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RELATIONS AMONG DERANGEMENT TYPE NUMBERS OF HIGHER ORDER, BERNOULLI, STIRLING AND COMBINATORIAL NUMBERS

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Abstract. We study various types of generating functions for higher-order derangement numbers. We present a generating function for a new class of binomial-type polynomials with coefficients given by higher-order derangement numbers. Using this generating function, we derive explicit formulas for these polynomials, as well as derivative and integral formulas involving Stirling numbers and Cauchy numbers of the first kind. By employing this generating function along with other special formulas, we obtain several new results involving higher-order derangement numbers, Bernoulli numbers, Stirling numbers, combinatorial numbers associated with Daehee numbers, Peters-type Simsek numbers, and special finite sums.

1. Introduction

In combinatorial mathematics, which is one of the important fields of algebra, probability, and statistics, it is well known that a permutation of the elements of a set in which no element appears in its original position is called a derangement. Thus, a derangement is a permutation with no fixed points. Derangements were first studied by the French mathematician Pierre Raymond de Montmort around 1690. Consequently, these numbers are also called the n-th de Montmort numbers, representing the number of derangements of a set of size n, also known as the subfactorial of n or the nth derangement number. The history of these numbers dates back to 1650, specifically to the Swiss mathematician Nicolaus Bernoulli, one of the leading mathematicians in the Bernoulli family. Derangement numbers are extensively used in various branches of number theory, probability, statistics, games of chance, partition theory, and more. With the discovery of generating functions for these numbers, their importance in function theory and related areas grew even more. These generating functions enabled the study of their fundamental properties

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through combinatorial and algebraic approaches, as well as the derivation of various recurrence relations and computational formulas [**3**, **25**].

The motivation of this article is to combine the generating functions for derangement numbers and polynomials with those of the Stirling numbers and other special numbers. Many new and interesting formulas and finite sums involving these numbers and polynomials are presented.

There are many notations for these numbers such as $!n$, D_n, and d_n. Here, we give some well-known (see e.g., [**3**, **17**, **24**]) generating functions, which will be used in the next sections. The Bernoulli numbers and polynomials are defined respectively by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

The Stirling numbers of the first kind are defined respectively by

$$F_{S_1}(t,k)=\frac{(\log(1+t))^k}{k!}=\sum_{n=0}^{\infty}S_1(n,k)\frac{t^n}{n!}, \tag{1.1}$$

or

$$(x)_n=x(x-1)\cdots(x-n+1)=n!\binom{x}{n}=\sum_{k=0}^{n}S_1(n,k)x^k, \tag{1.2}$$

where $n\in\mathbb{N}_0$

The Stirling numbers of the second kind determine the number of times the elements of a set with n elements can be partitioned into k non-empty subsets. Stirling numbers for partitioning a set of n elements into subsets were denoted as $S_2(m,n)$. The Stirling numbers of the second kind $S_2(m,n)$ are defined by means of the following generating function:

$$\frac{(e^t-1)^n}{n!}=\sum_{m=0}^{\infty}S_2(m,n)\frac{t^m}{m!}. \tag{1.3}$$

Some basic properties of these numbers are

$$S_2(m,n)=S_2(0,0)=1 \text{ if } m=n \text{ or } m=n=0,$$
$$S_2(m,n)=S_2(m,0)=S_2(0,n)=0 \text{ if } m\neq 0,\ n\neq 0 \text{ and } n>m.$$

The raising factorial to be used during the operations is defined by

$$x^{(n)}=(x)(x+1)\cdots(x+n-1).$$

We now give some properties of the derangement numbers.

Definition 1.1. Let $(j_1,j_2,\ldots,j_n)$ be a permutation of the set $\{1,2,\ldots,n\}$. The point j_r is called fixed of this permutation if $j_r=r$, for some fixed $r=1,2,\ldots,n$.

According to this definition, the problem of coincidences arising in the computation of permutations without a fixed point is particularly interesting and deserves special attention. To help the reader, we note that such permutations are called *derangements*, as discussed in [**3**].

EXAMPLE 1.1. Consider the permutations (j_1, j_2, j_3) of the set $(1, 2, 3)$. Clearly these 6 permutations can be classified according to the fixed points they have as follows.

