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CONVERGENCE OF OPERATORS BASED ON TWO DIFFERENT BASIS FUNCTIONS

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Abstract. We deal with the approximation operators which are generalization of the operators of exponential type. We construct some composition operators and discuss the convergence behaviour. Also we estimate some direct results with the aid of the characteristic functions of such operators. Also, difference estimates in terms of modulus of continuity are established.

1. Introduction

For any function $f \in C[0,\infty)$, the Szász–Mirakjan operators are defined by

$$(S_n f)(x) = \sum_{v=0}^{\infty} s_{n,v}(x) f\Big(\frac{v}{n}\Big), \tag{1.1}$$

where the Szász basis function is given by $s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}$. The generalization of such operators based on Charlier polynomials $C_k^{(a)}$, $a > 1$, $x \geqslant 0$ (see [**16**]) are defined by

$$(Q_m^a f)(x) = \sum_{k=0}^{\infty} c_{m,k}^a(x) f\Big(\frac{k}{m}\Big), \tag{1.2}$$

where the Charlier basis function is given by

$$c_{m,k}^a(x) = \frac{1}{e}\Big(1 - \frac{1}{a}\Big)^{(a-1)mx} \frac{C_k^{(a)}(-(a-1)mx)}{k!}.$$

Also $C_k^{(a)}(-(a-1)mx) = \sum_{r=0}^{k} \binom{k}{r} \frac{((a-1)mx)_r}{a^r}$, where the rising factorial is defined by $(b)_k = b(b+1)(b+2)\dots(b+k-1)$, $(b)_0 = 1$. In particular the Charlier and Szász basis functions are connected by the relation $\lim_{a\to\infty} c_{m,k}^a\big(x - \frac{1}{m}\big) = s_{m,k}(x)$.

Recently some approximation results have been discussed in [**5**–**7**, **9**–**14**] etc.

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Dedicated to well-known mathematician Professor Gradimir V. Milovanović.

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Throughout the article we take $C_b[0,\infty)$ as the class of bounded and continuous functions on $[0,\infty)$. Also, we denote $\exp_A(t) = e^{At}$ and $(L_n \exp_{is})$ denotes the characteristic function and is equal to $(L_n(\cos st + i\sin st))$.

2. Composition operators $Q_m^a \circ S_n$

Below we provide a new operator by the composition of the operators Q_m^a and S_n.

THEOREM 2.1. *The composition operator for $m,n \in N$ and $x \geqslant 0$ is defined by*

$$(Q_m^a \circ S_n f)(x) = \sum_{v=0}^{\infty} t_{v,n,m}^a(x) \frac{n^v}{m^v v!} f\Big(\frac{v}{n}\Big), \quad t_{v,n,m}^a(x) = \sum_{k=0}^{\infty} c_{m,k}^a(x) k^v e^{-nk/m}.$$

In particular

$$(Q_n^a \circ S_n f)(x) = \sum_{v=0}^{\infty} \frac{t_{v,n,n}^a(x)}{v!} f\Big(\frac{v}{n}\Big), \quad t_{v,n,n}^a(x) = \sum_{k=0}^{\infty} c_{m,k}^a(x) k^v e^{-k}.$$

PROOF. By definitions (1.2) and (1.1), we can write

$$\begin{aligned}(Q_m^a \circ S_n f)(x) &= \sum_{k=0}^{\infty} c_{m,k}^a(x) \sum_{v=0}^{\infty} \frac{n^v}{m^v v!} k^v e^{-nk/m} f\Big(\frac{v}{n}\Big)\\ &= \sum_{v=0}^{\infty} t_{v,n,m}^a(x) \frac{n^v}{m^v v!} f\Big(\frac{v}{n}\Big),\end{aligned}$$

where $t_{v,n,m}^a(x) = \sum_{k=0}^{\infty} c_{m,k}^a(x) k^v e^{-nk/m}$. The special case follows by considering $m = n$. □

