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A STEINHAUS-TYPE THEOREM FOR MULTI-DIMENSIONAL MATRIX TRANSFORMATIONS IN $(\mathcal{L}_1, \mathcal{L}_1)$

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Abstract. We prove a Steinhaus-type theorem for four-dimensional matrix transformations of double sequences, establishing that $(\mathcal{L}_1, \mathcal{L}_1, P) \cap (\mathcal{L}_s, \mathcal{L}_1) = \emptyset$, $s > 1$. This extends Fridy's classical result for single sequences. Our results hold for sequences of bounded variation, bounded, Pringsheim convergent, bounded Pringsheim convergent, and regularly convergent sequence spaces. We show this theorem fails in non-archimedean fields through a counterexample.

1. Introduction

In the study of matrix transformations of double sequences, Steinhaus-type theorems establish the impossibility of certain matrix methods to simultaneously preserve different types of summability. Building on the framework developed by Maddox [**8**] and Natarajan [**9**, Chapter 8] for analyzing two-dimensional matrix transformations, we consider the relationship between different sequence spaces. Specifically, given double sequence spaces X, Y and Z where $Z \supset X$, these theorems explore conditions under which $(X, Y; P) \cap (Z, Y) = \emptyset$, with P denoting preservation properties. In the context of absolute summability matrices, Fridy [**6**] established a key result showing $(\ell_1, \ell_1; P) \cap (\ell_s, \ell_1) = \emptyset$ for $s > 1$, where P represents sum preservation. For double sequences, using Pringsheim convergence, Patterson in [**11**] extended Steinhaus's result to the multidimensional case. In this paper, we extend Fridy's results [**6**] to four-dimensional matrices transforming double sequences.

The structure of this paper is as follows: Section 2 provides the foundational definitions, concepts, and preliminary results, including characterizations of the classes $(\mathcal{L}_1, \mathcal{L}_1)$ and $(\mathcal{L}_1, \mathcal{L}_1; P)$. In Section 3, we prove our main Steinhaus-type theorem, demonstrate through a counterexample that these results do not extend to

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non-archimedean valued fields, and establish the empty intersection with sequences of bounded variation. In Section 4, we extend these results to spaces of bounded, Pringsheim convergent, bounded Pringsheim convergent, and regularly convergent double sequences.

2. Preliminaries and definitions

In this section, we begin by recalling some fundamental concepts about double sequences and their convergence properties.

Definition 2.1. [**13**] A double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an x more briefly as "P-convergent".

Definition 2.2. [**10**] For a double sequence $x = (x_{k,l})$, let $\alpha_n = \sup_{n\in\mathbb{N}}\{x_{k,l} : k, l \geqslant n\}$. The Pringsheim limit superior of x is defined as

(i) if $\alpha_n = \infty$ for each n, then $P\text{-}\limsup x = \infty$,
(ii) if $\alpha_n < \infty$ for each n, then $P\text{-}\limsup x = \inf_{n\in\mathbb{N}} \alpha_n$.

Let Ω denote the set of all complex valued double sequences. The space $\mathcal{L}_s$ of *absolutely s-summable double sequences*, introduced by Başar and Sever [**4**], is defined as

$$\mathcal{L}_s := \left\{x = (x_{k,l}) \in \Omega : \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}|^s < \infty\right\}; \qquad (0 < s < \infty).$$

The special case $s = 1$ yields the space $\mathcal{L}_1$ of *absolutely summable double sequences.* The space $\mathcal{BV}$ of *double sequences of bounded variation*, is defined as

$$\mathcal{BV} := \left\{x = (x_{k,l}) \in \Omega : \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}| < \infty\right\}.$$

The A-transform of a double sequence is defined as follows:

Definition 2.3. Let $A = (a_{m,n,k,l})_{m,n,k,l}$ be a four-dimensional infinite matrix. The A-transform of a double sequence $x = (x_{k,l})_{k,l}$ is the double sequence $Ax = ((Ax)_{m,n})$, where

$$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}, \quad m, n \geqslant 0$$

assuming that the double series on the right exists.

The reader can refer to the recent textbooks [**3**] and [**5**] on the various kinds of convergence of double sequences and the characterizations of certain classes of four-dimensional matrix transformations, and related topics.

The class of four-dimensional matrices that map $\mathcal{L}_1$ into itself is defined as follows:

Definition 2.4. We write $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1)$ if for every $x = (x_{k,l}) \in \mathcal{L}_1$, the double sequence $Ax = ((Ax)_{m,n})$ belongs to the space $\mathcal{L}_1$.

