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AXIOMATIZATIONS OF NATURAL NUMBERS

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Abstract. We introduce two axiomatizations of natural numbers and place them in the context of the well-known formalizations of natural numbers by Frege, Dedekind, Peano, Russell, and Devide. To this end, we are developing a methodology and notations that allow a uniform presentation of these different formalizations. We prove that our axiomatizations categorically axiomatize the structure $(N, 0, \pi)$, where the predecessor relation π can be the immediate predecessor p or the general predecessor $<$. The first three axioms for the immediate and general predecessor are exactly the same, but the fourth axioms are specific for p and $<$. One postulates that the inverse of the immediate predecessor is a function, the other that the general predecessor is a total relation. We do not postulate that the inverse is an injection or that $<$ is an order. Finally, we discuss Henkin's analysis of Peano's axiomatization in the same context.

1. Preliminaries

In what follows we assume that U is the universe, i.e., all relations are defined on U and all sets are subsets of U. Let π be a binary relation (i.e., $\pi \subseteq U^2$). We call π the *predecessor relation* and define its inverse $\sigma = \pi^{-1}$ as the *successor relation*. If $x\ \pi\ y$, we say that x *precedes* y, or that y *succeeds* x.

The sets of elements that precede or succeed some elements of a set A are defined as:

$$\pi(A) \stackrel{\text{def}}{=} \{x \mid \exists a \in A\ (x\ \pi\ a)\} \quad \text{or} \quad \sigma(A) \stackrel{\text{def}}{=} \{x \mid \exists a \in A\ (a\ \pi\ x)\}.$$

A set A has a π-*extreme* if it has an element that is not preceded by any of its elements:

$$(\exists p)\ (p \in A \wedge p \notin \sigma(A)), \quad \text{i.e., } (A \not\subseteq \sigma(A)).$$

Definition 1.1 (Founded relation). A relation is *founded* if and only if every nonempty set has a π-extreme.

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Formally stated:

$$A \neq \emptyset \to A \not\subseteq \sigma(A), \quad \text{i.e., } A \subseteq \sigma(A) \to A = \emptyset.$$

Lemma 1.1. *A relation π is founded iff it has no π-descending sequences.*

Proof. The set of the members of a π-descending sequence has no π-extreme and a π-descending sequence is easily generated from a set with no π-extremes. □

Definition 1.2 (Induction principle). A relation π satisfies the induction principle iff

$$(\forall x)(\pi(x) \subseteq A \to x \in A) \to A = U.$$

Here $\pi(x)$ should be understood as $\pi(\{x\})$.

Theorem 1.1. *A relation π is founded iff it satisfies the induction principle.*

Proof.

$$\begin{aligned} \forall A(A \subseteq \sigma(A) \to A = \emptyset) &\iff \forall A(A^c \subseteq \sigma(A^c) \to A^c = \emptyset) \\ &\iff \forall A((\forall x)(x \in A^c \to x \in \sigma(A^c)) \to A = U) \\ &\iff \forall A((\forall x)(x \notin A \to \pi(x) \not\subseteq A) \to A = U) \\ &\iff \forall A((\forall x)(\pi(x) \subseteq A \to x \in A) \to A = U). \qquad \square \end{aligned}$$

Definition 1.3 (Ancestral). The ancestral (or the transitive colsure) of a relation π is the relation π^+ such that

$$x\ \pi^+\ y \overset{\text{def}}{\iff} x\ \pi\ y \lor (\exists z_1, \ldots, z_n)\ x\ \pi\ z_1\ \pi\ \ldots\ \pi\ z_n\ \pi\ y.$$

Lemma 1.2. *Relation π is founded iff π^+ is founded.*

Proof. Immediately follows from Lemma 1.1 □

In the defining formula of π^+ we are using natural numbers. Frege in 1879. has figured out how to define π^+ without using natural numbers.

