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THE ANGEL PROBLEM ON TRIANGULAR AND HEXAGONAL BOARDS

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Abstract. The Angel problem, introduced by Conway in 1982, is a two-player game played on an infinite board where an Angel of power k competes against the Devil. We examine variations of this game on triangular and hexagonal boards. Using Máthé's proof technique originally developed for the square board, we prove that the King (Angel of power 1) can win on a triangular board. Through a mapping between hexagonal and square boards, we then establish two results for the hexagonal board: first, that the Devil can defeat the King, and second, that an Angel of power 2 can win. Our proof for the triangular board involves analyzing the perimeter bounds of connected sets and developing a wall-following strategy for the King, while our hexagonal board results utilize transformations that map to known results from the square board case.

1. Introduction

The Angel problem was introduced by Conway in 1982 [**1**] and consists of the following. The Angel and the Devil are playing the following game on an infinite chessboard, where each square can be assigned a pair of integers (x, y). On its turn, the Devil can "eat" any square of the board, i.e., that square becomes forbidden for the Angel. The Angel is a piece that can move from (x, y) to any square (X, Y) such that $|X - x|$ and $|Y - y|$ are at most k, where k is a natural number called the Angel's power. Without loss of generality, let the Angel start the game at $(0, 0)$. The Devil wins if it traps the Angel, i.e., surrounds it with a barrier of eaten squares of thickness at least k. The Angel wins if it can move forever. [**3**]

The question arises: "Can an Angel of some power defeat the Devil?" The Angel problem was eventually solved in 2007 by four independent proofs: Máthé showed that the Angel of power 2 wins [**6**]; Gács established that sufficiently large power Angels can win [**4**]; Bowditch proved that the Angel of power 4 wins [**2**]; and Kloster provided another proof for the Angel of power 2 [**5**].

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The King is a player who, on its turn, can move only to cells that share a vertex with the current cell. Therefore, the King is the Angel of power 1.

In this paper, we will consider what happens when we relocate the players of this game to different boards: triangular and hexagonal. The rules remain almost the same. On its turn, the Devil “eats" one cell, while the Angel with power k can fly to a cell that is at most k cells away. That is, an Angel with power k can fly to any cell that the King could reach in at most k moves. The Angel loses if it is trapped, i.e., if the only cell it can fly to is its current cell.

By modifying Máthé's proof for the square board [**6**], we will prove that on the triangular board the King wins, and then, based on that, we will prove that on the hexagonal board, the Angel of power 2 wins.

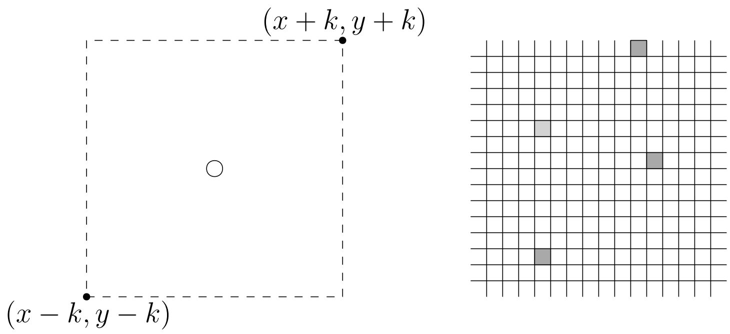

FIGURE 1. On the left, the Angel takes a move, while on the right, the Devil eats a square.

2. Triangular board

Máthé's proof geometrically relies on a certain inequality stating that a connected set of n cells cannot have an arbitrarily large perimeter. On the square board, the upper bound for the perimeter is $2n + 2$. Comparing this with the hexagonal and triangular boards, we realize that the coefficient 2 is actually the number of cell sides minus 2. On the hexagonal board, the corresponding upper bound is $4n + 2$, while on the triangular board it is $n + 2$. The fact that the bound is better for triangles suggests that a weaker winning Angel can be found on the triangular board. We will show that on the triangular board, the Angel in question (the King) wins.