LEMMA 2.1. *The composition operators $Q_m^a \circ S_n$ have the following moment generating function*

$$(Q_m^a \circ S_n \exp_A)(x) = \exp\Big(e^{\frac{n(e^{A/n}-1)}{m}} - 1\Big)\Bigg(\frac{a-1}{a - e^{\frac{n(e^{A/n}-1)}{m}}}\Bigg)^{(a-1)mx}.$$

In particular, we have

$$begineqnarray*(Q_n^a \circ S_n \exp_A)(x) = \exp(e^{e^{A/n}-1} - 1)\Big(\frac{a-1}{a - e^{e^{A/n}-1}}\Big)^{(a-1)nx}.$$

PROOF. First the moment generating function (abbr. m.g.f.) of the operators Q_m^a, can be obtained as follows:

$$(Q_m^a \exp_A)(x) = \frac{1}{e}\Big(1 - \frac{1}{a}\Big)^{(a-1)mx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)mx)}{k!} e^{Ak/m}.$$

As the Charlier polynomial $C_k^{(a)}(z)$ satisfies the following generating function

$$e^y\Big(1 - \frac{y}{a}\Big)^z = \sum_{k=0}^{\infty} \frac{C_k^{(a)}(z)}{k!} y^k, \quad |y| < a,$$

we have

$$(Q_m^a \exp_A)(x) = \frac{1}{e}\Big(1-\frac{1}{a}\Big)^{(a-1)mx} e^{e^{A/m}}\Big(1-\frac{e^{A/m}}{a}\Big)^{-(a-1)mx}$$
$$= \exp(e^{A/m}-1)\Big(\frac{a-1}{a-e^{A/m}}\Big)^{(a-1)mx}.$$

Also the Szász-Mirakyan operators satisfy $(S_n \exp_A)(x) = e^{nx(e^{A/n}-1)}$. Thus by simple computation for the composition operators, there hold

$$(Q_m^a \circ S_n \exp_A)(x) = (Q_m^a \exp_{n(e^{A/n}-1)})(x)$$
$$= \exp\big(e^{\frac{n(e^{A/n}-1)}{m}}-1\big)\Bigg(\frac{a-1}{a-e^{\frac{n(e^{A/n}-1)}{m}}}\Bigg)^{(a-1)mx}. \qquad \square$$

REMARK 2.1. We may remark here by the j-th order moment of an operator $(L_n e_j)$, $j = 0, 1, 2, \ldots$ connected to moment generating functions by the relation $(L_n e_j) = \big[\frac{\partial^j}{\partial A^j}(L_n \exp_A)(x)\big]_{A=0}$, thus using m.g.f. provided in Lemma 2.1, we get

$$(Q_m^a e_0)(x) = 1, \quad (Q_m^a e_1)(x) = x + \frac{1}{m}, \quad (Q_m^a e_2)(x) = x^2 + \frac{x(3a-2)}{m(a-1)} + \frac{2}{m^2};$$
$$(S_n e_0)(x) = 1, \quad (S_n e_1)(x) = x, \quad (S_n e_2)(x) = x^2 + \frac{x}{n}.$$

LEMMA 2.2. *The j-th order, $j = 0, 1, 2, \ldots$ moment of the composition operator denoted by $(Q_m^a \circ S_n e_j)(x)$ fulfills:*

$$(Q_m^a \circ S_n e_0)(x) = 1, \qquad (Q_m^a \circ S_n e_1)(x) = x + \frac{1}{m},$$
$$(Q_m^a \circ S_n e_2)(x) = x^2 + \frac{x(3an+am-2n-m)}{nm(a-1)} + \frac{m+2n}{nm^2}.$$

In particular for the composition operator $Q_n^a \circ S_n$ there hold:

$$(Q_n^a \circ S_n e_0)(x) = 1, \qquad (Q_n^a \circ S_n e_1)(x) = x + \frac{1}{n},$$
$$(Q_n^a \circ S_n e_2)(x) = x^2 + \frac{x(4a-3)}{n(a-1)} + \frac{3}{n^2}.$$

The proof of the above lemma follows by Lemma 2.1, we omit the details.