Definition 1.4 (Ancestral (after Frege)). The ancestral of a relation π is the relation π^+ such that $x\ \pi^+\ y$ iff every σ-closed set which contains x contains y. Formally stated,

$$x\ \pi^+\ y \overset{\text{def}}{\iff} x \neq y \land (\forall S \supseteq \sigma(S))(x \in S \to y \in S).$$

Quine in 1940 has proposed a modification which simplifies definition of natural numbers (cf. below).

Definition 1.5 (Ancestral (after Quine)). The ancestral of a relation π is the relation π^+ such that

$$x\ \pi^+\ y \overset{\text{def}}{\iff} x \neq y \land (\forall P \supseteq \pi(P))(y \in P \to x \in P).$$

Lemma 1.3. *The definitions of ancestral are equivalent.*

Proof. It is easy to prove that all tree definitions define the minimal transitive relation containing π. □

2. Axiomatizations

Dedekind has proved in 1888 that the following Dedekind–Peano axioms categorically axiomatize the structure $(N, 0, p)$; where $s = p^{-1}$.

(1) $-(\exists x) x \; p \; 0$ (0 has no immediate predecessors.)

(2) $(\forall x)(\exists y) x \; p \; y$ (Every number has an immediate sucessor.)

(3I) $0 \in A \wedge (\forall x)(x \in A \rightarrow s(x) \in A) \rightarrow A = U$ (Dedekind–Peano induction.)

(4) $x \; p \; y_1 \wedge x \; p \; y_2 \rightarrow y_1 = y_2$ (Immediate successor is functional.)

(5) $x_1 \; p \; y \wedge x_2 \; p \; y \rightarrow x_1 = x_2$ (Immediate successor is injective.)

Devide in 1955 proved that the following axioms categorically axiomatize the structure $(\mathbb{N}, 0, p)$:

(1′) $-(\exists x) x \; p \; y \leftrightarrow y = 0$ (Only 0 has no predecessors.)

(2) $(\forall x)(\exists y) x \; p \; y$ (Every number has an immediate successor.)

(3D) $A = s(A) \rightarrow A = \emptyset$ (Devide's axiom.)

(4) $x \; p \; y_1 \wedge x \; p \; y_2 \rightarrow y_1 = y_2$ (Immediate successor is functional.)

We propose the following two axiomatizations and prove that they categorically axiomatize the structure $(N, 0, \pi)$. The predecessor relation π can be the immediate predecessor p or the general predecessor $<$ (and their inverses are the immediate successor s and the general successor $>$). The first three axioms for the immediate and the general predecessor are completely the same:

(1) $-(\exists x) x \; \pi \; 0$ (0 has no immediate predecessors.)

(2) $(\forall x)(\exists y) x \; \pi \; y$ (Every number has a successor.)

(3F) $A \subseteq \sigma(A) \rightarrow A = \emptyset$ (π is founded.)

The fourth axioms are specific to p and $<$. One postulates that the immediate successor is functional, the other that the general successor is total:

(4p) $x \; p \; y_1 \wedge x \; p \; y_2 \rightarrow y_1 = y_2$ (Immediate successor is functional.)

(4<) $x < y \vee x = y \vee y < x$ (General successor is total.)

Notice that Devide and we postulate the functionality of the immediate successor, but not its injectivity and that we do not postulate that $<$ is an order relation.

Furthermore, notice that by induction $M \stackrel{\text{def}}{=} \{x : x = 0 \vee \exists y(y \; p \; x)\} = U$, which means that in the presence of induction axioms (1) and (1′) are equivalent.

THEOREM 2.1. *A total founded relation $<$ is an order relation.*

PROOF. The relation $<$ is irreflexive because $x < x$ would make it non-founded. The relation $<$ is transitive because from $x < y$, $y < z$ and foundedness of $<$ it follows that $x \neq z$ and $z \not< x$.

Namely, if $x = z$ then $z < y < z$, contrary to the foundedness of $<$. If $z < x$ then $x < y < z < x$, contrary to the foundedness of $<$. From totality we conclude that $x < z$. □

The immediate predecessor and the general predecessor are interdefinable:

$$<=p^+ \quad \text{and} \quad p=<^- .$$

Here $x <^- y \overset{\text{def}}{\iff} x < y \wedge -(\exists z) x < z < y$.