A particularly important subclass of $(\mathcal{L}_1, \mathcal{L}_1)$, denoted as $(\mathcal{L}_1, \mathcal{L}_1; P)$, which we define as

Definition 2.5. The class $(\mathcal{L}_1, \mathcal{L}_1; P)$ is defined as the set of all four-dimensional matrices $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1)$ that satisfy the additional property

$$\sum_{m,n=0,0}^{\infty,\infty} (Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} x_{k,l}$$

for all $x = (x_{k,l}) \in \mathcal{L}_1$.

We now present characterization theorems for the matrix classes $(\mathcal{L}_1, \mathcal{L}_1)$ and $(\mathcal{L}_1, \mathcal{L}_1; P)$.

Theorem 2.1. *[**12**, Theorem 6] A four-dimensional matrix* $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1)$ *if and only if there exists a positive constant* M_A *such that for each* k *and* l*,*

$$\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < M_A.$$

Theorem 2.2. *[**7**, Theorem 3.2] Let* $A = (a_{m,n,k,l})$ *be a four-dimensional matrix. Then* $A \in (\mathcal{L}_1, \mathcal{L}_1; P)$ *if and only if*

(i) *there exists* $M > 0$ *such that* $\sup_{k,l} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \leqslant M$*, and*
(ii) *for all* $k, l \geqslant 0$, $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$.

3. A Steinhaus type theorem for $(\mathcal{L}_1, \mathcal{L}_1)$

Following Fridy's approach for single sequences and using the characterizations of the matrix classes $(\mathcal{L}_1, \mathcal{L}_1)$ and $(\mathcal{L}_1, \mathcal{L}_1; P)$ given in Theorems 2.1 and 2.2, we investigate the intersections of various matrix classes that transform double sequences. Our results demonstrate the incompatibility of certain matrix properties in the four-dimensional case. The following result is needed.

Theorem 3.1. *Let* $A = (a_{m,n,k,l})$ *be a four-dimensional matrix in* $(\mathcal{L}_1, \mathcal{L}_1)$ *such that*

$$P\text{-}\limsup_{k,l\to\infty} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| > 0.$$

Then there exists a double sequence $x = (x_{k,l}) \in \mathcal{L}_s$*,* $s > 1$*, such that* $((Ax)_{m,n}) \notin \mathcal{L}_1$.

Proof. By the definition of P-limsup, let

$$\alpha_j = \sup_{k,l\geqslant j} \left\{ \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \right\}.$$

Then $\inf_{j\in\mathbb{N}} \alpha_j > 0$ and therefore, there exists $\varepsilon > 0$ such that $\inf_{j\in\mathbb{N}} \alpha_j \geqslant 2\varepsilon$. By hypothesis, for some $\varepsilon > 0$, there exists $(k(1), l(1))$ with $k(1), l(1) \geqslant 1$ such that

$$\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k(1),l(1)}| \geqslant 2\varepsilon.$$

Choose positive integers $m(1), n(1)$ such that

$$\sum_{m=m(1)+1}^{\infty} \sum_{n=n(1)+1}^{\infty} |a_{m,n,k(1),l(1)}| < \min\left\{\frac{1}{2}, \frac{\varepsilon}{6}\right\},$$
$$\sum_{m=0}^{m(1)} \sum_{n=n(1)+1}^{\infty} |a_{m,n,k(1),l(1)}| < \min\left\{\frac{1}{2}, \frac{\varepsilon}{6}\right\},$$
$$\sum_{m=m(1)+1}^{\infty} \sum_{n=0}^{n(1)} |a_{m,n,k(1),l(1)}| < \min\left\{\frac{1}{2}, \frac{\varepsilon}{6}\right\}.$$

Thus granting us

$$\sum_{m=0}^{m(1)} \sum_{n=0}^{n(1)} |a_{m,n,k(1),l(1)}| > \varepsilon.$$

In general, having chosen the positive integer pairs $(k(r), l(t))$, $(m(r), n(t))$, for $r \leqslant i-1$, $t \leqslant j-1$, choose positive integers $k(i), l(j)$ with $k(i) > k(i-1)$ and $l(j) > l(j-1)$ such that

$$\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k(i),l(j)}| \geqslant 2\varepsilon,$$
$$\sum_{m=0}^{m(i-1)} \sum_{n=0}^{n(j-1)} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\}.$$

