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TOPOLOGICALLY MULTIPLY $\mathcal{F}$-RECURRENT OPERATORS IN THE CONTEXT OF LOCALLY COMPACT HYPERGROUPS

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Abstract. Here we introduce topologically multiply $\mathcal{F}$-recurrent collections $\{T_t\}_{t\in S}$ of bounded linear operators on a Banach space X, where $\mathcal{F}$ is a Furstenberg family for a semigroup S. Then, we give some equivalent conditions for a collection $\{T_{a_t,w_t}\}_{t\in S}$ of weighted translation operators on an Orlicz space in the context of a locally compact hypergroup to be topologically multiply $\mathcal{F}$-recurrent.

1. Introduction and preliminaries

Recall that if X is a Banach space, an operator T on X is called *topologically multiply recurrent* if for each $N \in \mathbb{N}$ and every nonempty open set $\mathcal{O}$ in X there is an $m \in \mathbb{N}$ with

$$\mathcal{O} \cap T^{-m}(\mathcal{O}) \cap T^{-2m}(\mathcal{O}) \cap \cdots \cap T^{-Nm}(\mathcal{O}) \neq \emptyset,$$

where for any $k \in \mathbb{N}$ and $A \subseteq X$, $T^{-k}(A)$ denotes the inverse image of A by T^k, and $T^k := T \circ T \circ \cdots \circ T$ (k-times). If this condition holds for $N = 1$, then T is called *recurrent*; see [**17**] for more details. An operator T acting on a Banach space X is called *topologically transitive* whenever for any two nonempty open sets $\mathcal{O}, \mathcal{V} \subset X$, there is an $n \in \mathbb{N}$ such that $T^{-n}(\mathcal{O}) \cap \mathcal{V} \neq \emptyset$. Any topologically transitive operator on a separable Banach space X is recurrent. Recently, topologically transitive weighted translations on Lebesgue spaces and Orlicz spaces were studied in [**13–15**]. See also [**16, 25, 27**] for a characterization of topologically transitive weighted translations on a general Banach function space in the context of a measure space.

In [**8**, Proposition 5.3], Costakis and Parissis characterized topologically multiply recurrent bilateral weighted shifts. Moreover, Chen [**9**] explored topologically multiply recurrent weighted translations on L^p-spaces over locally compact groups.

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Furthermore, in [**10**], he characterized topologically multiply recurrent weighted translations on Orlicz spaces over locally compact groups.

Recently, in the motivating article [**11**] Chen gave an equivalent condition for a weighted translation operator $T_{a,w}$ on an Orlicz space in the context of a locally compact group G to be topologically multiple recurrent, where $a \in G$ is an aperiodic element and w is a bounded weight on G. Note that the Orlicz spaces are important generalization of the Lebesgue spaces. Also, aperiodicity of the fixed element $a \in G$ plays a key role in the proof of main result of the article [**11**].

On the other hand, the concepts $\mathcal{F}$-transitivity and d$\mathcal{F}$-transitivity, which recently were introduced in [**4**, **19**, **21**] are interesting generalizations of the usual topologically transitivity and disjoint topologically transitivity [**3**, **5**] related to a Furstenberg family $\mathcal{F}$. Recall that a collection $\{T_t\}_{t\in S}$ of operators on a Banach space X is called *$\mathcal{F}$-transitive* if for each nonempty open sets $\mathcal{O}, \mathcal{V} \subseteq X$,

$$\{t \in S : \mathcal{O} \cap T_t^{-1}(\mathcal{V}) \neq \emptyset\} \in \mathcal{F}.$$

The concept of topological $\mathcal{F}$-recurrence was defined in [**7**], where $\mathcal{F}$ is a Furstenberg family of subsets of $\mathbb{N}_0$, and it was also studied in [**1**, **2**].

In this paper, we first initiate the concept of topological multiply $\mathcal{F}$-recurrence for a collection $\{T_t\}_{t\in S}$ of bounded linear operators on a Banach space X, where $\mathcal{F}$ is a Furstenberg family of subsets of a semigroup S (see Definition 2.2). This notion would be a generalization of the current concept of topological multiple recurrence. Also, we introduce $\mathcal{F}$-aperiodic and strongly $\mathcal{F}$-aperiodic subsets $\{a_t\}_{t\in S}$ of a hypergroup $\mathcal{K}$, where S is a semigroup. Then, by some technical proofs, we give equivalent conditions for a collection $\{T_{a_t,w_t}\}_{t\in S}$ to be topologically multiply $\mathcal{F}$-recurrent, where any T_{a_t,w_t} is a weighted translation operator on an Orlicz space in the context of a *locally compact hypergroup*. So, in this work, we improve the main result of [**11**] in two directions: the context structure will be replaced by hypergroups which are extensions of locally compact groups, and also the usual notion topologically multiply recurrence will be replaced by the more general concept of topological multiply $\mathcal{F}$-recurrence. Hence, although the main results are given for the hypergroup case, they would be novel for the group case too.