We denote the class of continuous and real functions f denoted by $C^*[0,\infty)$, have finite limit, for x tending to ∞. Also, the modulus of continuity is defined by

$$\omega^*(f,\delta) := \sup_{\substack{|e^{-t}-e^{-u}|\leqslant\delta \\ t,u>0}} |f(u)-f(t)|.$$

THEOREM 2.2. *Let f and its second derivative belong to the class $C^*[0,\infty)$, then for any $x \geqslant 0$, there follows*

$$\Big|n[(Q_n^a \circ S_n f)(x) - f(x)] - f'(x) - \frac{(2a-1)x}{2(a-1)} f''(x)\Big|$$

$$\leqslant \frac{3}{2n}|f''(x)| + \Big[\frac{(2a-1)x}{2(a-1)} + \frac{3}{n} + a_n(x)\Big]\omega^*(f'', n^{-1/2}),$$

where

$$a_n(x) = n^2[(Q_n^a \circ S_n(\exp_{-1}(x) - \exp_{-1}(t))^4)(x)(Q_n^a \circ S_n(e_1 - xe_0)^4)(x)]^{1/2}.$$

PROOF. Applying Taylor's formula to $Q_n^a \circ S_n$, we have

$$\Big|(Q_n^a \circ S_n f)(x) - f(x) - (Q_n^a \circ S_n(e_1 - xe_0))(x)f'(x)\frac{(Q_n^a \circ S_n(e_1 - xe_0)^2)(x)}{2}f''(x)\Big| \leqslant |(Q_n^a \circ S_n(e_1 - xe_0)^2 h_{t,x})(x)|,$$

where $h_{t,x} = \frac{f''(\eta) - f''(x)}{2}$ and $x < \eta < t$. Using Lemma 2.2, we immediately have

$$\Big|n[(Q_n^a \circ S_n f)(x) - f(x)] - f'(x) - \frac{(2a-1)x}{2(a-1)}f''(x)\Big| \leqslant \frac{3}{2n}|f''(x)| + |n(Q_n^a \circ S_n(e_1 - xe_0)^2 h_{t,x})(x)|.$$

Next by the property used in [**1**, (3.1)], we can write

$$h_{t,x} \leqslant \Big(1 + \frac{(\exp_{-1}(x) - \exp_{-1}(t))^2}{\delta^2}\Big)\omega^*(f'', \delta), \quad \delta > 0.$$

Using said above and Cauchy–Schwarz inequality and selecting $\delta = n^{-1/2}$, we have

$$n(Q_n^a \circ S_n(e_1 - xe_0)^2 h_{t,x})(x) \leqslant n\omega^*(f'', \delta)(Q_n^a \circ S_n(e_1 - xe_0)^2)(x) + \frac{n}{\delta^2}\omega^*(f'', \delta)[(Q_n^a \circ S_n(\exp_{-1}(x) - \exp_{-1}(t))^4)(x)]^{1/2}[(Q_n^a \circ S_n(e_1 - xe_0)^4)(x)]^{1/2} = \Big[\frac{(2a-1)x}{2(a-1)} + \frac{3}{n} + a_n(x)\Big]\omega^*(f'', n^{-1/2}),$$

where

$$a_n(x) = n^2[(Q_n^a \circ S_n(\exp_{-1}(x) - \exp_{-1}(t))^4)(x)(Q_n^a \circ S_n(e_1 - xe_0)^4)(x)]^{1/2}. \qquad \square$$

COROLLARY 2.1. *We have the following asymptotic formula:*

$$\lim_{n\to\infty} n[(Q_n^a \circ S_n f - f)(x)] = f'(x) + \frac{(2a-1)x}{2(a-1)}f''(x).$$

REMARK 2.2. Further, by using Corollary 2.1 and Remark 2.1, one can get

$$\lim_{n\to\infty} n[(Q_n^a \circ S_n f - f)(x)] = \lim_{n\to\infty} n[((Q_n^a + S_n)f - 2f)(x)].$$