Then choose positive integers $m(i) > m(i-1)$ and $n(j) > n(j-1)$ such that

$$\sum_{m=m(i)+1}^{\infty} \sum_{n=n(j)+1}^{\infty} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\},$$
$$\sum_{m=0}^{m(i-1)} \sum_{n=n(j-1)+1}^{n(j)} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\},$$
$$\sum_{m=m(i-1)+1}^{m(i)} \sum_{n=0}^{n(j-1)} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\},$$
$$\sum_{m=m(i)+1}^{\infty} \sum_{n=0}^{n(j-1)} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\},$$

$$\sum_{m=m(i)+1}^{\infty} \sum_{n=n(j-1)+1}^{n(j)} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\},$$

$$\sum_{m=0}^{m(i-1)} \sum_{n=n(j)}^{\infty} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\},$$

$$\sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j)}^{\infty} |a_{m,n,k(i),l(j)}| < \min\left\{\frac{1}{2^{i+j}}, \frac{\varepsilon}{8}\right\}.$$

Therefore,

$$\sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} |a_{m,n,k(i),l(j)}| > \varepsilon.$$

Let $0 < \rho \leqslant 1$. For each $i, j = 1, 2, \ldots$, choose non-negative integers $\lambda_1(i)$ and $\lambda_2(j)$ such that $\rho^{\lambda_1(i)+1} \leqslant \frac{1}{i} < \rho^{\lambda_1(i)}$ and $\rho^{\lambda_2(j)+1} \leqslant \frac{1}{j} < \rho^{\lambda_2(j)}$. Define the double sequence $x = (x_{k,l})$ by

$$x_{k,l} = \begin{cases} \rho^{\lambda_1(i)+\lambda_2(j)}, & \text{if } (k,l) = (k(i), l(j)) \text{ for some } i, j \geqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

The double sequence $x = (x_{k,l})$ is an element in $\mathcal{L}_s \setminus \mathcal{L}_1$ since

$$\begin{aligned} \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}|^s &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |x_{k(i),l(j)}|^s = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\rho^{\lambda_1(i)+\lambda_2(j)}|^s \\ &\leqslant \frac{1}{\rho^{2s}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(ij)^s} = \frac{1}{\rho^{2s}} \left(\sum_{i=1}^{\infty} \frac{1}{i^s}\right)\left(\sum_{j=1}^{\infty} \frac{1}{j^s}\right) < \infty \end{aligned}$$

for $s > 1$, while

$$\sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |x_{k(i),l(j)}| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho^{\lambda_1(i)+\lambda_2(j)} > \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij} = \infty.$$

Define $m(0) = 0$ and $n(0) = 0$. Now,

$$\begin{aligned} \sum_{m=0}^{m(N)} \sum_{n=0}^{n(N)} |(Ax)_{m,n}| &\geqslant \sum_{i=1}^{N}\sum_{j=1}^{N} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \left| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \right| \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \left| \sum_{r=1}^{\infty}\sum_{t=1}^{\infty} a_{m,n,k(r),l(t)} x_{k(r),l(t)} \right| \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \left| \sum_{r=1}^{\infty}\sum_{t=1}^{\infty} a_{m,n,k(r),l(t)} \rho^{\lambda_1(r)+\lambda_2(t)} \right| \end{aligned}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \Big| a_{m,n,k(i),l(j)}\rho^{\lambda_1(i)+\lambda_2(j)} + \sum_{(r,t)\neq(i,j)} a_{m,n,k(r),l(t)}\rho^{\lambda_1(r)+\lambda_2(t)}\Big|$$

$$\geqslant \sum_{i=1}^{N}\sum_{j=1}^{N} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \Big\{ |a_{m,n,k(i),l(j)}\rho^{\lambda_1(i)+\lambda_2(j)}| - \sum_{(r,t)\neq(i,j)} |a_{m,n,k(r),l(t)}\rho^{\lambda_1(r)+\lambda_2(t)}| \Big\}.$$

Using the bounds

$$\rho^{\lambda_1(i)+\lambda_2(j)} > \frac{1}{ij} \quad \text{and} \quad \rho^{\lambda_1(r)+\lambda_2(t)} \leqslant \frac{1}{\rho^2(rt)} \leqslant \frac{1}{\rho^2}$$

we obtain

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \Big\{ \frac{|a_{m,n,k(i),l(j)}|}{ij} - \frac{1}{\rho^2} \sum_{(r,t)\neq(i,j)} |a_{m,n,k(r),l(t)}| \Big\}.$$