For the convenience of readers, in three subsections, we recall some basics regarding locally compact hypergroups, Orlicz spaces, and weighted translations on an Orlicz space in the context of a locally compact hypergroup.

1.1. Locally compact hypergroups. Locally compact hypergroups as generalizations of locally compact groups were introduced in [**18**, **20**, **26**]; see also [**6**] as a monograph for hypergroups. In contrast of the group case, necessarily there is no action between elements of a hypergroup while there exists some convolution among the regular measures of it.

We denote the space of all complex Radon measures on a locally compact Hausdorff space $\mathcal{X}$ by $\mathcal{M}(\mathcal{X})$. Also, the set of all nonnegative measures in $\mathcal{M}(\mathcal{X})$ by $\mathcal{M}^+(\mathcal{X})$. The support of each $\mu \in \mathcal{M}(\mathcal{X})$ is denoted by $\mathrm{supp}(\mu)$, and δ_x denotes the Dirac measure at a point $x \in \mathcal{X}$.

Definition 1.1. A locally compact Hausdorff space $\mathcal{K}$ equipped with a *convolution* product $*$ on $\mathcal{M}(\mathcal{K})$ and an *involution* $x \mapsto \check{x}$ is called a *locally compact hypergroup* if:

(1) $(\mathcal{M}(\mathcal{K}), +, *)$ is a Banach algebra.
(2) For each $x, y \in \mathcal{K}$, $\delta_x * \delta_y$ is a probability measure with compact support.
(3) The function $(x, y) \mapsto \delta_x * \delta_y$ from $\mathcal{K} \times \mathcal{K}$ into $\mathcal{M}^+(\mathcal{K})$ is continuous, where $\mathcal{M}^+(\mathcal{K})$ is equipped with the cone topology.
(4) The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $\mathcal{K} \times \mathcal{K}$ into the family of all nonempty compact subsets of $\mathcal{K}$, $\mathbf{C}(\mathcal{K})$, is continuous, where $\mathbf{C}(\mathcal{K})$ is equipped with the Michael topology.
(5) The involution $x \mapsto \check{x}$ is an involutive homeomorphism from $\mathcal{K}$ onto $\mathcal{K}$ such that for each $x, y \in \mathcal{K}$, $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$.
(6) There exists an element $e \in \mathcal{K}$ (called *identity*) such that for each $x \in \mathcal{K}$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$. Moreover, for each $x, y \in \mathcal{K}$, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

Any locally compact group, equipped with the usual convolution and the inverse mapping as involution, is a hypergroup.

A nonzero nonnegative regular measure λ on a hypergroup $\mathcal{K}$ is called a *(left) Haar measure* if for each $x \in \mathcal{K}$, $\delta_x * \lambda = \lambda$.

Every compact or commutative hypergroup admits a (left) Haar measure (see [**20**] and [**26**]).

In the sequel, $\mathcal{K}$ denotes a locally compact hypergroup and λ is a left Haar measure on $\mathcal{K}$.

The convolution of two subsets $A, B \subseteq \mathcal{K}$ is defined by

$$A * B := \bigcup_{x \in A,\, y \in B} \operatorname{supp}(\delta_x * \delta_y),$$

and also $\check{A} := \{\check{x} : x \in A\}$. For every $a \in \mathcal{K}$ we simply denote $a * A := \{a\} * A$.

The *center* of $\mathcal{K}$ is defined by

$$\operatorname{Ma}(\mathcal{K}) := \{x \in \mathcal{K} :\ \delta_x * \delta_{\check{x}} = \delta_{\check{x}} * \delta_x = \delta_e\},$$

which is the maximal subgroup of $\mathcal{K}$. For every $x \in \operatorname{Ma}(\mathcal{K})$ and $y \in \mathcal{K}$, $\delta_x * \delta_y$ is a Dirac measure; see [**20**, Section 10.4] and [**24**].