THEOREM 2.2. *If p satisfies p-axioms, then p^+ satisfies $<$-axioms. If $<$ satisfies $<$-axioms, then $<^-$ satisfies p-axioms.*

Furthermore, $(p^+)^- = p$ and $(<^-)^+ =<$.

PROOF. It is trivially true for axioms (1) and (2). For axiom (3F) it was proved in Lemma 1.2. For axioms ($4p$) and ($4<$) we have to prove the following.

If $<$ is total, then $<^-$ is right functional and if p is right functional, then p^+ is total.

Let us suppose that $<$ is total and $x <^- y_1 \wedge x <^- y_2$. Then we have the following.

$y_1 < y_2 \to x <^- y_1 < y_2 \to -(x <^- y_2)$, contrary to what we supposed. Hence, $-(y_1 < y_2)$.

$y_2 < y_1 \to x <^- y_2 < y_1 \to -(x <^- y_1)$, contrary to what we supposed. Hence, $-(y_2 < y_1)$.

But $<$ is total, hence $y_1 = y_2$.

Let us suppose that p is right functional. We first prove that 0 is p^+ connected with all the other numbers.

If this were not the case, there would be a p-minimal $m \neq 0$ to which 0 is not p^+-connected. Let n be such that $n\ p\ m$ (it exists by ($1'$)). This n must be p^+-connected with 0, i.e., there are $n_1, \dots, n_k$ such that

$$0\ p\ n_1 \dots n_k\ p\ m \vee 0\ p\ n \vee n\ p\ n_k \dots n_1\ p\ 0 \vee n\ p\ 0 \vee 0 = n.$$

In the first two cases it follows that $0\ p^+\ m$, which is a contradiction. In the third case it follows that $n_k = m$ (because p is right functional), so $m\ p^+\ 0$, which is again a contradiction. In the fourth case $m = 0$ (because p is right functional) which is again a contradiction. In the final case it follows that $0\ p\ m$, again a contradiction. Hence, 0 is p^+-connected with all the other numbers.

Now we prove that every $x \neq 0$ is p^+-connected with all the other numbers.

If there is a number $x \neq 0$ that is not p^+-connected with all the other numbers, then there is a p-minimal $m \neq 0$ that is not p^+-connected with all the other numbers. Let n be such that $n\ p\ m$ (it exists by ($1'$)). This n must be p^+-connected with every other number l, i.e., there are $n_1, \dots, n_k$ such that

$$l\ p\ n_1 \dots n_k\ p\ n \vee l\ p\ n \vee n\ p\ n_k \dots n_1\ p\ l \vee n\ p\ l \vee l = n.$$

In the first two cases it follows that $l\ p^+\ m$, which is a contradiction. In the third case it follows that $n_k = m$ (because p is right functional), so $m\ p^+\ l$, which is again a contradiction. In the fourth case $m = l$ (because p is right functional) which is again a contradiction. In the final case it follows that $l\ p\ m$, again a contradiction.

Hence, every number is p^+-connected with all the other numbers.

It is trivially true that $(p^+)^- = p$ and $(<^-)^+ =<$. □

THEOREM 2.3. *The immediate successor is injective, i.e.,*

$$x_1 \; p \; y \wedge x_2 \; p \; y \to x_1 = x_2.$$

PROOF. We have proved that p^+ is total. Hence,

$$x_1 = x_2 \vee x_1 \; p \; y_1 \; p \ldots p \; y_m \; p \; x_2 \vee x_2 \; p \; z_1 \; p \ldots p \; z_n \; p \; x_1.$$

In the second case $y = y_1$ (because the immediate successor is functional) and it follows that $y \; p^+ \; y$, which is a contradiction (because p^+ is founded).

