\lim_{i\to+\infty} \|u_{n_i} - u_{m_i}\| = \epsilon.$$

Following (2.6), observe that for every i, it holds that

$$\begin{aligned}\|u_{n_i} - u_{m_i}\| &\leqslant \|u_{n_i} - u_{n_i+1}\| + \|u_{m_i} - u_{m_i+1}\| + \|u_{n_i+1} - u_{m_i+1}\|\\ &\leqslant \|u_{n_i} - u_{n_i+1}\| + \|u_{m_i} - u_{m_i+1}\| + \phi(\|u_{n_i} - v_{m_i}\|)\\ &\quad + F(\lambda(u_{n_i})\|u_{n_i} - T_\lambda u_{n_i}\|, \lambda(v_{n_i})\|v_{n_i} - T_\lambda v_{n_i}\|).\end{aligned}$$

Now, using the properties of F, the upper semi-continuity of ϕ, the asymptotic regularity and taking $i \to +\infty$, we obtain

$$0 < \epsilon = \lim_{i\to+\infty} \|u_{n_i} - u_{m_i}\| \leqslant \limsup_{i\to+\infty} \phi(\|u_{n_i} - v_{m_i}\|) \leqslant \phi(\epsilon) < \epsilon$$

which leads to a contradiction. Thus, $\{u_n\}$ is a Cauchy sequence. Following the lines of proof of Theorem 2.2, we reach the desired conclusion. □

Example 2.2. Consider $U = [0, 2]$ along with the Euclidean norm $\|\cdot\|$. Let $T\colon U \to U$ be defined as $Tu = 4u - u^3/8$, for all $u \in [0, 1]$. Also take $\psi(u) = \frac{u^2}{2}, k = 2$, $\lambda(u) = \frac{4}{4+u^2}$, $\phi(t) = k{\cdot}t$ and $F(u, v) = \frac{u+v}{2}$. So, the contraction condition of Theorem 2.3 which is given by

$$\Big\|\frac{u\psi(u) + kTu}{k + \psi(u)} - \frac{v\psi(v) + kTv}{k + \psi(v)}\Big\| \leqslant \phi(\|u - v\|) + F\Big(\Big|\frac{k}{k + \psi(u)}\Big|{\cdot}\|u - Tu\|, \Big|\frac{k}{k + \psi(v)}\Big|{\cdot}\|v - Tv\|\Big),$$

becomes

$$\Big\|\frac{u}{2} - \frac{v}{2}\Big\| \leqslant 2(\|u - v\|) + F\Big(\Big|\frac{4}{4 + u^2}\Big|{\cdot}\Big\|\frac{u}{2}\Big\|, \Big|\frac{4}{4 + v^2}\Big|{\cdot}\Big\|\frac{v}{2}\Big\|\Big).$$

That is,

$$\frac{1}{2}\|u - v\| \leqslant 2(\|u - v\|) + \frac{2}{4 + u^2}{\cdot}\|u\| + \frac{2}{4 + v^2}{\cdot}\|v\|,$$

which it is satisfied for all $u, v \in U$. Thus, T is a (ψ, a, k)-SM-Bianchini type contraction. From Theorem 2.3, T possesses a unique fixed point, which is 0.

Let B be a class of functions containing $\beta\colon [0, +\infty) \to [0, +\infty)$ such that $\limsup_{t\to 0} \beta(t) < \infty$.

Corollary 2.4. *Consider $(U, \|\cdot\|)$ as a Banach space and a mapping $T\colon U \to U$ for which there exist $\phi \in Q$, $\psi \in \mho^*$, $0 \leqslant k < +\infty$, and $\beta_1, \beta_2 \in B$ such that*

$$\Big\|\frac{u\psi(u) + kTu}{k + \psi(u)} - \frac{v\psi(v) + kTv}{k + \psi(v)}\Big\| \leqslant \phi(\|u - v\|) + \beta_1\Big(\frac{k}{k + \psi(u)}\|u - Tu\|\Big)(\|u - Tu\|) + \beta_2\Big(\frac{k}{k + \psi(v)}\|v - Tv\|\Big)(\|v - Tv\|),$$

for all $u, v \in U$ and $T_{\frac{k}{k+\psi}}$ is an asymptotically regular mapping. If T is a continuous mapping, then T possesses a unique fixed point $u^* \in U$ and for every $u_0 \in U$, the sequence generated by $u_{n+1} = T_{\frac{k}{k+\psi}} u_n, n = 0, 1, 2, \ldots,$ converges to u^* as $n \to +\infty$.

Proof. Suppose $F(u, v) = \beta_1\big(\frac{ku}{k+\psi(u)}\big)u + \beta_2\big(\frac{kv}{k+\psi(v)}\big)v$. So $F(0, 0) = 0$ and $\lim_{u,v\to 0} F(u, v) = 0$ as $\lim_{t\to 0} \sup \beta_i(t) < +\infty$ for $i = 1, 2$. Hence, the required outcome is obtained from Theorem 2.3. □

3. Application to split feasibility problem

In 1994, Censor and Elfving [**11**] introduced the split feasibility problem (SFP). It has significant implications in various fields including signal processing, image reconstruction, and optimization problems. The ability to solve such problems efficiently has practical importance, especially when dealing with real-world applications where constraints are often split across different domains or spaces. It is described as follows:
(A): Find $v^* \in V$ such that $Av^* \in W$, where V and W are non-empty, convex and closed subsets within the Hilbert spaces H_1 and H_2, respectively, and T is a bounded operator mapping H_1 to H_2. Suppose that the SFP is consistent, that is, it possesses a solution, and denote the solution set by S. It can be shown [**16, 20**] that $v^* \in V$ solves that SFP if an only if it solves the following fixed point problem:

$$v = P_V(I - \gamma S^*(I - P_W)S)v.$$

Here, P_V and P_W represent the nearest point projections onto V and W, respectively, with $\gamma > 0$, and S^* is the adjoint operator of S. Byrne [**10**] showed that if $\gamma \in (0, \frac{2}{\delta})$ and δ is the spectral radius of S^*S, then the operator

$$T = P_V(I - \gamma S^*(I - P_W)S)$$

is both averaged and nonexpansive, and the iterative method

$$v_{n+1} = P_V(I - \gamma S^*(I - P_W)S)v_n, n \geqslant 0,$$

converges weakly to a solution of the SFP.

Theorem 3.1. *Consider that the SFP (A) possesses a solution, $\gamma \in (0, \frac{2}{\delta})$, and $P_V(I-\gamma S^*(I-P_W)S)$ is a (ψ, a, k)-MRB-Kannan type contraction, then there exists $\lambda \in \Omega^*$ such that the iterative algorithm $\{v_n\}$ defined by*

$$v_{n+1} = (1 - \lambda(u))v_n + \lambda(v)P_V(I - \gamma S^*(I - P_W)S)v_n, \quad n \geqslant 0,$$

converges strongly to the unique solution v^* of the SFP (A), for any $v_0 \in V$.

Proof. As V is closed, we consider $U = V$ and $T = P_V(I - \gamma S^*(I - P_W)S)$ and implement Corollary 2.1. □

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