1.2. Orlicz spaces on hypergroups.

Definition 1.2. Let Φ be a Young function. The set of all measurable functions $f\colon \mathcal{K} \to \mathbb{C}$ such that for some $c > 0$ we have $\int_{\mathcal{K}} \Phi(c\,|f|)\, d\lambda < \infty$, is denoted by $L^\Phi(\mathcal{K})$, and is called the *Orlicz space*. Two elements of $L^\Phi(\mathcal{K})$ are considered the same if they are equal almost everywhere. The Luxembourg norm of any $f \in L^\Phi(\mathcal{K})$ is defined by

$$\|f\|_\Phi := \inf\left\{\alpha > 0 : \int_{\mathcal{K}} \Phi\Big(\frac{1}{\alpha}|f|\Big)\, d\lambda \leqslant 1\right\}.$$

By [**22**], $(L^{\Phi}(\mathcal{K}), \|\cdot\|_{\Phi})$ is a Banach space. See the monographs [**22, 23**] for more details and facts regarding Orlicz spaces. For each $r > 0$ and $f \in L^{\Phi}(\mathcal{K})$ we denote

$$B(f; r) := \{g \in L^{\Phi}(\mathcal{K}) : \|f - g\|_{\Phi} < r\}.$$

1.3. Weighted translations on $L^{\Phi}(\mathcal{K})$. Any bounded continuous function $w\colon \mathcal{K} \to (0, \infty)$ is called a *weight* on $\mathcal{K}$.

DEFINITION 1.3. Let Φ be a Young function, $a \in \mathcal{K}$ and w be a bounded weight on $\mathcal{K}$. We define the mapping $T_{a,w}\colon L^{\Phi}(\mathcal{K}) \to L^{\Phi}(\mathcal{K})$ by

$$T_{a,w}(f)(x) := w(x)\, f(a * x), \text{ where } f(a * x) := \int_{\mathcal{K}} f\, d(\delta_a * \delta_x).$$

Then, $T_{a,w}$ is a bounded linear operator on $L^{\Phi}(\mathcal{K})$ with $\|T_{a,w}\| \leqslant \|w\|_{\infty}$, and is called a *weighted translation operator*. If $\frac{1}{w}$ is also bounded, then $T_{a,w}$ is invertible and its inverse is denoted by $S_{a,w}$.

2. Main results

DEFINITION 2.1. Let S be a nonempty set, and $\mathcal{F}$ be a proper nonempty collection of subsets of S. Then, $\mathcal{F}$ is called a *Furstenberg family on* S, if for every $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$ we have $B \in \mathcal{F}$.

DEFINITION 2.2. Let S be a semigroup, $\mathcal{F}$ be a proper Furstenberg family on S, X be a Banach space, and $\{T_t\}_{t \in S}$ be a collection of bounded linear operators from X into X. Then, $\{T_t\}_{t \in S}$ is called *topologically multiply $\mathcal{F}$-recurrent* if for every nonempty open subset $\mathcal{O} \subseteq X$ and $n \in \mathbb{N}$ we have

$$\big\{t \in S : \mathcal{O} \cap T_t^{-1}(\mathcal{O}) \cap T_{t^2}^{-1}(\mathcal{O}) \cap \cdots \cap T_{t^n}^{-1}(\mathcal{O}) \neq \emptyset\big\} \in \mathcal{F}. \tag{2.1}$$

DEFINITION 2.3. Let G be a locally compact group. An element $a \in G$ is called *compact* if the subgroup of G generated by a is relatively compact. Any noncompact element of G is called *aperiodic*.

If G is a second countable locally compact group, an element $a \in G$ is aperiodic if and only if for every compact subset $F \subseteq G$ there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$, $a^n F \cap F = \emptyset$.

Note that if G is a noncompact locally compact group, then it contains some aperiodic elements because the set of all compact elements of a locally compact group G is a compact subset of G.

DEFINITION 2.4. Assume that $\mathcal{F}$ is a Furstenberg family on a semigroup S. Then, $\{a_t\}_{t \in S} \subseteq \mathcal{K}$ is called

1. *$\mathcal{F}$-aperiodic* if for every nonempty compact subset $E \subseteq \mathcal{K}$, $F \in \mathcal{F}$ and $n \in \mathbb{N}$, there exists some $t \in F$ such that $(a_{t^j} * E) \cap E = \emptyset$ for all $j = 1, 2, \ldots, n$.
2. *strongly $\mathcal{F}$-aperiodic* if for every nonempty compact subset $E \subseteq \mathcal{K}$, every sequence $(F_k)_{k \in \mathbb{N}}$ in $\mathcal{F}$ and every $n \in \mathbb{N}$, there exists some $t_k := t_{E,k,n} \in F_k$ such that $(a_{t_k^j} * E) \cap E = \emptyset$ for all $j = 1, 2, \ldots, n$, and also the set $\{t_k : k \in \mathbb{N}\}$ belongs to $\mathcal{F}$.