In the third case $y = z_1$ (because the immediate successor is functional) and it follows that $y \; p^+ \; y$, which is a contradiction (because p^+ is founded). Hence, $x_1 = x_2$. □

THEOREM 2.4. *The structure* $(\mathbb{N}, 0, p)$ *is categorically axiomatized by*

(1) $-(\exists x) \; x \; p \; 0$

(2) $(\forall x)(\exists y) \; x \; p \; y$

(3F) $A \subseteq s(A) \to A = \emptyset$

(4p) $x \; p \; y_1 \wedge x \; p \; y_2 \to y_1 = y_2$.

PROOF. Dedekind proved that his axioms (1)–(5) are categorical. Hence, it is enough to prove that his axioms follow from our axioms. We have proved (in Theorem 2.3.) that (5) follows from our axioms. Now we prove that (3F) $\iff$ (3I), given (1), (4p) and (5).

In (3I) we may write $y = s(x)$, because there is the unique such y, by (5). But then $x = p(y)$, for $y \neq 0$, because there is the unique such x, by (4p) and (1$'$) which is equivalent to (1). Hence, (3I) is equivalent to

$$0 \in A \wedge (\forall y \neq 0)(p(y) \in A \to y \in A) \to A = U.$$

But $0 \in A$ is equivalent to $p(0) \subseteq A \to 0 \in A$ because $p(0) = \emptyset$. Hence, (3I) is equivalent to

$$(\forall y)(p(y) \subseteq A \to x \in A) \to A = U.$$

But this is equivalent to (3F) by Therem 1.1. □

THEOREM 2.5. *The structure* $(\mathbb{N}, 0, <)$ *is categorically axiomatized by*

(1) $-(\exists x) \; x < 0$

(2) $(\forall x)(\exists y) \; x < y$

(3F) $A \subseteq \; >(A) \to A = \emptyset$

(4<) $x < y \vee x = y \vee y < x$.

PROOF. It follows from Theorem 2.2 and Theorem 2.4. □

3. History

Within a universe of concepts, Frege in 1879. defined the number 0 and the relation p which satisfy (1), (2), (4) and (5). Then he defined p^+ as in Frege's definition of ancestral (cf. above). Finally he defined natural numbers as

$$\mathbb{N}(x) \overset{\text{def}}{\iff} x = 0 \vee 0 \; p^+ \; x.$$

Dedekind in 1888 defined a relation p which satisfies (1), (2), (4) and (5) on an arbitrary universe U containing a prominent element 0. Then he defined the set of natural numbers as

$$\mathbb{N} \overset{\text{def}}{=} \bigcap\{S : 0 \in S \wedge S \supseteq s(S)\}.$$

This is one and the same definition from which (3) immediately follows. Namely, cf. Frege's definition of ancestral,

$$\mathbb{N}(x) \overset{\text{def}}{\iff} x = 0 \vee 0\, p^+\, x \overset{\text{def}}{\iff} (\forall S)(0 \in S \wedge S \supseteq s(S) \to x \in S),$$

$$x \in \mathbb{N} \overset{\text{def}}{\iff} x \in \bigcap\{S : 0 \in S \wedge S \supseteq s(S)\} \overset{\text{def}}{\iff} (\forall S)(0 \in S \wedge S \supseteq s(S) \to x \in S).$$

Frege–Dedekind's definition of natural numbers is not vacuous only if there are sets (in Frege's case concepts) satisfying $S \supseteq s(S)$. Such sets are necessarily infinite (because there are infinitely many numbers that succeed 0). Hence, Frege-Dedekind's definition of natural numbers presupposes the existence of infinite sets.

Quine in 1940 defined the number 0 and the relation p which satisfy (1), (2), (4) and (5). Then he defined p^+ as in Quine's definition of ancestral, cf. above. Finally he defined natural numbers as

$$N(x) \overset{\text{def}}{\iff} x = 0 \vee 0\, p^+\, x.$$

It is easy to prove that Quine's definition is equivalent with Frege's:

$$\begin{aligned} &0 \in M \wedge s(M) \subseteq M \overset{\text{ind}}{\Longrightarrow} M = N \\ &\equiv 0 \notin M^c \wedge p(M^c) \subseteq M^c \Rightarrow M^c = \emptyset \\ &\equiv 0 \notin M \wedge p(M) \subseteq M \Rightarrow M = \emptyset \\ &\equiv M \neq \emptyset \wedge p(M) \subseteq M \overset{\text{Quine}}{\Longrightarrow} 0 \in M. \end{aligned}$$

Quine's definition of natural numbers is not vacuous only if there are sets satisfying $P \supseteq p(P)$. Such sets are finite (because there are only finitely many numbers that precede any number). Hence, Quine's definition of natural numbers does not presuppose the existence of infinite sets (although the universe has to be infinite because it contains arbitrarily large finite sets).