(a) The derangements are the following 2:

$$(2,3,1),\ (3,1,2)$$

(b) The permutations with one fixed point are the following 3:

$$(1,3,2),\ (3,2,1),\ (2,1,3)$$

(c) There are no permutations with two fixed points, while

$$(1,2,3)$$

is the only permutation with three fixed points.

The generating function for the derangement numbers d_m are given by

$$F_d(t) = \frac{e^{-t}}{1-t} = \sum_{m=0}^{\infty} d_m \frac{t^m}{m!}, \tag{1.4}$$

for detail, see [**3**, p. 171] and [**25**, p. 97].

Using (1.4), for $n \in \mathbb{N}_0$, we have

$$d_m = m! \sum_{j=0}^{m} \frac{(-1)^j}{j!}, \tag{1.5}$$

for detail, see [**3**, p. 171] and [**25**, p. 97].

By using (1.4), a few values of the derangement numbers d_m are given by the following table:

m	0	1	2	3	4	5	6	7	8	9	10
d_m	1	0	1	2	9	44	265	1854	14833	133496	1334961

Generating function for the r-derangement numbers was given by Kim et al. [**7**].

$$\frac{t^r e^{-t}}{(1-t)^{r+1}} = \sum_{m=0}^{\infty} d_r(m) \frac{t^m}{m!}.$$

When $r = 0$, $d_0(m)$ reduces to the numbers d_m. That is $d_m := d_0(m)$.

Recently, many papers and books have been published with these numbers [**1, 6, 15, 25**].

By using (1.4), the authors have studied the generating function

$$F(t,k) = \left(\frac{e^{-t}}{1-t}\right)^k = \sum_{m=0}^{\infty} d_m^{(k)} \frac{t^m}{m!}. \tag{1.6}$$

The authors [**1**] gave some formulas and open problems for the numbers $d_m^{(k)}$, which are the so-called derangement numbers of order k, see for details [**1**]. Here we note that the generating function in (1.6) with degenerate type generating functions for $d_m^{(k)}$ were also studied in different topics by Kim et al. [**7**]. Modification of the degenerate generating function of Kim et al. [**7**] was studied by Kim [**4**] with the umbral calculus method.

When $k=1$, $d_m^{(1)}$ reduces to the numbers d_m. That is $d_m := d_m^{(1)}$.

THEOREM 1.1. [**1**] *Let* $m \in \mathbb{N}$ *and* $n_m \in \mathbb{N} \cup \{0\}$*. Then we have*

$$\sum_{k=0}^{n_m} (-1)^k \binom{n_m}{k} \frac{(-m)_k}{(n_m-k)!} d_{n_m-k} = \sum_{k_m=0}^{n_m} \sum_{k_{m-1}=0}^{n_{m-1}} \cdots \sum_{k_2=0}^{n_2} \sum_{k_1=0}^{n_1} d_{k_1} \frac{1}{k_1!},$$

where $x^{(n)}$ *denotes the falling factorials.*

THEOREM 1.2. [**1**] *Let* $n \in \mathbb{N}_0$ *and* $k \in \mathbb{N}$*. Then we have*

$$(k)_n = n! \sum_{j=0}^{n} \frac{k^{n-j}}{(n-j)!j!} d_j^{(k)}.$$

Simsek [**21**] defined the following generating functions:

$$\frac{2}{\lambda^2 t + 2(\lambda - 1)} = \sum_{n=0}^{\infty} Y_{n,2}(\lambda) \frac{t^n}{n!}, \tag{1.7}$$

which yields

$$Y_{n,2}(\lambda) = 2^{-n} n! \frac{\lambda^{2n}}{(1-\lambda)^{n+1}}. \tag{1.8}$$

Here we note that the numbers $Y_{n,2}(\lambda)$ are called the Peters-type Simsek numbers and polynomials of the second kind by Kucukoglu [**14**], and Kilar [**11**] see also [**9**, **10**, **12**, **13**, **23**].