EXAMPLE 2.1. Assume that G is a locally compact group. Let $a \in G$ be an aperiodic element. Then, by [**12**, Lemma 2.1], for every $E \in \mathcal{C}(G)$ there is some $N(E) \in \mathbb{N}$ such that for each $n \geqslant N(E)$, $a^n E \cap E = \emptyset$, where $\mathcal{C}(G)$ is a collection of some nonempty compact subsets of G such that for every compact set $F \subseteq G$, there is an element $E \in \mathcal{C}(G)$ with $E \subseteq F$. The minimum of the numbers $N(E)$ satisfying the above condition is denoted by $\alpha(E)$. Put $B := \{\alpha(E) : E \in \mathcal{C}(G)\}$. Assume that $\mathcal{F}$ denoted the collection of all subsets A of $\mathbb{N}$ with $B \subseteq A$. Now, one can easily see that the sequence $\{a^n\}_{n=1}^{\infty}$ is $\mathcal{F}$-aperiodic.

EXAMPLE 2.2. Assume that $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ with $k_n > 2$, is a strictly increasing sequence. Thanks to the last remark, setting $\mathcal{C}(\mathbb{R})$ as the collection of all segments $[-k_n, k_n]$, where $n \in \mathbb{N}$, the sequence $\{2n\}_{n=1}^{\infty}$ is $\mathcal{F}$-aperiodic, where $\mathcal{F}$ is the family of all subsets E of $\mathbb{N}$ with $\{k_n + 1 : n \in \mathbb{N}\} \subseteq E$.

THEOREM 2.1. *Let $\mathcal{K}$ be a hypergroup equipped with a left Haar measure. Assume that Φ is a Young function. Let S be a semigroup and $\mathcal{F}$ be a proper Furstenberg family on S. Let $\{a_t\}_{t\in S}$ be an $\mathcal{F}$-aperiodic family of elements of* $\mathrm{Ma}(\mathcal{K})$ *and $\{w_t\}_{t\in S}$ be a collection of weights on $\mathcal{K}$ such that for every $t \in S$, $\|1/w_t\|_\infty < \infty$. For every $t \in S$, let T_t denote the weighted translation operator T_{a_t,w_t} on $L^\Phi(\mathcal{K})$. Then,* (i) $\Rightarrow$ (ii)*:*

- (i) *The collection $\{T_t\}_{t\in S}$ is topologically multiply $\mathcal{F}$-recurrent.*
- (ii) *For each $n \in \mathbb{N}$ and compact $E \subseteq \mathcal{K}$, there exist a sequence $(E_{k,n})_{k=1}^{\infty}$ of Borel sets and a collection $\{t_k\}_{k\in\mathbb{N}} \subseteq S$ such that for all $k \in \mathbb{N}$, $E_{k,n} \subseteq E$,*

$$\lim_{k\to\infty} \|\chi_{E\setminus E_{k,n}}\|_\Phi = 0,$$

and for any $j \in \{1, 2, \dots, n\}$,

$$\lim_{k\to\infty} \left\|\chi_{\check{a}_{t_k^j} * E_{k,n}}\, w_{t_k^j}\right\|_\Phi = \lim_{k\to\infty} \left\|\frac{\chi_{E_{k,n}}}{w_{t_k^j}}\right\|_\Phi = 0.$$

PROOF. Suppose that $\{T_t\}_{t\in S}$ is topologically multiply $\mathcal{F}$-recurrent. Let $E \subseteq \mathcal{K}$ be compact set. Then, we have $\chi_E \in L^\Phi(\mathcal{K})$. Fix $n \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$ we have

$$F_{k,n} := \{t \in S : \mathcal{V}_k \cap T_t^{-1}(\mathcal{V}_k) \cap T_{t^2}^{-1}(\mathcal{V}_k) \cap \cdots \cap T_{t^n}^{-1}(\mathcal{V}_k) \neq \emptyset\} \in \mathcal{F},$$

where $\mathcal{V}_k := B\big(\chi_E; \frac{1}{4^k}\big)$. Simply, we denote $F_k := F_{k,n}$. Hence, since $\{a_t\}_{t\in S} \subseteq \mathcal{K}$ is an $\mathcal{F}$-aperiodic collection, for every $k \in \mathbb{N}$ there is an element $t_k \in F_k$ such that $E \cap (a_{t_k^j} * E) = \emptyset$ for all $j = 1, 2, \dots, n$. For every $k \in \mathbb{N}$, since