DEFINITION 2.1. The triangle $(x, y) \in \mathbb{Z}^2$ corresponds to a triangle in the plane with side length 1, whose center of the height to the horizontal side is at the point $\left(\frac{1}{2}x, \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}y\right) \in \mathbb{R}^2$. For even $x + y$, the triangle points downward, and for odd $x + y$, it points upward.

LEMMA 2.1. *Let S be a set of n triangles from $\mathbb{Z}^2$ ($n \geqslant 1$). If S is connected, then the boundary of the set $\bigcup S$ consists of at most $n + 2$ edges (unit segments).*

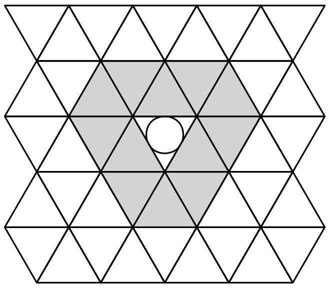

FIGURE 2. The King on the triangular board.

PROOF. It can easily be proven by induction. For $n = 1$, it is obviously true. Let us consider a set of $n+1$ connected triangles. Then, there must be a triangle whose removal leaves the remaining set connected (if we consider it as a graph, we can choose any leaf of the spanning tree). By the inductive hypothesis, the remaining set has at most $n+2$ edges. By returning the previously removed triangle to the connected set of triangles, we delete (at least) one edge and add at most two new ones. Thus, the set of $n+1$ connected triangles has at most $n+2-1+2 = (n+1)+2$ edges, thereby proving the lemma. □

DEFINITION 2.2. A move by the Devil to a cell where the Angel previously stood, or a cell the Angel could have jumped to but did not, is called an ambush.

DEFINITION 2.3. The Nice Devil is a Devil who does not set ambushes.

As stated in Theorem 8.1 of [**3**] and used in [**6**], the Angel's prospects do not worsen if, after each move, every cell he could have moved to but did not choose is regarded as removed from further play; this observation is also the motivation for introducing the Nice Devil. In particular, we will apply the same idea on the new boards: we minimally restrict the Angel's movement rules so that he never returns to a cell that was previously reachable, and hence no back-and-forth movement is possible.

We consider the game of the Nice Devil and the King, who applies a strategy similar to Máthé's Runner. First, we will prove analogue of Theorem 2.7 from [**6**] (regarding the equivalence of whether the Angel plays against the Devil or the Nice Devil) in a more *general* way and reformulate $B(N)$ into any finite set of cells S, so that we can apply it to the triangular board as well. If S is a finite set of cells, then a Devil's strategy that ensures the Angel never leaves S is called an S-winning strategy. It is clear that the Devil has a winning strategy if and only if, for some finite set S, he has an S-winning strategy (as stated in Claim 2.1 of [**6**]).

LEMMA 2.2 (Wästlund [**7**]). *Let S be a finite set of cells. If the Devil has an S-winning strategy, then the Nice Devil also has one.*

PROOF. We will prove the lemma by induction. Suppose the Devil has an S-winning strategy Θ that does not set ambushes in the first n moves. Let us then prove that there exists an S-winning strategy that does not set ambushes in the first $n+1$ moves.

Let the Devil play according to Θ for the first n moves. Suppose that in the $(n+1)$-th move, Θ requires the Devil to set an ambush on cell a (otherwise, there is nothing to prove). In that case, we modify the strategy so that in that move, the Devil makes any move that is not an ambush. Afterward, the Devil plays according to Θ as long as the Angel does not move to a. Whenever the Angel moves to cell a, the Devil switches to playing as if it were following Θ and the Angel had moved to cell a the first time it could have.

We conclude that there is no n for which the Devil is forced to set an ambush during the first n moves to keep the Angel within S. Since the Devil has only a finite number of *reasonable* moves (cells in S and those the Angel can reach in one move from S), the lemma follows. □

Before the game begins, we ask the Nice Devil to eat all the triangles to the left of the line $y = -x\sqrt{3}$ (i.e., triangles $\{(x, y) \mid y < -x\}$). This certainly won't harm him. For our purposes, it is sufficient that the Nice Devil doesn't eat the triangles that the King uses.

