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A NOTE ON FIBONACCI–HERMITE POLYNOMIALS

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Abstract. We first review and analyze the golden integral and its definitions and some properties. Then we introduce a new generalization of the Hermite polynomials via the golden exponential function (called Fibonacci–Hermite polynomials) and investigate several properties and relations. We derive some explicit and implicit summation formulas for mentioned polynomials. Then, we analyze derivative properties and provide a higher-order difference equation of the Fibonacci–Hermite polynomials. Moreover, we examine a recurrence relation and integral representation. In addition, we obtain some properties of Fibonacci-Bernstein polynomials. Lastly, we obtain a correlation between the Fibonacci–Hermite polynomials and the Fibonacci–Bernstein polynomials.

1. Introduction

Throughout the paper, let $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{R}$ denote, respectively, the set of all natural numbers, the set of all nonnegative integers, and the set of all real numbers.

The golden ratio is frequently used in many branches of science as well as mathematics. Interestingly, this mysterious number also appears in architecture and art. Miscellaneous properties of golden calculus (or F-calculus) have been introduced and investigated in detail by Nalci and Pashaev [**12**, **15**], which are the key references for golden calculus. Also, for more information, readers can refer to the references [**2**, **5**, **6**, **8**, **10**–**12**, **14**–**17**, **20**]. Here, we mention some definitions and properties related to golden calculus.

The Fibonacci sequence, taking its name from Leonardo Fibonacci (1170–1250), is defined, for $n \geqslant 2$, by the recurrence relation $F_{n+1} = F_n + F_{n-1}$ with initial values $F_0 = 0$ and $F_1 = 1$. The first few terms of this sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$ (cf. [**2**, **5**, **6**, **8**–**17**, **20**–**22**]). The Binet formula of the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.1}$$

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where $\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618033\ldots$ called as the golden ratio and $\beta = \frac{1-\sqrt{5}}{2} \approx -0.618033\ldots$ called as the silver ratio. Also, it is known that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$.

Fibonomials (golden binomial coefficients) are defined, for $1 \leqslant k \leqslant n$, by

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!}, \tag{1.2}$$

where F-factorial (or say, golden factorial) is given as

$$F_n! := F_n F_{n-1} F_{n-2} \ldots F_2 F_1,$$

with $F_0! := 1$. Here $\binom{n}{0}_F = 1$ and $\binom{n}{k}_F = 0$ for $k > n$. Many properties of $\binom{n}{k}_F$ are listed in [**17**].

F-analog of $(x+y)^n$ (golden binomial theorem) is provided, for $n \in \mathbb{N}$, by (cf. [**12**, **15**])

$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\binom{k}{2}} x^k y^{n-k}$$

with $(x+y)_F^0 := 1$. Also, it is denoted that $(x+y)_F := (x+y)_F^1$.

The F-analogs of the usual exponential function (say, also the golden exponential functions) are defined by (cf. [**2**, **5**, **6**, **8**, **11**, **14**–**17**, **20**])

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!}, \tag{1.3}$$

$$E_F^t = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{t^n}{F_n!}. \tag{1.4}$$

It is noted that $e_F^{xt} E_F^{yt} = e_F^{(x+y)_F t}$ and $e_F^{xt} e_F^{yt} = e_F^{(x+_F y)t}$, where (cf. [**20**])

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}.$$

The golden F-derivative operator is defined, for $f(x)$ being any function, by (cf. [**2**, **8**, **12**, **14**–**17**, **20**])

$$D_F^x[f(x)] = \frac{f(\alpha x) - f(\beta x)}{x(\alpha - \beta)}, \tag{1.5}$$

which is linear, namely it satisfies $D_F^x[af(x) + bg(x)] = aD_F^x[f(x)] + bD_F^x[g(x)]$, for a, b being two scaler and $f(x), g(x)$ being any functions. It can be readily seen from (1.3) and (1.5) that

$$D_F^x[x^n] = F_n x^{n-1},\ D_F^x[e_F^{xt}] = te_F^{xt} \text{ and } D_F^x[E_F^{xt}] = tE_F^{-xt}. \tag{1.6}$$