$$\mathcal{V}_k \cap T_{t_k}^{-1}(\mathcal{V}_k) \cap T_{t_k^2}^{-1}(\mathcal{V}_k) \cap \cdots \cap T_{t_k^n}^{-1}(\mathcal{V}_k) \neq \emptyset,$$

there is some $f_{t_k} \in L^\Phi(\mathcal{K})$ such that

$$\|f_{t_k} - \chi_E\|_\Phi < \frac{1}{4^k} \quad \text{and} \quad \left\|T_{t_k^j}(f_{t_k}) - \chi_E\right\|_\Phi < \frac{1}{4^k}$$

for all $j = 1, 2, \dots, n$. For each $k \in \mathbb{N}$, we put $A_k := \big\{x \in E : |f_{t_k}(x) - 1| \geqslant \frac{1}{2^k}\big\}$. Then for every $x \in E \setminus A_k$, $|f_{t_k}(x) - 1| < \frac{1}{2^k}$. We claim that $\|\chi_{A_k}\|_\Phi < \frac{1}{2^k}$ for all

$k \in \mathbb{N}$. We have $0 \leqslant \frac{1}{2^k}.\chi_{A_k} \leqslant |f_{t_k} - 1|\chi_{A_k} \leqslant |f_{t_k} - \chi_E|$. Since $f_{t_k} - \chi_E \in L^\Phi(\mathcal{K})$,

$$\Big\|\frac{1}{2^k}\chi_{A_k}\Big\|_\Phi \leqslant \|f_{t_k} - \chi_E\|_\Phi < \frac{1}{4^k},$$

So $\frac{1}{2^k}\|\chi_{A_k}\|_\Phi < \frac{1}{4^k}$. Hence, $\|\chi_{A_k}\|_\Phi < \frac{1}{2^k}$. Now, for every $j = 1, 2, \dots, n$ and $k \in \mathbb{N}$, set

$$B_{j,k} := \Big\{x \in E : |T_{t_k^j}(f_{t_k})(x) - 1| \geqslant \frac{1}{2^k}\Big\}.$$

Then $|T_{t_k^j}(f_{t_k})(x)| > 1 - \frac{1}{2^k}$ $(x \in E \setminus B_{j,k})$ and

$$\Big\|\frac{1}{2^k}\chi_{B_{j,k}}\Big\|_\Phi \leqslant \big\|T_{t_k^j}(f_{t_k}) - \chi_E\big\|_\Phi < \frac{1}{4^k}.$$

This implies that $\|\chi_{B_{j,k}}\|_\Phi < \frac{1}{2^k}$. For each $k \in \mathbb{N}$, put

$$E_{k,n} := E \setminus (A_k \cup B_{1,k} \cup \cdots \cup B_{n,k}).$$

Then,

$$\begin{aligned}
\frac{1}{4^k} &\geqslant \big\|T_{t_k^j}(f_{t_k}) - \chi_E\big\|_\Phi \geqslant \big\|\chi_{\check{a}_{t_k^j} * E_{k,n}}\big(T_{t_k^j}(f_{t_k}) - \chi_E\big)\big\|_\Phi \\
&= \big\|\chi_{\check{a}_{t_k^j} * E_{k,n}} T_{t_k^j}(f_{t_k})\big\|_\Phi = \big\|\chi_{\check{a}_{t_k^j} * E_{k,n}} w_{t_k^j} f_{t_k}(a_{t_k^j} * \cdot)\big\|_\Phi \\
&= \big\|\chi_{\check{a}_{t_k^j} * E_{k,n}}(\check{a}_{t_k^j} * \cdot) w_{t_k^j}(\check{a}_{t_k^j} * \cdot) f_{t_k}\big\|_\Phi \\
&= \big\|\chi_{E_{k,n}} w_{t_k^j}(\check{a}_{t_k^j} * \cdot) f_{t_k}\big\|_\Phi \\
&\geqslant \Big\|\big(\chi_{E_{k,n}} w_{t_k^j}(\check{a}_{t_k^j} * \cdot)\big)\Big(1 - \frac{1}{2^k}\Big)\Big\|_\Phi \\
&= \Big(1 - \frac{1}{2^k}\Big)\big\|\chi_{\check{a}_{t_k^j} * E_{k,n}} w_{t_k^j}\big\|_\Phi.
\end{aligned}$$

So,

$$\big\|\chi_{\check{a}_{t_k^j} * E_{k,n}}\ w_{t_k^j}\big\|_\Phi \leqslant \frac{1/4^k}{1 - 1/2^k}.$$