Let us now imagine our board as a maze: the eaten triangles are high walls that the King cannot pass through. At every moment, we imagine that the King is in a triangle and facing a certain direction. (We will adopt some somewhat informal terminology, such as *the King's left hand*, but we hope this will be much clearer to the reader than any formal version.) Before making his first move, the King is positioned at $(0, 0)$, facing up and to the left, with his left hand touching the wall (there is a wall because the triangle $(-1, 0)$ has already been eaten). He could even close his eyes: he only needs his left hand to find the way. Since the triangles in $\{(x, y) \mid y < -x\}$ are eaten before the game begins, the King will always have a wall on his left side. Let us imagine that the King also holds green paint in his left hand and paints a line along the walls he touches.

The King will behave similarly to the Máthé's Runner [**6**]. The changes are that the King can move to any cell that shares a vertex with the current one (e.g., from $(0, 0)$ to $(-1, 1)$), and of course, only one cell per move (power 1). Naturally, the King is always on a cell that shares an edge with an eaten cell, while his left hand touches and colors the wall.

It is easy to verify that the King colors at least one wall in every move. Thus, in t moves, he colors at least t walls (counting multiple colors if necessary). Since the *lowest* vertex of the triangle $(0, 0)$ corresponds to the origin of the coordinate system, the King's green line starts from $(0, 0) \in \mathbb{R}^2$.

Note that if we see a painted wall, we know which direction the King was moving, since the walls are always on his left side. For simplicity, let us call the triangles the King has (already) run through blue triangles.

Lemma 2.3. *If the King uses the strategy defined above, the following holds true:*

(1) *In each step, the green (directed) line consists of directed segments of length one. For each directed green segment, there is an eaten triangle on the left side and a blue (uneaten) triangle on the right side.*
(2) *Even if the King paints the same wall again, he always paints it in the same direction.*
(3) *If there is a wall that has been painted twice, then the first wall that the King painted twice is the first wall that the King ever painted. Therefore, in this case, the green line is closed.*

Proof. The first is trivial since the Nice Devil does not eat blue triangles. The second follows from the fact that the King only paints on the left side.

To prove the third statement, let us consider the first time the King paints a previously painted edge and denote this edge as e. Clearly, if edge e is the first edge ever painted, then the King has created a loop before painting e for the second time, and he will run infinitely along this loop (in fact, along its inner side, in a clockwise direction).

Assume that e is not the first edge ever painted. Then the King must have painted edge f immediately before he first painted e. From the first two parts of this lemma, it follows that he must have first painted f twice, which is a contradiction. □

Therefore, as long as the King's green line does not return to the x-axis, the King cannot color the same wall twice that he colored at the very beginning of the game, and moreover, he colors each wall at most once. We will use this fact later.

Theorem 2.1. *The King's green line will never again reach the x-axis.*

Proof. Let's consider the first time the green line reached the x-axis. Suppose this happened in the t-th move of the King. Thus, the Nice Devil has already eaten (at most) t triangles, and the King has already colored at least t walls. From the previous information, it is easy to conclude that these are different walls. Let d be the number of triangles that the Nice Devil has eaten in the region of triangles $\{(x, y) \mid y \geqslant 0 \wedge y \geqslant -x\}$. Let a be the number of different walls that the King has colored before his green line reached the x-axis. We have

$$d \leqslant t \quad \text{and} \quad a \geqslant t. \tag{2.1}$$

Let uss stop the game at this point and erase everything from the lower half of the board: that is, we will erase all eaten triangles and all triangles eaten before the start of the game in $\{(x, y) \mid y < 0\}$. (There is no green line in this half-plane.) We will also erase all triangles eaten before the start of the game except for $\{(-x-1, x) \mid 0 \leqslant x \leqslant N-1\}$ and $\{(-x-2, x) \mid 0 \leqslant x \leqslant N-1\}$ for some sufficiently large integer N (much larger than the coordinates of any triangle where either of the players was). Thus, there are exactly $2N + d$ eaten cells remaining on the board.