Also, the multiplication rule and quotient rule, for $f(x)$ and $g(x)$, of the golden derivative are given as

$$D_F^x[f(x)g(x)] = g(\alpha x) D_F^x[f(x)] + f(\beta x) D_F^x[g(x)], \tag{1.7}$$

$$D_F^x\Big[\frac{f(x)}{g(x)}\Big] = \frac{g(\alpha x) D_F^x[f(x)] - f(\beta x) D_F^x[g(x)]}{g(\alpha x) g(\beta x)}. \tag{1.8}$$

The golden trigonometric functions, Fibonacci sine, and cosine functions are defined as follows

$$\sin_F(t) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{F_{2n+1}!}, \quad \cos_F(t) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{F_{2n}!}, \tag{1.9}$$

respectively. It can be observed from (1.5), (1.9) that

$$D_F^x[\sin_F(xt)] = t\cos_F(xt) \text{ and } D_F^x[\cos_F(xt)] = -t\sin_F(xt).$$

The fibonomial convolution of two sequences is defined by Krot [**10**] as follows

$$c_n = a_n * b_n = \sum_{k=0}^{n}\binom{n}{k}_F a_k b_{n-k},$$

where a_n and b_n are two sequences provided by

$$A_F(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{F_n!} \text{ and } B_F(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{F_n!}.$$

Therefore, it can be written that

$$C_F(t) = A_F(t)B_F(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{F_n!},$$

which resembles the golden form of the usual Cauchy product.

Using (1.3), Pashaev et al. [**16**] defined the Bernoulli–Fibonacci polynomials and related numbers. Then the Euler–Fibonacci numbers and polynomials and the Apostol–Bernoulli–Fibonacci and Apostol–Euler–Fibonacci of order α were introduced in [**6**, **8**, **11**, **20**, **21**], and also some identities and matrix representations for Bernoulli–Fibonacci polynomials and Euler–Fibonacci polynomials were provided. The Fibonacci–Bernoulli polynomials and Fibonacci–Euler polynomials are defined by

$$\sum_{n=0}^{\infty} B_{n,F}(x)\frac{t^n}{F_n!} = \frac{te_F^{xt}}{(e_F^t - 1)}, \qquad \sum_{n=0}^{\infty} E_{n,F}(x)\frac{t^n}{F_n!} = \frac{2e_F^{xt}}{(e_F^t + 1)}. \tag{1.10}$$

The numbers of $B_{n,F}(x)$ and $E_{n,F}(x)$ are determined as $B_{n,F}(0) := B_{n,F}$ and $E_{n,F}(0) := E_{n,F}$, respectively. Several properties and relations of the polynomials in (1.10) have been investigated and analyzed in the papers [**2**, **6**, **8**, **11**, **16**, **20**, **21**], also see the references cited therein.

The classical Hermite polynomials are defined by the following exponential generating function to be (see [**3**, **4**, **7**, **12**])

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt - t^2}.$$

For a long time, the Hermite polynomials and their various generalizations have been extensively studied and investigated by many mathematicians and physicists (see [**3**, **4**, **7**, **12**, **19**] and cited references therein).

2. A Review for golden Integral (F-Integral)

In this section, we focus on the golden integral to review and investigate its definition and some of its properties by the (p,q)-integral (or say, post quantum integral). By substituting $p \to \alpha = \frac{1+\sqrt{5}}{2}$ and $q \to \beta = \frac{1-\sqrt{5}}{2}$ in the definitions and formulas of (p,q)-calculus (or say, post quantum calculus) from [**18**], we now subsequently define some corresponding definitions and terms of golden calculus as follows.

The function $G(x)$ is a golden antiderivative of $g(x)$ if $D_F G(x) = g(x)$. It is denoted by $\int g(x)\, d_F(x)$ or $\int g(x)\, d_{\alpha,\beta}(x)$. Note that we say "a" golden antiderivative instead of "the" golden antiderivative, because, as in ordinary calculus, an antiderivative is not unique. In ordinary calculus, the uniqueness is up to a constant since the derivative of a function vanishes if and only if it is a constant. The situation in the golden calculus is more subtle. $D_F g(x)=0$ if and only if $g(\alpha x) = g(\beta x)$, which does not necessarily imply g a constant. If we require g to be a formal power series, the condition $g(\alpha x) = g(\beta x)$ implies $\alpha^n c_n = \beta^n c_n$ for each n, where c_n is the coefficient of x^n. It is possible only when $c_n = 0$ for any $n \geqslant 1$, that is, g is constant.