This implies that $\lim_{k\to\infty} \chi_{\check{a}_{t_k^j} * E_{k,n}}\ w_{t_k^j} = 0$ in $L^\Phi(\mathcal{K})$. For every $t \in S$, let S_t denote S_{a_t, w_t}. Similarly, for each $k \in \mathbb{N}$ and $j = 1, 2, \dots, n$,

$$\begin{aligned}
\frac{1}{4^k} &\geqslant \|f_{t_k} - \chi_E\|_\Phi \\
&= \big\|S_{t_k^j}\big(T_{t_k^j} f_{t_k}\big) - \chi_E\big\|_\Phi = \Big\|\frac{1}{w_{t_k^j}(\check{a}_{t_k^j} * \cdot)} T_{t_k^j} f_{t_k}(\check{a}_{t_k^j} * \cdot) - \chi_E\Big\|_\Phi \\
&\geqslant \Big\|\chi_{a_{t_k^j} * E_{k,n}}\Big[\frac{1}{w_{t_k^j}(\check{a}_{t_k^j} * \cdot)} T_{t_k^j} f_{t_k}(\check{a}_{t_k^j} * \cdot) - \chi_E\Big]\Big\|_\Phi \\
&= \Big\|\frac{\chi_{a_{t_k^j} * E_{k,n}}}{w_{t_k^j}(\check{a}_{t_k^j} * \cdot)}\ T_{t_k^j} f_{t_k}(\check{a}_{t_k^j} * \cdot)\Big\|_\Phi = \Big\|\frac{\chi_{E_{k,n}}}{w_{t_k^j}}\ T_{t_k^j} f_{t_k}\Big\|_\Phi \\
&\geqslant \Big(1 - \frac{1}{2^k}\Big)\Big\|\frac{\chi_{E_{k,n}}}{w_{t_k^j}}\Big\|_\Phi,
\end{aligned}$$

so, $\lim_{k\to\infty} \chi_{E_{k,n}}/w_{t_k^j} = 0$ in $L^\Phi(\mathcal{K})$. Also,

$$\begin{aligned}\|\chi_{E\smallsetminus E_{k,n}}\|_\Phi &= \|\chi_{A_k\cup B_{1,k}\cup B_{2,k}\cup\cdots\cup B_{n,k}}\|_\Phi \\ &\leqslant \|\chi_{A_k}\|_\Phi + \|\chi_{B_{1,k}}\|_\Phi + \|\chi_{B_{2,k}}\|_\Phi + \cdots + \|\chi_{B_{n,k}}\|_\Phi \\ &< (n+1)\frac{1}{2^k},\end{aligned}$$

and hence, $\lim_{k\to\infty} \chi_{E\smallsetminus E_{k,n}} = 0$ in $L^\Phi(\mathcal{K})$. This completes the proof. □

COROLLARY 2.1. *Assume that Φ is a Young function. Let S be a semigroup and $\mathcal{F}$ be a proper Furstenberg family on S. Let $\{a_t\}_{t\in S}$ be an $\mathcal{F}$-aperiodic collection of elements of a locally compact group G and $\{w_t\}_{t\in S}$ be a collection of weights on G such that for every $t \in S$, $\|1/w_t\|_\infty < \infty$. For every $t \in S$, let T_t denote the weighted translation operator T_{a_t,w_t} on $L^\Phi(G)$. Then,* (i) $\Rightarrow$ (ii)*:*

(i) *The collection $\{T_t\}_{t\in S}$ is topologically multiply $\mathcal{F}$-recurrent.*

(ii) *For each $n \in \mathbb{N}$ and compact $E \subseteq G$, there exist a sequence $(E_{k,n})_{k=1}^\infty \subseteq \mathcal{B}$, and $\{t_k\}_{k\in\mathbb{N}} \subseteq S$ such that for all $k \in \mathbb{N}$, $E_{k,n} \subseteq E$,*

$$\lim_{k\to\infty} \|\chi_{E\smallsetminus E_{k,n}}\|_\Phi = 0,$$

and for any $j \in \{1, 2, \dots, n\}$,

$$\lim_{k\to\infty} \big\|\chi_{a_{t_k^j}^{-1}E_{k,n}}\ w_{t_k^j}\big\|_\Phi = \lim_{k\to\infty} \Big\|\frac{\chi_{E_{k,n}}}{w_{t_k^j}}\Big\|_\Phi = 0.$$