Now, we ask the King to continue his path and coloring as he did before. However, we will not allow the Nice Devil to eat any more triangles. Sooner or later, the King will return to his initial position in the triangle $(0, 0)$, and the green line will return to the point $(0, 0)$. Therefore, the green line will form a loop. Let's denote its length by l (this is also the total number of colored walls).

LEMMA 2.4. *For the length l defined above, the following holds:*

$$l \geqslant a + 2N + 3.$$

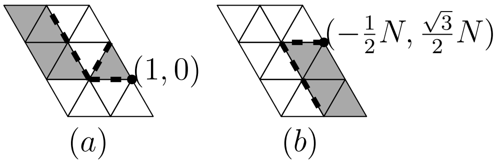

FIGURE 3. Key points of the green line.

PROOF. During the regular game, the Angel colored different walls until the green line reached the x-axis, say at the point $(x_0, 0) \in \mathbb{R}^2$. Thus, the smallest value that x_0 can take is 0 or 1, but in both cases, the green line passes through $(1, 0)$ (see Figure 3). Before the green line reaches $\big(-\frac{1}{2}N, \frac{\sqrt{3}}{2}N\big)$, it must travel a path of at least $N + 1$ in length. Then, the green line covers the segments $\big[\big(-\frac{1}{2}N, \frac{\sqrt{3}}{2}N\big), \big(-\frac{1}{2}N - 1, \frac{\sqrt{3}}{2}N\big)\big]$, $\big[\big(-\frac{1}{2}N - 1, \frac{\sqrt{3}}{2}N\big), (-1, 0)\big]$, and $[(-1, 0), (0, 0)]$. Therefore, $l \geqslant a + N + 1 + 1 + N + 1 = a + 2N + 3$. □

It is easy to verify that the eaten triangles within the green line are connected. This connected set has at least l edges on its boundary (possibly many more). Thus, the number of triangles in this set is at least $l - 2$ according to Lemma 2.1. On the other hand, there are at most $2N + d$ triangles in this set, so

$$2N + d \geqslant l - 2.$$

From Lemma 2.4, we further have

$$2N + d \geqslant a + 2N + 3 - 2 = 2N + a + 1.$$

From (2.1), it follows that

$$2N + t \geqslant 2N + t + 1,$$

which is a contradiction. Therefore, the green line cannot reach the x-axis again during the game. □

We have given a strategy for the King such that the imagined green line will never reach the x-axis again, and thus the King will never paint the same wall again. Since in a bounded region there are only paths of finite length where the green line could go, it is evident that the green line will eventually leave any bounded region; therefore, the King will leave any bounded region, no matter how the Nice Devil plays. Based on Lemma 2.1, we conclude:

THEOREM 2.2. *The King can defeat the Devil on a triangular board.*

3. Hexagonal board

Let the Angel have a *strength* j if their maximum number of legal moves from any position is j. With the 1-1 mapping between the triangular and square boards shown in Figure 4, the King of the triangular board transforms into an Angel of strength 12 on the square grid. The movement pattern of this Angel is restricted but not translation invariant. With the standard chessboard coloring, this Angel will move as shown in Figure 5(a) from squares of one color, and as shown in Figure 5(b) from squares of the other color. It follows that the Angel in Figure 5(c), who can move from the center square to any other square within a 5 by 3 rectangle, can escape from the Devil. This Angel of strength 14 is the weakest known translation-invariant winning Angel [**7**].

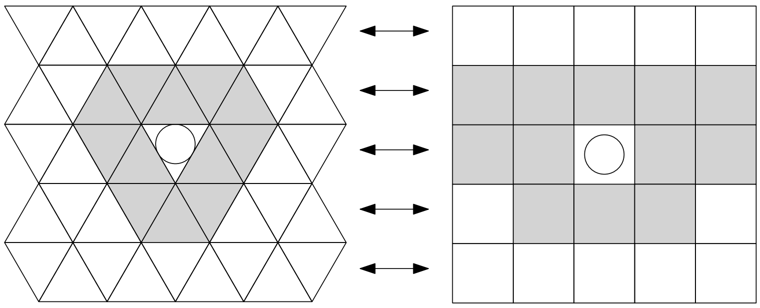

FIGURE 4. 1–1 mapping.