THEOREM 2.3. *For $f \in C_b[0, +\infty)$ there holds*

$$\lim_{n\to\infty}(Q_n^a \circ S_n f)(x) = f(x),$$
$$\lim_{n\to\infty}(Q_m^a \circ S_n f)(x) = (Q_m^a f)(x),$$
$$\lim_{m\to\infty}(Q_m^a \circ S_n f)(x) = (S_n f)(x).$$

Proof. Let us denote by $g^x_{m,n}(s)$ and $g^x(s)$ as the characteristic functions respectively of the operators $Q^a_m \circ S_n$ and identity operators. Then by using Lemma 2.1, we have

$$\begin{aligned}\lim_{n\to\infty} g^x_{n,n}(s) &= \lim_{n\to\infty} (Q^a_n \circ S_n \exp_{is})(x)\\ &= \lim_{n\to\infty} \exp\left(e^{e^{is/n}-1}-1\right)\Big(\frac{a-1}{a-e^{e^{is/n}-1}}\Big)^{(a-1)nx}\\ &= e^{isx} = (Id\exp_{is})(x) = g^x(s).\end{aligned}$$

Next, in the same manner we have

$$\begin{aligned}\lim_{n\to\infty} g^x_{m,n}(s) &= \lim_{n\to\infty} (Q^a_m \circ S_n \exp_{is})(x)\\ &= \lim_{n\to\infty} \exp\left(e^{\frac{n(e^{A/n}-1)}{m}}-1\right)\left(\frac{a-1}{a-e^{\frac{n(e^{A/n}-1)}{m}}}\right)^{(a-1)mx}\\ &= \exp(e^{A/m}-1)\Big(\frac{a-1}{a-e^{A/m}}\Big)^{(a-1)mx} = (Q^a_m \exp_{is})(x).\end{aligned}$$

Finally

$$\begin{aligned}\lim_{m\to\infty} g^x_{m,n}(s) &= \lim_{m\to\infty} (Q^a_m \circ S_n \exp_{is})(x)\\ &= \lim_{m\to\infty} \exp\left(e^{\frac{n(e^{A/n}-1)}{m}}-1\right)\left(\frac{a-1}{a-e^{\frac{n(e^{A/n}-1)}{m}}}\right)^{(a-1)mx}\\ &= e^{nx(e^{A/n}-1)} = (S_n \exp_{is})(x).\end{aligned}$$

Therefore by [**2**, Th. 1.1] and [**3**, Th. 2.1], we get the desired result. □

Theorem 2.4. *If $f \in C_b[0,\infty)$, then we have the following difference estimates:*

$$|(Q^a_m \circ S_n f)(x) - (Q^a_m f)(x)| \leqslant 2\omega\left(f, \sqrt{\frac{x}{n}+\frac{1}{mn}}\right),$$

$$|(Q^a_m \circ S_n f)(x) - (S_n f)(x)| \leqslant 2\big(2-e^{-n(\frac{ax}{(a-1)m}+\frac{2}{m^2})^{-1/2}}\big)\omega\left(f, \sqrt{\frac{ax}{(a-1)m}+\frac{2}{m^2}}\right).$$

Proof. We start by considering the following:

$$|(Q^a_m \circ S_n f)(x) - (Q^a_m f)(x)| \leqslant \sum_{k\geqslant 0} c^a_{m,k}(x)\Big|(S_n f)\Big(\frac{k}{m}\Big) - f\Big(\frac{k}{m}\Big)\Big|.$$

Using the following inequality and by Remark 2.1, we can write

$$|(S_n f)(x) - f(x)| \leqslant \Big(1+\frac{(S_n(e_1-xe_0)^2)(x)}{\delta^2}\Big)\omega(f,\delta) = \Big(1+\frac{x}{n\delta^2}\Big)\omega(f,\delta).$$

Finally by Remark 2.1, we have

$$\begin{aligned}|(Q^a_m \circ S_n f)(x) - (Q^a_m f)(x)| &\leqslant \sum_{k\geqslant 0} c^a_{m,k}(x)\Big(1+\frac{k}{mn\delta^2}\Big)\omega(f,\delta)\\ &= \Big[1+\Big(x+\frac{1}{m}\Big)\frac{1}{n\delta^2}\Big]\omega(f,\delta),\end{aligned}$$

choosing $\delta = \left(\frac{x}{n} + \frac{1}{mn}\right)^{-1/2}$, the result follows.