By (1.8), first a few values of the numbers $Y_{n,2}(\lambda)$ are

$$Y_{0,2}(\lambda) = \frac{1}{(1-\lambda)}, \quad Y_{1,2}(\lambda) = -\frac{\lambda^2}{2(1-\lambda)^2}, \quad Y_{2,2}(\lambda) = \frac{\lambda^4}{2(1-\lambda)^3},$$
$$Y_{3,2}(\lambda) = -\frac{3\lambda^6}{4(1-\lambda)^4}, \quad Y_{4,2}(\lambda) = \frac{3\lambda^8}{2(1-\lambda)^5}, \quad Y_{5,2}(\lambda) = -\frac{15\lambda^{10}}{4(1-\lambda)^6},$$

and so on.

2. Generating Function for a New Class of Binomial Type Polynomials with the Derangement Numbers of Higher Order Coefficients

In this section, we give a unified generating function a for new class of with binomial type polynomials with the derangement numbers of higher order coefficients as follows:

$$G(t,x;k) = (1-t)^{x-k} e^{-kt} = \sum_{m=0}^{\infty} P_m(x;k) \frac{t^m}{m!}. \tag{2.1}$$

When $x = 0$, we have $P_m(0;k) = d_m^{(k)}$. For $|t| < 1$, applying the binomial theorem to (2.1), we obtain

$$\sum_{m=0}^{\infty} P_m(x;k)\frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{(-k)^m}{m!}t^m \sum_{m=0}^{\infty} \frac{(x-k)_m}{m!}(-1)^m t^m.$$

By applying the Cauchy product rule for series to the above equation yields

$$\sum_{m=0}^{\infty} P_m(x;k)\frac{t^m}{m!} = \sum_{m=0}^{\infty} (-1)^m \left(\sum_{j=0}^{m} \binom{m}{j} k^{m-j}(x-k)_j\right)\frac{t^m}{m!}.$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.1. *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$P_m(x;k) = \sum_{j=0}^{m} (-1)^m \binom{m}{j} k^{m-j}(x-k)_j.$$

Combining (1.6) with (2.1), we get

$$\sum_{m=0}^{\infty} P_m(x;k)\frac{t^m}{m!} = \left(\sum_{m=0}^{\infty} \frac{d_m^{(k)}}{m!}t^m\right)\left(\sum_{m=0}^{\infty} \frac{(x)_m}{m!}t^m\right).$$

By applying the Cauchy product rule for series to the above equation, we get

$$\sum_{m=0}^{\infty} P_m(x;k)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{j=0}^{m} \binom{m}{j}(x)_{m-j} d_j^{(k)}\frac{t^m}{m!}.$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.2. *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$P_m(x;k) = \sum_{j=0}^{m} \binom{m}{j}(x)_{m-j} d_j^{(k)}. \tag{2.2}$$

Combining (2.2) with (1.2), we also get the following relation between the polynomials $P_m(x;k)$ and the Stirling numbers of the first kind:

Theorem 2.3. *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$P_m(x;k) = \sum_{j=0}^{m}\sum_{e=0}^{m-j} \binom{m}{j} S_1(m-j,e) d_j^{(k)} x^e. \tag{2.3}$$

3. Derivative Formula of the Polynomials $P_m(x;k)$

Here, we give derivative formulas of the polynomials $P_m(x;k)$. By applying the derivative operator $\frac{d^v}{dx^v}$ to (2.1), we get the following derivative equation

$$\frac{d^v}{dx^v}\{G(t,x;k)\} = \ln(1-t)^v G(t,x;k).$$

By combining the above equation with (1.1), we get

$$\sum_{m=0}^{\infty}\frac{d^v}{dx^v}\{P_m(x;k)\}\frac{t^m}{m!}=v!\sum_{m=0}^{\infty}(-1)^m S_1(m,v)\frac{t^m}{m!}\sum_{m=0}^{\infty}P_m(x;k)\frac{t^m}{m!}.$$

Therefore

$$\sum_{m=0}^{\infty}\frac{d^v}{dx^v}\{P_m(x;k)\}\frac{t^m}{m!}=v!\sum_{m=0}^{\infty}\sum_{j=0}^{m-j}\binom{m}{j}(-1)^{m-j}S_1(m-j,v)P_j(x;k)\frac{t^m}{m!}.$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

THEOREM 3.1. *Let $m,v\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Then we have*

$$\frac{d^v}{dx^v}\{P_m(x;k)\}=v!\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}S_1(m-j,v)P_j(x;k).$$