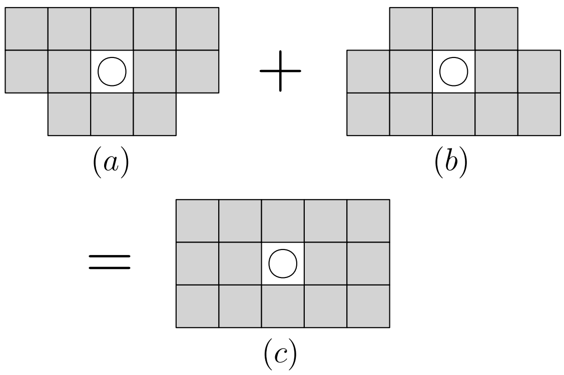

FIGURE 5. Angel of strength 14.

DEFINITION 3.1. The hexagon $(x, y) \in \mathbb{Z}^2$ corresponds to a hexagon in the plane with side length 1, whose center is at the point $\left(\frac{3}{2}x, \operatorname{odd}(x) \cdot \frac{\sqrt{3}}{2} + \sqrt{3}y\right) \in \mathbb{R}^2$, where $\operatorname{odd}(x) = \frac{1-(-1)^x}{2}$.

Berlekamp [**1**] showed that the Devil can defeat the Angel of power one (the King) on a square board of at least 32×33, and we state this theorem without proof.

THEOREM 3.1. *The Devil can defeat the King on a square board.*

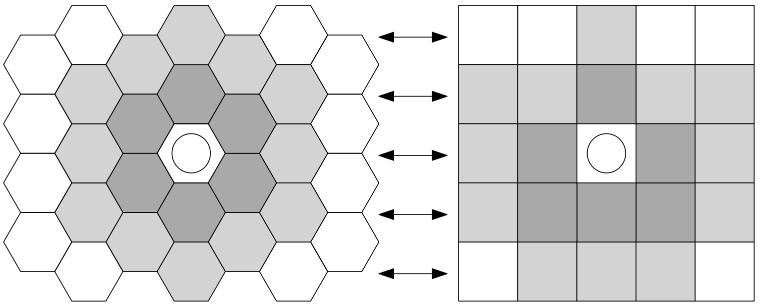

FIGURE 6. 1–1 mapping.

By applying the one-to-one mapping between the hexagonal and square boards shown in Figure 6, the King of the hexagonal board is transformed into an Angel of strength 6 on the square grid. It follows that he is weaker than the King of the square board, so from the previous theorem, we directly obtain the following.

THEOREM 3.2. *The Devil can defeat the King on a hexagonal board.*

Using this same mapping, the Angel of power 2 on the hexagonal board is mapped to an Angel of strength 18 on the square board (Figure 6). Notice that he is then strictly stronger than the Angel of strength 14 shown in Figure 5(c), and thus we obtain the following.

THEOREM 3.3. *The Angel of power* 2 *can defeat the Devil on a hexagonal board.*

References

1. E. R. Berlekamp, R. K. Guy, J. H. Conway, *Winning Ways for Your Mathematical Plays*, Vol. 3. 2nd ed., CRC Press, 2003.
2. B. H. Bowditch, *The angel game in the plane*, Comb. Probab. Comput. **16**(3) (2007), 345–362.
3. J. H. Conway, *The angel problem*, in: *Games of no chance* **29** (1996), 3–12.
4. P. Gács, *The angel wins*, preprint arXiv:0706.2817, 2007.
5. O. Kloster, *A solution to the angel problem*, Theor. Comput. Sci. **389**(1-2) (2007), 152–161.
6. A. Máthé, *The angel of power* 2 *wins*, Comb. Probab. Comput. **16**(3) (2007), 363–374.
7. J. Wästlund, *A weaker winning angel*, 2008, available at https://www.math.chalmers.se/~wastlund/weakerAngel.pdf

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