Strictly speaking neither Frege nor Dedekind or Quine axiomatized natural numbers. They defined them within a set theory (theory of concepts in Frege's case). In Dedekind's case it was an informal set theory, in Quine's case it was a first order set theory $ZF \setminus \omega$ (i.e., a theory of hereditary finite sets).

The first axiomatization was Peano's one in 1889. He used Dedekind's defining properties as his axioms. In fact, his nine axioms axiomatized four notions: "*number*", "*one*", "*successor*" and "*is equal to*". Today we would consider that his axioms 2, 3, 4 and 5, which deal with identity, belong to the underlying logic. That leaves the five axioms that have become universally known as Peano axioms or Dedekind–Peano axioms. This modern version was introduced by Russell in 1903 ([**7**, on page 181] with 0 instead of Peano's 1 as the first natural number).

Devide in 1955 proved injectivity of successor in a different way than we did (i.e., without p^+). He also proved that his axiom (3D) implies (3F) without realising that this is the foundedness axiom. We noticed this in [**8**] and this motivated our two axiomatizations.

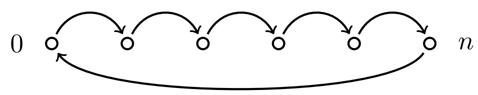
The best modern analysis of Peano's axiomatization was given by Henkin in 1960. He called structures $(M, 0, p)$ that satisfy (2), (4) and (3I) *inductive models*. So, inductive models are structures $(M, 0, s)$ which satisfy Dedekind–Peano's induction principle (3I), with successor $s = p^{-1}$ being a total function. If an inductive model satisfies (1) 0 has no predecessors and (5) the successor is injective, Henkin calls it *Peano model*.

Henkin proved that homomorphisms from the Peano models to inductive models are unique. There follows, as a corollary, that all Peano models are isomorphic (that is really Dedekind's categoricity theorem). He also proved that inductive models are exactly the homomorphic images of Peano models and introduced the appropriate congruencies.

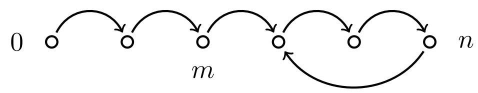
The intuitive reasoning behind Henkin's arguments is as follows. Inductive models are sequences of successors emanating from 0. They contain 0, successor of 0, successor of successor of 0, etc. And they contain nothing else (this follows from induction). In other words, in an inductive model everything is s-reachable from 0.

If the successor function is injective and 0 is not a successor, then every successor in the sequence should be new and such inductive model is isomorphic to the standard Peano model:

If the successor function is injective and 0 is a successor, then there is the unique n such that $0 = s(n)$ and such inductive model is a cycle going from 0 to 0 through n (it is a congruence modulo n):



If successor function is not injective then there is the first pair $0 \neq m < n$ such that $s(m) = s(n)$. Such inductive model is a straight sequence from 0 to $s(m) \neq 0$ with a cycle from $s(m)$ to $s(m)$ through n (we call it a congruence modulo m/n):



It is easy to see that the second and the third cases are homomorphic images of the first case. Hence, congruencies modulo n and modulo m/n are exactly the congruencies of Peano structure $(N, 0, s)$. Note that the n-type is the standard notion of congruence in number theory, while the n/m-type is never discussed in number theory.

Henkin also defined recursive models as models on which recursive functions are uniquely defined and proved that the Peano model is the only recursive model. Hence, we have three types of inductive models and only one recursive model.

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