By applying the derivative operator $\frac{d^v}{dx^v}$ to (2.3), we get

$$\frac{d^v}{dx^v}\{P_m(x;k)\}=\sum_{j=0}^{m}\sum_{e=0}^{m-j}\binom{m}{j}S_1(m-j,e)d_j^{(k)}\frac{d^v}{dx^v}\{x^e\}.$$

Since

$$\frac{d^v}{dx^v}\{x^e\}=(e)_{v-1}x^{e-v},$$

with $e\geqslant v$, otherwise $\frac{d^v}{dx^v}\{x^e\}=0$, we arrive at the following theorem:

THEOREM 3.2. *Let $m,v\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Then we have*

$$\frac{d^v}{dx^v}\{P_m(x;k)\}=\sum_{j=0}^{m}\sum_{e=0}^{m-j}\binom{m}{j}S_1(m-j,e)d_j^{(k)}(e)_v x^{e-v}.$$

4. Integral Formula of the Polynomials $P_m(x;k)$

Here, we give some integral formulas of the polynomials $P_m(x;k)$. Integrating (2.2) from 0 to 1, we get

$$\int_0^1 P_m(x;k)dx=\sum_{j=0}^{m}\binom{m}{j}d_j^{(k)}\int_0^1(x)_{m-j}dx.$$

By combining the above equation with the definition of the Cauchy numbers of the first kind: $\mathcal{C}_m=\int_0^1(x)_m dx$, [**8**, **18**], we get the following theorem:

THEOREM 4.1. *Let $m\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Then we have*

$$\int_0^1 P_m(x;k)dx=\sum_{j=0}^{m}\binom{m}{j}\mathcal{C}_{m-j}d_j^{(k)}. \tag{4.1}$$

Integrating (1.2) from 0 to 1, we get

$$\int_0^1 P_m(x;k)dx = \sum_{j=0}^{m}\sum_{d=0}^{m-j}\binom{m}{j}S_1(m-j,d)d_j^{(k)}\int_0^1 x^d dx.$$

Thus, we get another integral formula for the polynomials $P_m(x;k)$ as follows:

THEOREM 4.2. *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\int_0^1 P_m(x;k)dx = \sum_{j=0}^{m}\sum_{d=0}^{m-j}\binom{m}{j}\frac{S_1(m-j,d)d_j^{(k)}}{d+1}. \tag{4.2}$$

With the aid of (4.2) and (4.1), we get the following finite sum:

$$\sum_{j=0}^{m}\sum_{d=0}^{m-j}\binom{m}{j}\frac{S_1(m-j,d)d_j^{(k)}}{d+1} = \sum_{j=0}^{m}\binom{m}{j}\mathcal{C}_{m-j}d_j^{(k)}.$$

Which gives us the following novel formula:

THEOREM 4.3. *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{m}\sum_{d=0}^{m-j}\binom{m}{j}d_j^{(k)}\left(\frac{S_1(m-j,d)}{d+1} - \mathcal{C}_{m-j}\right) = 0. \tag{4.3}$$

Observe that using (4.3), we also arrive at the following well-known formula:

$$\sum_{d=0}^{m-j}\frac{S_1(m-j,d)}{d+1} - \mathcal{C}_{m-j} = 0. \tag{4.4}$$

There are many different proofs of (4.4), for details see also [**8**, **16**, **18**, **20**].

5. Relation Between the Numbers d_n and $Y_{n,2}(\lambda)$

In this section, using the generating function, we derive a relation between the numbers d_n and $Y_{n,2}(\lambda)$.

Firstly in [**2**], we investigate a relation between the numbers $Y_{n,2}(\lambda)$ and d_n. Using (1.7), we now give this interesting relations as follows:

$$(1-\lambda)\sum_{n=0}^{\infty}Y_{n,2}(\lambda)\frac{t^n}{n!} = -\sum_{n=0}^{\infty}\left(\frac{\lambda^2}{2-2\lambda}\right)^n\frac{d_n t^n}{n!}\sum_{n=0}^{\infty}\frac{\left(\lambda^2 t\right)^n}{\left(2(1-\lambda)\right)^n n!}.$$