From our construction, each fixed ordered pair (i,j) grants us

$$\sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} |a_{m,n,k(i),l(j)}| > \varepsilon.$$

Therefore

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \frac{|a_{m,n,k(i),l(j)}|}{ij} > \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\varepsilon}{ij}.$$

Let's analyze $\frac{1}{\rho^2}\sum_{(r,t)\neq(i,j)} |a_{m,n,k(r),l(t)}|$ by separating into our regions. Note that

$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \sum_{m=m(i1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \sum_{r<i}\sum_{t<j} |a_{m,n,k(r),l(t)}|$$
$$= \frac{1}{\rho^2}\sum_{r=1}^{\infty}\sum_{t=1}^{\infty} \sum_{m=m(r)+1}^{\infty} \sum_{n=n(t)+1}^{\infty} |a_{m,n,k(r),l(t)}|$$
$$\leqslant \frac{1}{\rho^2}\sum_{r=1}^{\infty}\sum_{t=1}^{\infty} \min\Big\{\frac{1}{2^{r+t}}, \frac{\varepsilon}{8}\Big\} \leqslant \frac{1}{\rho^2}\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{2^{r+t}} = \frac{1}{\rho^2},$$

$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \sum_{m=m(i-1)+1}^{m(i)} \sum_{n=n(j-1)+1}^{n(j)} \sum_{r>i}\sum_{t>j} |a_{m,n,k(r),l(t)}|$$

$$= \frac{1}{\rho^2}\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\sum_{m=1}^{m(s-1)}\sum_{n=1}^{n(t-1)}|a_{m,n,k(r),l(t)}|$$
$$\leqslant \frac{1}{\rho^2}\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\min\left\{\frac{1}{2^{r+t+1}},\frac{\varepsilon}{8}\right\} \leqslant \frac{1}{\rho^2}\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{2^{r+t+1}} = \frac{1}{2\rho^2}.$$

In a similar manner, the following inequalities hold

$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}\sum_{r\leqslant i}\sum_{t>j}|a_{m,n,k(r),l(t)}| \leqslant \frac{1}{2\rho^2},$$
$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}\sum_{r>i}\sum_{t\leqslant j}|a_{m,n,k(r),l(t)}| \leqslant \frac{1}{2\rho^2},$$
$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}\sum_{r=i}\sum_{t>j}|a_{m,n,k(r),l(t)}| \leqslant \frac{1}{2\rho^2},$$
$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}\sum_{r>i}\sum_{t=j}|a_{m,n,k(r),l(t)}| \leqslant \frac{1}{2\rho^2},$$
$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}\sum_{r<i}\sum_{t=j}|a_{m,n,k(r),l(t)}| \leqslant \frac{1}{2\rho^2},$$
$$\frac{1}{\rho^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}\sum_{r=i}\sum_{t<j}|a_{m,n,k(r),l(t)}| \leqslant \frac{1}{2\rho^2}.$$

Thus, from the above bounds we are granted

$$\sum_{m=0}^{m(N)}\sum_{n=0}^{n(N)}|(Ax)_{m,n}| \geqslant \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}\left\{\frac{|a_{m,n,k(i),l(j)}|}{ij}-\frac{7}{2\rho^2}\right\}.$$

From our construction

$$\sum_{m=m(i-1)+1}^{m(i)}\sum_{n=n(j-1)+1}^{n(j)}|a_{m,n,k(i),l(j)}| > \varepsilon.$$

Therefore

$$\sum_{m=0}^{m(N)}\sum_{n=0}^{n(N)}|(Ax)_{m,n}| > \varepsilon\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{ij}-\frac{7}{2}$$

and since $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij} = \infty$, it follows that $((Ax)_{m,n}) \notin \mathcal{L}_1$, completing the proof. □

We now prove the following Steihaus-type theorem.

Theorem 3.2. *For four-dimensional matrices, the following equality holds*

$$(\mathcal{L}_1, \mathcal{L}_1; P) \cap (\mathcal{L}_s, \mathcal{L}_1) = \emptyset, \quad 1 < s < \infty.$$

Proof. Let $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1; P) \cap (\mathcal{L}_s, \mathcal{L}_1)$, $1 < s < \infty$. Since $A \in (\mathcal{L}_1, \mathcal{L}_1; P)$, then $\sup_{k,l} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < \infty$ and $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$ for all $k, l \geqslant 0$. From these conditions, we can state

$$\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \geqslant \left| \sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} \right| = 1, \quad k, l = 0, 1, 2, \ldots$$