To prove the second inequality, we consider $S_n f = g$ and using Remark 2.1, we proceed as follows

$$\begin{aligned}|(Q_m^a \circ S_n f)(x) - (S_n f)(x)| &= |(Q_m^a \circ g)(x) - g(x)| \\ &\leqslant \left(1 + \frac{(Q_m^a(e_1 - xe_0)^2)(x)}{\delta^2}\right)\omega(g,\delta) \\ &= \left(1 + \frac{ax}{(a-1)m\delta^2} + \frac{2}{m^2\delta^2}\right)\omega(g,\delta).\end{aligned}$$

Also, by [**4**, Ex. (D)], we have $\omega(S_n f, \delta) \leqslant (2 - e^{-n\delta})\omega(f,\delta)$, thus

$$|(Q_m^a \circ f)(x) - (S_n f)(x)| \leqslant (2 - e^{-n\delta})\left(1 + \frac{ax}{(a-1)m\delta^2} + \frac{2}{m^2\delta^2}\right)\omega(f,\delta).$$

choosing $\delta = \left(\frac{ax}{(a-1)m} + \frac{2}{m^2}\right)^{-1/2}$, the result follows. □

3. Composition operators $S_n \circ Q_m^a$

This section deals with the reverse order composition of operators S_n and Q_m^a in order. The new operator takes the following form.

THEOREM 3.1. *The composition operators for $n, m \in N$ and $x \geqslant 0$ are defined by*

$$(S_n \circ Q_m^a f)(x) = \sum_{k=0}^{\infty} \frac{u_{k,m,n}^a(x)}{k!} f\left(\frac{k}{m}\right),$$

where

$$u_{k,m,n}^a(x) = \sum_{v=0}^{\infty} e^{-nx-1} \frac{\left(nx\left(1-\frac{1}{a}\right)^{(a-1)\frac{m}{n}}\right)^v}{v!} C_k^{(a)}\left(-\frac{mv(a-1)}{n}\right).$$

Also

$$(S_n \circ Q_n^a f)(x) = \sum_{k=0}^{\infty} \frac{u_{k,n,n}^a(x)}{k!} f\left(\frac{k}{n}\right),$$

where

$$u_{k,n,n}^a(x) = \sum_{v=0}^{\infty} e^{-nx-1} \frac{\left(nx\left(1-\frac{1}{a}\right)^{(a-1)}\right)^v}{v!} C_k^{(a)}(-v(a-1)).$$

PROOF. By definitions (1.2) and (1.1), we have

$$\begin{aligned}(S_n \circ Q_m^a f)(x) &= \sum_{v=0}^{\infty} s_{n,v}(x) \sum_{k=0}^{\infty} c_{m,k}^a\left(\frac{v}{n}\right) f\left(\frac{k}{m}\right) \\ &= \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!} f\left(\frac{k}{m}\right)\sum_{v=0}^{\infty} s_{n,v}(x) C_k^{(a)}\left(-\frac{mv(a-1)}{n}\right)\left(1-\frac{1}{a}\right)^{(a-1)\frac{mv}{n}} \\ &= \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!} f\left(\frac{k}{m}\right)\sum_{v=0}^{\infty} e^{-nx}\frac{(nx)^v}{v!} C_k^{(a)}\left(-\frac{mv(a-1)}{n}\right)\left(1-\frac{1}{a}\right)^{(a-1)\frac{mv}{n}}\end{aligned}$$

$$:= \sum_{k=0}^{\infty} \frac{u^a_{k,m,n}(x)}{k!} f\Big(\frac{k}{m}\Big).$$

The other consequence is immediate. □

LEMMA 3.1. *We have*

$$(S_n \circ Q^a_m \exp_A)(x) = \exp\Big(e^{A/m} - 1 + nx\Big(\frac{a-1}{a-e^{A/m}}\Big)^{\frac{m(a-1)}{n}} - nx\Big).$$