For f being an arbitrary function, the (p,q)-integral of f is defined as follows (cf. [**18**]):

$$\int f(x)\, d_{p,q}(x) = (p-q)x \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\Big(\frac{q^n}{p^{n+1}} x\Big). \tag{2.1}$$

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a formal power series. Applying (p,q)-integral to both sides of $f(x)$ yields

$$\int f(x)\, d_{p,q}x = \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{[k+1]_{p,q}} + C, \tag{2.2}$$

where C is constant and $[k]_{p,q} := \frac{p^k - q^k}{p-q}$ (cf. [**18**]).

Let f be an arbitrary function and let a and b be two natural numbers such that $a < b$. Thereafter, see [**18**], we note that the definite (p,q)-integral is provided by

$$\int_0^b f(x)\, d_{p,q}x = (p-q)b \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\Big(\frac{q^k}{p^{k+1}} b\Big) \text{ if } \Big|\frac{q}{p}\Big| < 1, \tag{2.3}$$

$$\int_a^b f(x)\, d_{p,q}x = \int_0^b f\Big(\frac{x}{p}\Big)\, d_{p,q}x - \int_0^a f\Big(\frac{x}{p}\Big)\, d_{p,q}x. \tag{2.4}$$

Also, the more general formula for the definite (p,q)-integral is provided by

$$\begin{aligned} \int_0^b f(x) D_{p,q} g(x)\, d_{p,q}x &= \int_0^b f(x)\, d_{p,q} g(x) \\ &= \sum_{k=0}^{\infty} f\Big(\frac{q^k}{p^{k+1}} b\Big)\Big(g\Big(\frac{q^k}{p^k} b\Big) - g\Big(\frac{q^{k+1}}{p^{k+1}} b\Big)\Big), \end{aligned} \tag{2.5}$$

where $D_{p,q}$ is the (p,q)-derivative operator defined by

$$D_{p,q;x}f(x) := D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and $(D_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0.

Tuglu et al. [**20**] considered golden integral (F-integral) by choosing $p \to \alpha = \frac{1+\sqrt{5}}{2}$ and $q \to \beta = \frac{1-\sqrt{5}}{2}$ in (2.1) as follows

$$\int f(x)\, d_{\alpha,\beta}(x) := \int f(x)\, d_F(x) = (\alpha - \beta)x \sum_{n=0}^{\infty} \frac{\beta^n}{\alpha^{n+1}} f\Big(\frac{\beta^n}{\alpha^{n+1}}x\Big). \tag{2.6}$$

We note that this is a formal definition since we do not care about the convergence of the right hand side of (2.6). As the same motivation of constructing the golden integral in (2.6), using (2.4), the definite golden integral of $f(x)$ is defined as

$$\int_0^b f(x)\, d_F(x) = (\alpha - \beta)b \sum_{n=0}^{\infty} \frac{\beta^n}{\alpha^{n+1}} f\Big(\frac{\beta^n}{\alpha^{n+1}}b\Big), \tag{2.7}$$

which is always convergent because of $\big|\frac{\beta}{\alpha}\big| < 1$. Utilizing (2.5), the definite golden integral satisfies the following property

$$\int_a^b f(x)\, d_F(x) = \int_0^b f(x)\, d_F(x) - \int_0^a f(x)\, d_F(x),$$

for $a,b \in \mathbb{R}$ with $a < b$. For example, by (2.7), we see that $\int_0^1 x^n d_F(x) = \frac{1}{F_{n+1}}$. The golden analog of (2.5) can be given as

$$\begin{aligned}\int_0^b f(x) D_F^x g(x)\, d_F x &= \int_0^b f(x)\, d_F g(x)\\ &= \sum_{k=0}^{\infty} f\Big(\frac{\beta^k}{\alpha^{k+1}}b\Big)\Big(g\Big(\frac{\beta^k}{\alpha^k}b\Big) - g\Big(\frac{\beta^{k+1}}{\alpha^{k+1}}b\Big)\Big).\end{aligned}$$

For f being an arbitrary function and b being a nonnegative real number, the improper golden integral of $f(x)$ is defined to be

$$\int_b^{\infty} f(x)\, d_F(x) = (\alpha - \beta)b \sum_{n=0}^{\infty} \frac{\beta^{-n}}{\alpha^{-(n+1)}} f\Big(\frac{\beta^{-n}}{\alpha^{-(n+1)}}b\Big),$$

which is always convergent because of $\big|\frac{\beta}{\alpha}\big| < 1$. Similarly to the ordinary and (p,q) cases, we have the following fundamental theorem or golden Newton–Leibniz formula. Moreover, we can also give the fundamental theorem of the golden calculus as follows.