This implies $P\text{-}\limsup_{k,l\to\infty} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \geqslant 1 > 0$. By Theorem 3.1, there exists a double sequence $x = (x_{k,l}) \in \mathcal{L}_s$ such that $((Ax)_{m,n}) \notin \mathcal{L}_1$. However, since $A \in (\mathcal{L}_s, \mathcal{L}_1)$, we should have $((Ax)_{m,n}) \in \mathcal{L}_1$ for all $x \in \mathcal{L}_s$. This contradiction proves that $(\mathcal{L}_1, \mathcal{L}_1; P) \cap (\mathcal{L}_s, \mathcal{L}_1) = \emptyset$ for $1 < s < \infty$. □

Remark 3.1. The result in Theorem 3.2 does not directly extend to the case where the field K is a complete, non-trivially valued, non-archimedean field. Before presenting Example 3.1 to illustrate this fact, it is crucial to understand that for any prime number p, we can construct a complete, non-trivially valued, non-archimedean field $\mathbb{Q}_p$ called the p-adic numbers. Each such field $\mathbb{Q}_p$ comes equipped with a p-adic absolute value $|\cdot|_p$ that satisfies not just the usual properties (positivity, multiplicativity, and triangle inequality), but also the stronger ultrametric inequality: $|x + y|_p \leqslant \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}_p$. In $\mathbb{Q}_p$, the absolute value measures "divisibility by the prime p", so numbers with more factors of p have smaller absolute value. For example, in $\mathbb{Q}_3$, we have $|3|_3 = \frac{1}{3}$ (since 3 has one factor of 3), and $\left|\frac{1}{16}\right|_3 = 1$ (since neither 1 nor 16 contain any factors of 3). For a comprehensive introduction to p-adic numbers and their properties, we refer the reader to [**2**].

Example 3.1. Let $K = \mathbb{Q}_3$ (the 3-adic field) and define a four-dimensional matrix $A = (a_{m,n,k,l})$ where $a_{m,n,k,l} = \frac{1}{16}\left(\frac{3}{4}\right)^{m+n}$, $m, n, k, l = 0, 1, 2, \ldots$ First, we verify that $A \in (\mathcal{L}_1, \mathcal{L}_1; P)$: (i) For any fixed k, l,

$$\begin{aligned}
\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}|_3 &= \sum_{m,n=0,0}^{\infty,\infty} \left| \frac{1}{16}\left(\frac{3}{4}\right)^{m+n} \right|_3 \\
&= \left|\frac{1}{16}\right|_3 \sum_{m,n=0,0}^{\infty,\infty} \left|\frac{3}{4}\right|_3^{m+n} \\
&= \sum_{m,n=0,0}^{\infty,\infty} \lambda^{m+n} \quad \text{where} \quad \lambda = |3|_3 = \frac{1}{3} \\
&= \frac{1}{(1-\lambda)^2} < \infty.
\end{aligned}$$

(ii) For all $k,l \geqslant 0$,

$$\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = \frac{1}{16}\sum_{m,n=0,0}^{\infty,\infty}\Big(\frac{3}{4}\Big)^{m+n} = 1.$$

Now, for $s > 1$, we verify that $A \in (\mathcal{L}_s, \mathcal{L}_1)$. Let $x = (x_{k,l}) \in \mathcal{L}_s$. Then

$$\begin{aligned}\sum_{m,n=0,0}^{\infty,\infty} |(Ax)_{m,n}|_3 &= \sum_{m,n=0,0}^{\infty,\infty}\Big|\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}\Big|_3\\ &= \sum_{m,n=0,0}^{\infty,\infty}\Big|\frac{1}{16}\Big(\frac{3}{4}\Big)^{m+n}\Big|_3\Big|\sum_{k,l=0,0}^{\infty,\infty} x_{k,l}\Big|_3\\ &= \Big|\sum_{k,l=0,0}^{\infty,\infty} x_{k,l}\Big|_3 \cdot \frac{1}{(1-\lambda)^2} < \infty.\end{aligned}$$

The last inequality holds because: (1) $(x_{k,l}) \in \mathcal{L}_s$ implies that $P - \lim x_{kl} = 0$. (2) Since $K = \mathbb{Q}_3$ is complete, $\sum_{k,l=0,0}^{\infty,\infty} x_{k,l}$ converges, and (3) $\lambda = |3|_3 = \frac{1}{3} < 1$ ensures the convergence of the double geometric series. Therefore,

$$A \in (\mathcal{L}_1, \mathcal{L}_1; P) \cap (\mathcal{L}_s, \mathcal{L}_1),$$

showing that this intersection is non-empty in the four-dimensional case over $\mathbb{Q}_3$.