THEOREM 2.2. *Let $\mathcal{K}$ be a hypergroup equipped with a left Haar measure. Assume that Φ is a Δ_2-regular Young function. Let S be a semigroup and $\mathcal{F}$ be a proper Furstenberg family on S. Let $\{a_t\}_{t\in S}$ be a subset of* $\mathrm{Ma}(\mathcal{K})$ *and $\{w_t\}_{t\in S}$ be a collection of weights on $\mathcal{K}$ such that for every $t \in S$, $\|1/w_t\|_\infty < \infty$. For every $t \in S$, let T_t denote the weighted translation operator T_{a_t,w_t} on $L^\Phi(\mathcal{K})$. Then,* (ii) $\Rightarrow$ (i)*:*

(i) *The collection $\{T_t\}_{t\in S}$ is topologically multiply $\mathcal{F}$-recurrent.*

(ii) *For each $n \in \mathbb{N}$ and compact $E \subseteq \mathcal{K}$, there exist a sequence $\{E_{k,n}\}_{k=1}^\infty \subseteq \mathcal{B}$, and $\{t_k\}_{k\in\mathbb{N}} \in \mathcal{F}$ such that for all $k \in \mathbb{N}$, $E_{k,n} \subseteq E$,*

$$\lim_{k\to\infty} \|\chi_{E\smallsetminus E_{k,n}}\|_\Phi = 0,$$

for every distinct $i, j \in \{1, 2, \dots, n\}$,

$$\lim_{k\to\infty} \Big\|\frac{1}{w_{t_k^i}(\check{a}_{t_k^i} * a_{t_k^j} * \cdot)}\chi_{\check{a}_{t_k^j} * a_{t_k^i} * E_{k,n}}\ w_{t_k^j}\Big\|_\Phi = 0,$$

and for any $j \in \{1, 2, \dots, n\}$,

$$\lim_{k\to\infty} \Big\|\frac{\chi_{E_{k,n}}}{w_{t_k^j}}\Big\|_\Phi = 0.$$

PROOF. Assume that (ii) holds. Let $\mathcal{O}$ be a nonempty open subset of $L^\Phi(\mathcal{K})$, and $n \in \mathbb{N}$. Since $C_c(\mathcal{K})$ is dense in $L^\Phi(\mathcal{K})$, we have $\mathcal{O} \cap C_c(\mathcal{K}) \neq \emptyset$, so one can pick some $f \in \mathcal{O} \cap C_c(\mathcal{K})$. Put $E := \mathrm{supp}(f)$. Hence E is compact. Then by (ii) there exist $(E_{k,n})_{k=1}^{\infty}$ and $\{t_k : k \in \mathbb{N}\} \in \mathcal{F}$, where $E_{k,n} \subseteq E$ are Borel sets, satisfying condition (ii). Hence, for every distinct $i, j \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}
\left\|T_{t_k^j}(S_{t_k^i}(f\chi_{E_{k,n}}))\right\|_{\Phi} &= \left\|w_{t_k^j}(\cdot)(S_{t_k^i}(f\chi_{E_{k,n}}))(a_{t_k^j} * \cdot)\right\|_{\Phi} \\
&= \left\|w_{t_k^j}(\check{a}_{t_k^j} * \cdot)(S_{t_k^i}(f\chi_{E_{k,n}}))(\cdot)\right\|_{\Phi} \\
&= \left\|w_{t_k^j}(\check{a}_{t_k^j} * \cdot)\frac{1}{w_{t_k^i}(\check{a}_{t_k^i} * \cdot)}(f\chi_{E_{k,n}})(\check{a}_{t_k^i} * \cdot)\right\|_{\Phi} \\
&= \left\|w_{t_k^j}(\check{a}_{t_k^j} * a_{t_k^i} * \cdot)\frac{1}{w_{t_k^i}(\cdot)}(f\chi_{E_{k,n}})(\cdot)\right\|_{\Phi} \\
&\leqslant \|f\|_{\infty}\left\|w_{t_k^j}(\check{a}_{t_k^j} * a_{t_k^i} * \cdot)\frac{1}{w_{t_k^i}(\cdot)}\chi_{E_{k,n}}(\cdot)\right\|_{\Phi} \\
&= \|f\|_{\infty}\left\|\frac{1}{w_{t_k^i}(\check{a}_{t_k^i} * a_{t_k^j} * \cdot)}\, w_{t_k^j}\, \chi_{\check{a}_{t_k^j} * a_{t_k^i} * E_{k,n}}\right\|_{\Phi} \to 0
\end{aligned}$$