Theorem 2.1 (Fundamental theorem of golden calculus). *If $G(x)$ is a golden antiderivative of $g(x)$ and $G(x)$ is continuous at $x = 0$, we have*

$$\int_a^b g(x)\, d_F(x) = G(b) - G(a),$$

where $0 \leqslant a \leqslant b \leqslant \infty$.

Also, if $g(x)$ exists in a neighborhood of $x = 0$ and is continuous at $x = 0$, where $g'(x)$ denotes the ordinary derivative of $g(x)$, we have

$$\int_a^b D_F^x[g(x)]\, d_F(x) = g(b) - g(a). \tag{2.8}$$

As the (p, q)-integral, an important difference between the golden integral and its ordinary counterpart is that even if we are integrating a function on an interval like $[3, 4]$, we have to care about the behavior at $x = 0$. This has to do with the definition of the definite golden integral and the condition for the convergence of the golden integral.

Let $f(x)$ and $g(x)$ are two functions whose ordinary derivatives exists in a neighborhood of $x = 0$. Using the product rule (1.7), it is seen from (2.2) that

$$\int_a^b f(\alpha x) D_F^x[g(x)]\, d_F(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(\beta x) D_F^x[f(x)]\, d_F(x),$$

that is the formula of golden integration by part. Note that $b = \infty$ is allowed.

3. On the Fibonacci–Hermite Polynomials

In this part, we aim to define F-extension of the Hermite polynomials and to derive some of their properties and relations.

We introduce F-extension of the Hermite polynomials via the golden exponential function as follows:

$$\sum_{n=0}^{\infty} H_{n,F}(x) \frac{t^n}{F_n!} = e_F^{2tx} e_F^{-t^2}. \tag{3.1}$$

Now we give a fundamental property (known also as explicit formula) of the Fibonacci–Hermite polynomials $H_{n,F}(x)$ by the following theorem.

Theorem 3.1. *The explicit formula for $H_{n,F}(x)$ is given by:*

$$H_{n,F}(x) = F_n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2x)^{n-2k}(-1)^k}{F_{n-2k}!F_k!}, \tag{3.2}$$

where $\lfloor \cdot \rfloor$ means the greatest integer function.

Proof. Using (cf. [**4**])

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} A(m, n-2m) \tag{3.3}$$

and (3.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} H_{n,F}(x) \frac{t^n}{F_n!} = e_F^{2tx} e_F^{-t^2} &= \left(\sum_{n=0}^{\infty} (2x)^n \frac{t^n}{F_n!} \right) \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{F_n!} \right) \\ &= \sum_{n=0}^{\infty} \left(F_n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2x)^{n-2k}(-1)^k}{F_{n-2k}!F_k!} \right) \frac{t^n}{F_n!},\end{aligned}$$

which gives the asserted formula (3.2) by comparing the coefficients $t^n/F_n!$ of both sides above. □

The first few Fibonacci–Hermite polynomials are listed via (3.2) below:

$$\begin{aligned}
H_{0,F}(x) &= 1,\\
H_{1,F}(x) &= 2x,\\
H_{2,F}(x) &= 4x^2-1,\\
H_{3,F}(x) &= 8x^3-4x,\\
H_{4,F}(x) &= 16x^4-12x^2+1,\\
H_{5,F}(x) &= 32x^5-120x^3+60!x,\\
H_{6,F}(x) &= 64x^6-480x^4+960x^2-120,\\
H_{7,F}(x) &= 128x^7-3328x^5+12480x^3-3120x,\\
H_{8,F}(x) &= 256x^8-17472x^6+174720x^4-131040x^2+10920,\\
H_{9,F}(x) &= 512x^9-91392x^7+2376192x^5-4455360x^3+742560x,\\
H_{10,F}(x) &= 1024x^{10}-478720x^8+32672640x^6-163363200x^4+81681600x^2-4084080.
\end{aligned}$$

From (3.2), we get the following corollary.