By applying the Cauchy product rule for series to the above equation, we get

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{d_k}{(1-\lambda)^{n+1}}\left(\frac{\lambda^2}{2}\right)^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}Y_{n,2}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get

$$Y_{n,2}(\lambda) = \left(\frac{\lambda^2}{2}\right)^n\frac{1}{(1-\lambda)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}d_k.$$

Combining the above equation with (1.8) yields

$$\sum_{k=0}^{n} k! \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{n}{k} = n!. \tag{5.1}$$

Joining (5.1) with (1.5), we obtain the following theorem:

Theorem 5.1. [**2**], [**25**, Problem 4, p. 101] *Let* $n \in \mathbb{N}_0$. *Then we have*

$$\sum_{k=0}^{n} \binom{n}{k} d_k = n!. \tag{5.2}$$

We note that there are many different proofs of (5.2). One of them is now given in this paper. That is, using the proof of (5.1) and (5.2), we give a solution to Problem 4 which was given in the book of Wallis and George [**25**, Problem 4, p. 101].

6. Relations Among the Numbers d_n, $d^{(k)}$, $S_1(m,j)$, $S_2(m,j)$, Bernoulli Numbers, and Daehee Numbers

In this section, we give some formulas and relations, which involve some relations among the numbers d_n, $d^{(k)}$, $S_1(m,j)$, $S_2(m,j)$, the Bernoulli numbers, and the Daehee numbers.

We now give an explicit formula for the numbers $d_m^{(k)}$. Multiplying the function $F_d(t)$ by itself k times yields $F(t,k)$, and using binomial series for $|t| < 1$, we get

$$\sum_{m=0}^{\infty} d_m^{(k)} \frac{t^m}{m!} = \left(\sum_{m=0}^{\infty} (-1)^m \binom{-k}{m} t^m \right) \left(\sum_{m=0}^{\infty} (-k)^m \frac{t^m}{m!} \right).$$

By using the Cauchy rule for the product of series in the above equation, we obtain

$$\sum_{m=0}^{\infty} d_m^{(k)} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} (-1)^m \binom{-k}{j} k^{m-j} \frac{t^m}{(m-j)!}.$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 6.1. *Let* $m \in \mathbb{N}_0$ *and* $k \in \mathbb{N}$. *Then we have*

$$d_m^{(k)} = (-1)^m \sum_{j=0}^{m} \binom{-k}{j} \binom{m}{j} j! k^{m-j}.$$

Applying the binomial theorem to equation (1.6), we get

$$\sum_{n=0}^{\infty} d_n^{(k)} \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \binom{-k}{n} t^n \right) \left(\sum_{n=0}^{\infty} (-1)^n \frac{k^n t^n}{n!} \right).$$

By using the Cauchy product rule formula on the right-hand side, we get

$$\sum_{n=0}^{\infty} d_n^{(k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{-k}{n-j} \frac{k^j}{j!} t^n.$$

Comparing the coefficient t^n on both sides of the previous equation gives the following theorem:

THEOREM 6.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$d_n^{(k)} = n! \sum_{j=0}^{n} (-1)^j \binom{-k}{n-j} \frac{k^j}{j!}.$$

Applying binomial theorem to equation (1.6), we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{k^n t^n}{n!} = \left(\sum_{n=0}^{\infty} (-1)^n \binom{k}{n} t^n \right) \left(\sum_{n=0}^{\infty} d_n^{(k)} \frac{t^n}{n!} \right).$$

By using the Cauchy product rule formula on the right-hand side, we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{k^n t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \binom{k}{n-j} d_j^{(k)} \frac{t^n}{j!}.$$

Comparing coefficient t^n on both sides of the previous equation yields the desired theorem:

THEOREM 6.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} (-1)^j \binom{k}{n-j} d_j^{(k)} = k^n.$$

Applying the binomial theorem to (1.6), after some calculations, we get

$$\sum_{n=0}^{\infty} \binom{-k}{n} (-1)^n t^n = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \frac{k^{n-j}}{(n-j)! j!} d_j^{(k)} t^n.$$

Comparing the coefficient t^n on the both sides of the previous equation yields desired of the theorem:

THEOREM 6.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$\binom{-k}{n} = \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} k^{n-j} d_j^{(k)}.$$