As a special case for $S_n \circ Q^a_n$, we have

$$(S_n \circ Q^a_n \exp_A)(x) = \exp\Big(e^{A/n} - 1 + nx\Big(\frac{a-1}{a-e^{A/n}}\Big)^{(a-1)} - nx\Big).$$

PROOF. By using the moment generating function of Q^a_m and S_n from Lemma 2.1, we have

$$\begin{aligned}(S_n \circ Q^a_m \exp_A)(x) &= \exp(e^{A/m} - 1)\big(S_n \exp_{(a-1)m \log\left(\frac{a-1}{a-e^{A/m}}\right)}\big)(x)\\ &= \exp\Big(e^{A/m} - 1 + nx\Big(\frac{a-1}{a-e^{A/m}}\Big)^{\frac{m(a-1)}{n}} - nx\Big).\end{aligned}$$ □

LEMMA 3.2. *The j-th order moment of the composition operator $S_n \circ Q^a_m$ fulfills the identity*

$$(S_n \circ Q^a_m e_0)(x) = 1, \qquad (S_n \circ Q^a_m e_1)(x) = x + \frac{1}{m},$$
$$(S_n \circ Q^a_m e_2)(x) = x^2 + \frac{(3na + ma - 2n - m)x}{mn(a-1)} + \frac{2}{n^2}.$$

In particular, the moments of the composition operator $S_n \circ Q^a_n$ are given by

$$(S_n \circ Q^a_n e_0)(x) = 1, \qquad (S_n \circ Q^a_n e_1)(x) = x + \frac{1}{n},$$
$$(S_n \circ Q^a_n e_2)(x) = x^2 + \frac{x(4a-3)}{n(a-1)} + \frac{2}{n^2}.$$

THEOREM 3.2. *Let f and its second derivative belong to the class $C^*[0,\infty)$, then for any $x \geqslant 0$, there follows*

$$\begin{aligned}&\Big|n[(S_n \circ Q^a_n f)(x) - f(x)] - f'(x) - \frac{(2a-1)x}{2(a-1)} f''(x)\Big|\\ &\qquad\qquad \leqslant \frac{1}{n}|f''(x)| + \Big[\frac{(2a-1)x}{2(a-1)} + \frac{2}{n} + b_n(x)\Big]\omega^*(f'', n^{-1/2}),\end{aligned}$$

where

$$b_n(x) = n^2\big[(S_n \circ Q^a_n(\exp_{-1}(x) - \exp_{-1}(t))^4)(x)(S_n \circ Q^a_n(e_1 - xe_0)^4)(x)\big]^{1/2}.$$

The proof of the above theorem follows along the lines of Theorem 2.2, we omit the details.

COROLLARY 3.1. *We have the following asymptotic formula:*

$$\lim_{n\to\infty} n[(S_n \circ Q_n^a f - f)(x)] = f'(x) + \frac{(2a-1)x}{2(a-1)} f''(x).$$

REMARK 3.1. Analogous to Corollary 3.1, and by Remark 2.1, we immediately have:

$$\begin{aligned}\lim_{n\to\infty} n[(S_n \circ Q_n^a f - f)(x)] &= \lim_{n\to\infty} n[((Q_n^a + S_n)f - 2f)(x)]\\ &= f'(x) + \frac{(2a-1)x}{2(a-1)} f''(x).\end{aligned}$$

THEOREM 3.3. *If $f \in C_b[0,+\infty)$, then*

$$\begin{aligned}\lim_{n\to\infty} (S_n \circ Q_n^a f)(x) &= f(x),\\ \lim_{n\to\infty} (S_n \circ Q_m^a f)(x) &= (Q_n^a f)(x),\\ \lim_{m\to\infty} (S_n \circ Q_m^a f)(x) &= (S_n f)(x).\end{aligned}$$