as $k \to \infty$. Also,

$$\begin{aligned}
\left\|S_{t_k^j}(f\chi_{E_{k,n}})\right\|_{\Phi} &= \left\|\frac{1}{w_{t_k^j}(\check{a}_{t_k^j} * \cdot)}(f\chi_{E_{k,n}})(\check{a}_{t_k^j} * \cdot)\right\|_{\Phi} \\
&= \left\|\frac{f\chi_{E_{k,n}}}{w_{t_k^j}}\right\|_{\Phi} \leqslant \|f\|_{\infty}\left\|\frac{\chi_{E_{k,n}}}{w_{t_k^j}}\right\|_{\Phi} \to 0
\end{aligned}$$

for all $j = 1, 2, \ldots, n$. Also, $\lim_{k\to\infty} f - f\chi_{E_{k,n}} = 0$ in $L^{\Phi}(\mathcal{K})$. Now, for each $k \in \mathbb{N}$ put

$$v_k := f\chi_{E_{k,n}} + S_{t_k}(f\chi_{E_{k,n}}) + S_{t_k^2}(f\chi_{E_{k,n}}) + \cdots + S_{t_k^n}(f\chi_{E_{k,n}}).$$

We have

$$\|v_k - f\|_{\Phi} \leqslant \|f\chi_{E_{k,n}} - f\|_{\Phi} + \sum_{i=1}^{n}\left\|S_{t_k^i}(f\chi_{E_{k,n}})\right\|_{\Phi}$$

and for every $j = 1, 2, \ldots, n$,

$$\left\|T_{t_k^j}v_k - f\right\|_{\Phi} \leqslant \left\|f\chi_{E_{k,n}} - f\right\|_{\Phi} + \sum_{\substack{i=1\\ i\neq j}}^{n}\left\|T_{t_k^j}S_{t_k^i}(f\chi_{E_{k,n}})\right\|_{\Phi}$$

which implies $\lim_{k\to\infty}\|v_k - f\|_{\Phi} = \lim_{k\to\infty}\left\|T_{t_k^j}v_k - f\right\|_{\Phi} = 0$ for all $j = 1, 2, \ldots, n$. So, $\mathcal{O} \cap \bigcap_{j=1}^{n} T_{t_k^j}^{-1}(\mathcal{O}) \neq \emptyset$. This implies that

$$\{t_k : k \in \mathbb{N}\} \subseteq \left\{t \in S : \mathcal{O} \cap \bigcap_{r=1}^{N} T_{t^j}^{-1}(\mathcal{O}) \neq \emptyset\right\},$$

hence, since $\mathcal{F}$ is a Furstenberg family, $\left\{t \in S : \mathcal{O} \cap \bigcap_{j=1}^{n} T_{t^j}^{-1}(\mathcal{O}) \neq \emptyset\right\} \in \mathcal{F}$, so (i) holds. □

Corollary 2.2. *Assume that Φ is a Δ_2-regular Young function. Let S be a semigroup and $\mathcal{F}$ be a proper Furstenberg family on S. Let $\{a_t\}_{t\in S}$ be a subset of a locally compact group G and $\{w_t\}_{t\in S}$ be a collection of weights on G such that for* *every* $t \in S$, $\|1/w_t\|_\infty < \infty$. *For every* $t \in S$, *let* T_t *denote the weighted translation operator* T_{a_t,w_t} *on* $L^\Phi(G)$. *Then,* (ii) $\Rightarrow$ (i)*:*

(i) *The collection* $\{T_t\}_{t\in S}$ *is topologically multiply* $\mathcal{F}$*-recurrent.*

(ii) *For each* $n \in \mathbb{N}$ *and compact* $E \subseteq G$, *there exist a sequence* $\{E_{k,n}\}_{k=1}^\infty \subseteq \mathcal{B}$, *and* $\{t_k\}_{k\in\mathbb{N}} \in \mathcal{F}$ *such that for all* $k \in \mathbb{N}$, $E_{k,n} \subseteq E$, $\lim_{k\to\infty} \|\chi_{E\smallsetminus E_{k,n}}\|_\Phi = 0$, *for every distinct* $i, j \in \{1, 2, \ldots, n\}$,

$$\lim_{k\to\infty} \left\| \frac{1}{w_{t_k^i}(a_{t_k^i}^{-1} a_{t_k^j}\cdot)} \chi_{a_{t_k^j}^{-1} a_{t_k^i} E_{k,n}} w_{t_k^j} \right\|_\Phi = 0,$$

and for any $j \in \{1, 2, \ldots, n\}$,

$$\lim_{k\to\infty} \left\| \frac{\chi_{E_{k,n}}}{w_{t_k^j}} \right\|_\Phi = 0.$$

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