Observe that using

$$\begin{aligned} \binom{-k}{j} &= \frac{(-k)(-k-1)\cdots(-k-j+1)}{j!} \\ &= (-1)^k \frac{k(k+1)\cdots(k+j-1)}{j!} = (-1)^k \binom{k+j-1}{j}, \end{aligned}$$

we [**1**] also gave the following result,

$$d_m^{(k)} = (-1)^m m! \sum_{j=0}^{m} \binom{k+m-j-1}{m-j} \frac{k^j}{j!}.$$

Since

$$(-1)^m m! = (m+1)\sum_{j=0}^{m} S_1(m,j)B_j,$$

[**5**,**17**,**19**,**22**], and $(-1)^m m! = (m+1)D_m$, where D_m denotes the Daehee numbers [**5**], we get a relation between the numbers $d_m^{(k)}$ and D_m, with the following theorem:

THEOREM 6.5. *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$d_m^{(k)} = (m+1)D_m \sum_{j=0}^{m} \binom{k+m-j-1}{m-j}\frac{k^j}{j!}. \tag{6.1}$$

We now give a relation among the numbers $d_m^{(k)}$, $S_1(m,n)$ and B_m. Combining (6.1) with the above equation

$$D_m = \sum_{v=0}^{m} S_1(m,v)B_v,$$

[**3**, **5**, **17**, **22**], we arrive at the following theorem:

THEOREM 6.6. *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$d_m^{(k)} = (m+1)\sum_{v=0}^{m}\sum_{j=0}^{m}\binom{k+m-j-1}{m-j}S_1(m,v)B_v\frac{k^j}{j!}.$$

We now give a relation between the numbers $S_2(m,n)$ and d_m. Using (1.4) and the negative binomial theorem, we get

$$\sum_{m=0}^{\infty} d_m \frac{t^m}{m!} = \frac{1}{1-t}\sum_{n=0}^{\infty}\binom{-1}{n}(e^t-1)^n.$$

Combining the above equation with (1.3), after some elementary calculations, since $S_2(m,n) = 0$ when $n-m>0$, we obtain

$$\sum_{m=0}^{\infty} d_m \frac{t^m}{m!} = \left(\sum_{k=0}^{\infty} t^k\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^{(n)}S_2(m,n)\frac{t^m}{m!}\right).$$

By using the Cauchy product rule formula in the right-hand side, we get

$$\sum_{m=0}^{\infty} d_m \frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{n=0}^{k}(-1)^{(n)}\frac{S_2(k,n)}{k!}t^m.$$

Comparing the coefficients of t^m on both sides of the aforementioned equations yields the following theorem:

THEOREM 6.7. *Let $n \in \mathbb{N}_0$. Then we have*

$$d_m = m!\sum_{k=0}^{m}\sum_{n=0}^{k}\frac{(-1)^{(n)}S_2(k,n)}{k!}.$$

We now give a relation between the numbers $Y_{n,2}(\lambda)$ and d_m. By using (1.7), we get

$$\frac{(-1)}{(1-\frac{\lambda^2}{2(1-\lambda)}t)} = (1-\lambda)\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!}.$$

After some calculations in the above equation gives

$$\frac{(-1)}{(1-\frac{\lambda^2}{2(1-\lambda)}t)}\frac{e^{\frac{-\lambda^2}{2(1-\lambda)}}}{e^{\frac{-\lambda^2}{2(1-\lambda)}}} = (1-\lambda)\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!}.$$

Combining the above equation with (6.2), we obtain

$$-\sum_{n=0}^{\infty}\frac{\lambda^{2n}}{2^n(1-\lambda)^n}d_n\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{\lambda^{2n}}{2^n(1-\lambda)^n}\frac{t^n}{n!} = (1-\lambda)\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!}.$$

By using the Cauchy product rule on the left-hand side of the above equation, we also get

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}d_k\frac{\lambda^{2n}}{2^n(1-\lambda)^{n+1}}\frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficient $\frac{t^n}{n!}$ on both sides of the previous equation yields the desired theorem:

Theorem 6.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}d_k = -\left(\frac{2}{\lambda^2}\right)^n (1-\lambda)^{n+1} Y_{n,2}(\lambda). \tag{6.2}$$