In view of Theorem 3.2, we have the following Steinhaus-type theorem.

THEOREM 3.3. *For four-dimensional matrices, the following equality holds*

$$(\mathcal{L}_1, \mathcal{L}_1; P) \cap (\mathcal{BV}, \mathcal{L}_1) = \emptyset.$$

PROOF. Let $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1; P) \cap (\mathcal{BV}, \mathcal{L}_1)$. Since $A \in (\mathcal{L}_1, \mathcal{L}_1; P)$, then $\sup_{k,l}\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < \infty$ and $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$ for all $k,l \geqslant 0$. Consider the double sequence $x = (x_{k,l})$ defined by $x_{k,l} = 1$ for all $k,l \geqslant 0$. Then $x \in \mathcal{BV}$ since

$$\sum_{k,l=0,0}^{\infty,\infty} |x_{k,l} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}| = 0 < \infty.$$

Since $A \in (\mathcal{BV}, \mathcal{L}_1)$, the A-transform of x, defined as

$$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}$$

must be in $\mathcal{L}_1$. However, since $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$ for all $k,l \geqslant 0$, this implies

$$\sum_{m,n=0,0}^{\infty,\infty} |(Ax)_{m,n}| = \sum_{m,n=0,0}^{\infty,\infty}\Big|\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\Big| = \infty.$$

This contradicts $Ax \in \mathcal{L}_1$, proving that $(\mathcal{L}_1, \mathcal{L}_1; P) \cap (\mathcal{BV}, \mathcal{L}_1) = \emptyset$. □

4. Extensions to other sequence spaces

In this section, we extend our Steinhaus-type theorems to other double sequence spaces characterized by different kinds of convergence. We show that the impossibility results established in Section 3 naturally extend to spaces of bounded, Pringsheim convergent, bounded Pringsheim convergent, and regularly convergent sequences. The necessary definitions and Theorem 4.1 that follow are established in [**1**].

DEFINITION 4.1. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u := \Big\{x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty\Big\}.$$

DEFINITION 4.2 ($\mathcal{C}_p$-Pringsheim convergent sequences). Let $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $L \in \mathbb{C}$ such that $|x_{mn} - L| < \varepsilon$ for all $m, n > n_0$, then we call that the double sequence x is convergent in Pringsheim's sense to the limit L and write $P - \lim x_{mn} = L$, where $\mathbb{C}$ denotes the complex field. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in Pringsheim's sense.

DEFINITION 4.3 ($\mathcal{C}_{bp}$-bounded Pringsheim convergent sequences). The space $\mathcal{C}_{bp}$ consists of all double sequences which are both convergent in Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$ where $\mathcal{M}_u$ is the space of bounded double sequences.

DEFINITION 4.4 ($\mathcal{C}_r$-regularly convergent sequences). A sequence in the space $\mathcal{C}_p$ is said to be regularly convergent if it is a single convergent sequence with respect to each index. We denote the set of all such sequences by $\mathcal{C}_r$.

The relationship between these sequence spaces and the space $\mathcal{BV}$ of sequences of bounded variation is particularly interesting, as shown in the following fundamental result.

THEOREM 4.1. *[**1**] Let $\vartheta \in \{p, bp, r\}$. Then, the inclusions $\mathcal{BV} \subset \mathcal{C}_\vartheta$ and $\mathcal{BV} \subset \mathcal{M}_u$ strictly hold.*

Using the inclusion relationships established in Theorem 4.1, we obtain analogous results about the emptiness of intersections $(\mathcal{L}_1, \mathcal{L}_1; P) \cap (X, \mathcal{L}_1)$ for these broader sequence spaces X.

COROLLARY 4.1. *Since $\mathcal{BV} \subset \mathcal{M}_u$ and $\mathcal{BV} \subset \mathcal{C}_\vartheta$ where $\vartheta \in \{p, bp, r\}$, we have $(\mathcal{M}_u, \mathcal{L}_1) \subset (\mathcal{BV}, \mathcal{L}_1)$, and $(\mathcal{C}_\vartheta, \mathcal{L}_1) \subset (\mathcal{BV}, \mathcal{L}_1)$. In view of Theorem* 3.3*, we have $(\mathcal{L}_1, \mathcal{L}_1; P) \cap (X, \mathcal{L}_1) = \emptyset$ when $X = \mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$.*

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