PROOF. Let $f_{n,m}^x(s)$ and $f^x(s)$ denote respectively the characteristic functions of $S_n \circ Q_m^a$ and L. Then applying Lemma 3.1, we at once have

$$\begin{aligned}\lim_{n\to\infty} f_{n,n}^x(s) &= \lim_{n\to\infty} (S_n \circ Q_n^a \exp_{is})(x)\\ &= \lim_{n\to\infty} \exp\left(e^{is/n} - 1 + nx\left(\frac{a-1}{a-e^{is/n}}\right)^{(a-1)} - nx\right)\\ &= e^{isx} = (Id \exp_{is})(x) = f^x(s).\end{aligned}$$

Next, in the same manner we have

$$\begin{aligned}\lim_{n\to\infty} f_{n,m}^x(s) &= \lim_{n\to\infty} (S_n \circ Q_m^a \exp_{is})(x)\\ &= \lim_{n\to\infty} \exp\left(e^{is/m} - 1 + nx\left(\frac{a-1}{a-e^{is/m}}\right)^{\frac{m(a-1)}{n}} - nx\right)\\ &= \exp(e^{A/m} - 1)\left(\frac{a-1}{a-e^{A/m}}\right)^{(a-1)mx} = (Q_m^a \exp_{is})(x).\end{aligned}$$

Finally

$$\begin{aligned}\lim_{m\to\infty} f_{n,m}^x(s) &= \lim_{m\to\infty} (S_n \circ Q_m^a \exp_{is})(x)\\ &= \lim_{m\to\infty} \exp\left(e^{is/m} - 1 + nx\left(\frac{a-1}{a-e^{is/m}}\right)^{\frac{m(a-1)}{n}} - nx\right)\\ &= e^{nx(e^{is/n}-1)} = (S_n \exp_{is})(x).\end{aligned}$$

Therefore by [**2**, Th. 1.1] and [**3**, Th. 2.1], we get the desired result. □

THEOREM 3.4. *If $f \in C_b[0,\infty)$, then we get*

$$|(S_n \circ Q_m^a f)(x) - (S_n f)(x)| \leqslant 2\omega\left(f, \sqrt{\frac{ax}{(a-1)m} + \frac{2}{m^2}}\right),$$

$$|(S_n \circ Q_m^a f)(x) - (Q_m^a f)(x)| \leqslant 2\Big[2 - \frac{1}{e}\Big(1 - \frac{1}{a}\Big)^{(a-1)m\sqrt{x/n}}\Big]\omega\big(f, \sqrt{x/n}\big).$$

PROOF. Firstly, we have

$$|(S_n \circ Q_m^a f)(x) - (S_n f)(x)| \leqslant \sum_{v \geqslant 0} s_{n,v}(x)\Big|(Q_m^a f)\Big(\frac{v}{n}\Big) - f\Big(\frac{v}{n}\Big)\Big|.$$

Using the methods as in Theorem 2.4 by Remark 2.1, we can write

$$|(Q_m^a f)(x) - f(x)| \leqslant \Big(1 + \frac{ax}{(a-1)m\delta^2} + \frac{2}{m^2\delta^2}\Big)\omega(f, \delta).$$

Finally by Remark 2.1, we have

$$\begin{aligned}|(S_n \circ Q_m^a f)(x) - (S_n f)(x)| &\leqslant \sum_{v \geqslant 0} s_{n,v}(x)\Big(1 + \frac{av}{(a-1)nm\delta^2} + \frac{2}{m^2\delta^2}\Big)\omega(f, \delta)\\ &= \Big[1 + \frac{ax}{(a-1)m\delta^2} + \frac{2}{m^2\delta^2}\Big]\omega(f, \delta),\end{aligned}$$

choosing $\delta = \big(\frac{ax}{(a-1)m} + \frac{2}{m^2}\big)^{-1/2}$, the result follows. To prove second inequality let $Q_m^a f = g$, we consider

$$\begin{aligned}|(S_n \circ Q_m^a f)(x) - (Q_m^a f)(x)| &= |(S_n g)(x) - g(x)|\\ &\leqslant \sum_{v \geqslant 0} s_{n,v}(x)\Big(1 + \frac{x}{n\delta^2}\Big)\omega(g, \delta) = \Big[1 + \frac{x}{n\delta^2}\Big]\omega(g, \delta),\end{aligned}$$