References

1. E. Bozo, Y. Simsek, *Some new formulas derived from generating functions for derangement numbers*, in: *Proc. 13th Symp. on Generating Functions of Special Numbers and Polynomials (GFSNP 2023)*, Antalya, Turkey, 2023, pp. 157–160.
2. ———, *Formulas derived from relationships between derangement numbers and Peters-type Simsek numbers of the second kind*, in: *Proc. 6th Mediterr. Int. Conf. Pure Appl. Math. (MICOPAM 2023)*, Paris, France, 2023, pp. 150–153.
3. C. A. Charalambides, *Enumerative Combinatorics*, Chapman and Hall CRC, London/New York, 2002.
4. H. K. Kim, *Some identities of the degenerate higher order derangement polynomials and numbers*, Symmetry **13**(2) (2021), 176.
5. D. S. Kim, T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. **7**(117) (2013), 5969–5976.
6. T. Kim, D. S. Kim, *Probabilistic derangement numbers and polynomials*, arXiv.2401.03986.
7. T. Kim, D. S. Kim, H.-I. Kwon, L.-C. Jang, *Fourier series of sums of products of r-derangement functions*, J. Nonlinear Sci. Appl. **11**(4) (2018), 575–590.
8. T. Komatsu, Y. Simsek, *Third and higher order convolution identities for Cauchy numbers*, Filomat **30**(4) (2016), 1053–1060.
9. S. Khan, T. Nahid, M. Riyasat, *Partial derivative formulas and identities involving* 2*-variable Simsek polynomials*, Bol. Soc. Mat. Meh. **26** (2020), 1–13.
10. S. Khan, T. Nahid, M. Riyasat, *Properties and graphical representations of the* 2*-variable form of the Simsek polynomials*, Vietnam J. Math. **50**(1) (2022), 95–109.

11. N. Kilar, *Building generating functions for degenerate Simsek-type numbers and polynomials of higher order*, Montes Taurus J. Pure Appl. Math., **6**(3) (2024).
12. N. Kilar, Y. Simsek, *Combinatorial sums involving Fubini type numbers and other special numbers and polynomials: Approach trigonometric functions and p-adic integrals*, Adv. Stud. Contemp. Math. **31**(1) (2021), 75–87.
13. I. Kucukoglu, *Unification of the generating functions for Sheffer type sequences and their applications*, Montes Taurus J. Pure Appl. Math. **5**(2) (2023), 71–88.
14. I. Kucukoglu, Y. Simsek, *New formulas and numbers arising from analyzing combinatorial numbers and polynomials*, Montes Taurus J. Pure Appl. Math., **3**(3) (2021), 238–259.
15. M. Ma, D. Lim, *Degenerate derangement polynomials and numbers*, Fractal Fract. **5**(3) (2021).
16. D. Merlini, R. Sprugnoli, M. C. Verri, *The Cauchy numbers*, Discrete Math., **306**(16) (2006), 1906–1920.
17. J. Riordan, *Introduction to Combinatorial Analysis*, Princeton University Press, 1958.
18. S. Roman, *The Umbral Calculus*, Dover, 1984.
19. Y. Simsek, *Analysis of the p-adic q-Volkenborn integrals: an approach to generalized Apostol-type special numbers and polynomials and their applications*, Cogent Math., **3**(1) (2016), 1–17.
20. ———, *Peters type polynomials and numbers and their generating functions: Approach with p-adic integral method*, Math. Meth. Appl. Sci., **42**(3) (2019), 7030–7046.
21. ———, *A new family of combinatorial numbers and polynomials associated with Peters polynomials and numbers*, Appl. Anal. Discrete Math., **14**(3) (2020), 627–640.
22. ———, *Formulas for p-adic q-integrals including falling-rising factorials, combinatorial sums and special numbers*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **118**(92) (2024).
23. Y. Simsek, I. Kucukoglu, *Some classes of combinatorial numbers and polynomials attached to Dirichlet characters: Their construction by p-adic integration and applications to probability distribution functions*, in: I. N. Parasidis et al. (Eds), *Mathematical Analysis in Interdisciplinary Research*, Springer Optim. Appl., Springer, **179** (2021), 795–857.
24. R. P. Stanley, S. P. Fomin, *Enumerative Combinatorics*, 2nd ed., Cambridge University Press, **2** (2011).
25. W. D. Wallis, J. C. George, *Introduction to Combinatorics*, CRC Press, 2011.

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