Corollary 3.1. *For $n\in\mathbb{N}_0$, we have*

$$H_{2n,F}(0)=(-1)^n\frac{F_{2n}!}{F_n!}\ \textit{and}\ H_{2n+1,F}(0)=0.$$

The following formula is a symmetric property for $H_{n,F}(x)$.

Theorem 3.2. *For $n\in\mathbb{N}_0$, we have*

$$H_{n,F}(-x)=(-1)^nH_{n,F}(x). \tag{3.4}$$

Proof. We readily obtain that

$$\sum_{n=0}^{\infty}H_{n,F}(-x)\frac{t^n}{F_n!}=e_F^{2t(-x)}e_F^{-t^2}=e_F^{2(-t)x}e_F^{-(-t)^2}=\sum_{n=0}^{\infty}(-1)^nH_{n,F}(x)\frac{t^n}{F_n!},$$

which yields (3.4). □

Theorem 3.3. *We have*

$$H_{n,F}(x_1+x_2)=\sum_{k=0}^{n}\binom{n}{k}_F H_{n-k,F}(x_1)(x_2)^k. \tag{3.5}$$

Proof. It is readily seen from (3.1) that

$$\begin{aligned}
\sum_{n=0}^{\infty}H_{n,F}(x_1+x_2)\frac{t^n}{F_n!}&=e_F^{2t(x_1+x_2)}e_F^{-t^2}=e_F^{2tx_2}e_F^{2tx_1}e_F^{-t^2}\\
&=\sum_{n=0}^{\infty}H_{n,F}(x)\frac{t^n}{F_n!}\sum_{n=0}^{\infty}(2x_2)^n\frac{t^n}{F_n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_F H_{n-k,F}(x)(2x_2)^k \frac{t^n}{F_n!},$$

which gives (3.5). □

Now we research some behaviors of $H_{n,F}(x)$ by applying the golden derivative operator with respect to x and t, respectively.

THEOREM 3.4. *We have*

$$D_F^x[H_{n,F}(x)] = 2F_n H_{n-1,F}(x). \tag{3.6}$$

PROOF. Applying the golden derivative operator D_F^x (1.5) to both sides of (3.1) with respect to x and using (1.2), we acquire

$$\sum_{n=0}^{\infty} D_F^x[H_{n,F}(x)] \frac{t^n}{F_n!} = D_F^x[e_F^{2tx} e_F^{-t^2}] = D_F^x[e_F^{2tx}] e_F^{-t^2} = 2te_F^{2tx} e_F^{-t^2}.$$

By comparing the coefficients $\frac{t^n}{F_n!}$ of both sides above, we get (3.6). □

The immediate results of (3.6) are stated below:

$$D_F^x[H_{2n,F}(0)] = 0 \quad \text{and} \quad D_F^x[H_{2n+1,F}(x)] = 2(-1)^n \frac{F_{2n+1}!}{F_n!}.$$

Another result of (3.6) is given for $m < n$ as follows:

$$D_F^{x,(m)}[H_{n,F}(x)] = \frac{2^m F_n!}{F_{n-m}!} H_{n-m,F}(x), \tag{3.7}$$

where the notation $D_F^{x,(m)}$ shows the golden derivative operator of order m as $D_F^{x,(m)} = D_F^{x,(m-1)} D_F^x$.

THEOREM 3.5. *We give the higher-order differential equation of the Fibonacci–Hermite polynomials:*

$$\begin{aligned} \frac{y^n}{F_n!} D_F^{x,(n)}[H_{n,F}(x)] &+ \frac{y^{n-1}}{F_{n-1}!} D_F^{x,(n-1)}[H_{n,F}(x)] \\ &+ \frac{y^{n-2}}{F_{n-2}!} D_F^{x,(n-2)}[H_{n,F}(x)] + \cdots + y^2 D_F^{x,(2)}[H_{n,F}(x)] \\ &+ yD_F^x[H_{n,F}(x)] + H_{n,F}(x) - H_{n,F}(x+y) = 0. \end{aligned} \tag{3.8}$$

PROOF. Using (3.7) as

$$H_{n-k,F}(x) = \frac{F_{n-k}!}{2^k F_n!} D_F^{x,(k)}[H_{n,F}(x)],$$

it can be observed from (3.5) that

$$H_{n,F}(x+y) = \sum_{k=0}^{n} \binom{n}{k}_F H_{n-k,F}(x)(2y)^k = \sum_{k=0}^{n} \binom{n}{k}_F (2y)^k \frac{F_{n-k}!}{2^k F_n!} D_F^{x,(k)}[H_{n,F}(x)].$$

Thus, we get (3.8). □

In order to state Theorem 3.6, we need the following lemma.