Also, the standard Charlier distribution [**15**], with applications of [**4**], is recently discussed in [**8**] satisfy

$$\omega(Q_m^a f, h) \leqslant \Big[2 - \frac{1}{e}\Big(1 - \frac{1}{a}\Big)^{(a-1)mh}\Big]\omega(f, h). \tag{3.1}$$

by (3.1) and selecting $\delta = \sqrt{\frac{x}{n}}$, the result follows. □

4. Durrmeyer Variant

The operator Q_m^a is not suitable to approximate Lebesgue integrable functions, to approximate them we consider below the Durrmeyer type modification of the operators Q_m^a with weights of the Szász basis function:

$$(D_m^a f)(x) = m \sum_{k=0}^{\infty} c_{m,k}^a(x) \int_0^{\infty} s_{m,k}(y) f(y)\,dy,$$

where $c_{m,k}^a$ and $s_{m,k}$ respectively the Charlier and Szász basis functions. It is observed here that the Durrmeyer type operator D_m^a can be decomposed into Charlier and modified Rathore operators namely $D_m^a = Q_m^a \circ R_m$, where the modified Rathore operators R_m are defined by

$$(R_m f)(x) = \frac{m^{mx+1}}{\Gamma(mx+1)} \int_0^{\infty} e^{-my} y^{mx} f(y)\,dy, \quad x \in [0, \infty)$$

Lemma 4.1. *We have*

$$(D_m^a \exp_A)(x) = \Big(\frac{m}{m-A}\Big) \exp\Big(\frac{A}{m-A}\Big)\Big(\frac{(a-1)(m-A)}{a(m-A)-m}\Big)^{(a-1)mx}.$$

Proof. By simple computation, we have

$$(R_m \exp_A) = \Big(\frac{m}{m-A}\Big)^{mx+1}.$$

Using said above and the moment generating function of Q_m^a given in the proof of Lemma 2.1, we get

$$\begin{aligned}(D_m^a \exp_A)(x) &= (Q_m^a \circ R_m \exp_A)(x)\\ &= \Big(\frac{m}{m-A}\Big)(Q_m^a \exp_{m\log(\frac{m}{m-A})})(x)\\ &= \Big(\frac{m}{m-A}\Big) \exp\Big(\frac{A}{m-A}\Big)\Big(\frac{(a-1)(m-A)}{a(m-A)-m}\Big)^{(a-1)mx}. \qquad \square\end{aligned}$$

Remark 4.1. From the moment generating function given in Lemma 4.1, we have

$$\begin{gathered}(D_m^a e_0)(x) = 1, \qquad (D_m^a(e_1 - xe_0))(x) = \frac{2}{m},\\ (D_m^a(e_1 - xe_0)^2)(x) = \frac{7(a-1)+(2a-1)mx}{(a-1)m^2}.\end{gathered}$$

It can be observed that the asymptotic formula for the integral modification of Charlier based polynomials with weight of the Szász basis function is given by

$$\lim_{n\to\infty} n[(D_n^a f - f)(x)] = 2f'(x) + \frac{(2a-1)x}{2(a-1)} f''(x).$$

Remark 4.2. From Corollaries 2.1 and 3.1, it is observed that the Durrmeyer variant is connected with the composition operators leading to the following relation:

$$\begin{aligned}\lim_{n\to\infty} n[(Q_n^a \circ S_n f - f)(x)] + f'(x) &= \lim_{n\to\infty} n[(S_n \circ Q_n^a f - f)(x)] + f'(x)\\ &= \lim_{n\to\infty} n[(D_n^a f - f)(x)].\end{aligned}$$

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