LEMMA 3.1. *We have*

$$D_F^t[e_F^{-t^2}] = -t(\alpha e_F^{-\alpha t^2} + \beta e_F^{-\beta t^2}). \tag{3.9}$$

PROOF. We observe from (1.1), (1.3) and (1.6) that

$$\begin{aligned} D_F^t[e_F^{-t^2}] &= \sum_{n=0}^{\infty}(-1)^n \frac{D_F^t[t^{2n}]}{F_n!} = \sum_{n=1}^{\infty}(-1)^n \frac{F_{2n}t^{2n-1}}{F_n!} \\ &= -\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{F_n!}\frac{F_{2n+2}}{F_{n+1}} = -t\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{F_n!}(\alpha^{n+1}+\beta^{n+1}) \\ &= -\alpha t\sum_{n=0}^{\infty}(-1)^n \frac{(t\sqrt{\alpha})^{2n}}{F_n!} - t\beta\sum_{n=0}^{\infty}(-1)^n \frac{(t\sqrt{\beta})^{2n}}{F_n!} \\ &= -\alpha t e_F^{-\alpha t^2} - \beta t e_F^{-\beta t^2}. \end{aligned}$$

□

Now, we give the following relation for the Fibonacci–Hermite polynomials.

THEOREM 3.6. *We have*

$$H_{n+1,F}(x) = 2x\alpha^n H_{n,F}\Big(\frac{x}{\alpha}\Big) - \alpha^{\frac{n+2}{2}}F_n H_{n-1,F}\Big(\frac{x\beta}{\sqrt{\alpha}}\Big) - \beta^{\frac{n+2}{2}}F_n H_{n-1,F}\big(x\sqrt{\beta}\big). \tag{3.10}$$

PROOF. Applying (1.5) to the both sides of (3.1), we obtain

$$\text{LHS} = \sum_{n=0}^{\infty} H_{n,F}(x)\frac{D_F^t[t^n]}{F_n!} = \sum_{n=1}^{\infty} H_{n,F}(x)\frac{t^{n-1}}{F_{n-1}!} = \sum_{n=0}^{\infty} H_{n+1,F}(x)\frac{t^n}{F_n!}$$

and, by using (1.7) and (3.9),

$$\begin{aligned} \text{RHS} &= D_F^t[e_F^{2tx}e_F^{-t^2}] \\ &= e_F^{-(\alpha t)^2}2xe_F^{2tx} + e_F^{2\beta tx}(-t\alpha e_F^{-\alpha t^2} - t\beta e_F^{-\beta t^2}) \\ &= 2xe_F^{-(\alpha t)^2}e_F^{2tx} - t\alpha e_F^{2\beta tx}e_F^{-\alpha t^2} - t\beta e_F^{2\beta tx}e_F^{-\beta t^2} \\ &= 2x\sum_{n=0}^{\infty} H_{n,F}\Big(\frac{x}{\alpha}\Big)\frac{\alpha^n t^n}{F_n!} - \alpha t\sum_{n=0}^{\infty} H_{n,F}\Big(\frac{x\beta}{\sqrt{\alpha}}\Big)\frac{(\sqrt{\alpha})^n t^n}{F_n!} \\ &\quad - \beta t\sum_{n=0}^{\infty} H_{n,F}(x\sqrt{\beta})\frac{(\sqrt{\beta})^n t^n}{F_n!}. \end{aligned}$$

Comparing LHS and RHS gives (3.10). □

As a result of (3.6) and (3.10), we give the following differential equation for the Fibonacci–Hermite polynomials.

COROLLARY 3.2. *We have*

$$H_{n+1,F}(x) = 2x\alpha^n H_{n,F}\Big(\frac{x}{\alpha}\Big) - \frac{\alpha^{\frac{n+3}{2}}}{2\beta}D_F^x\Big[H_{n,F}\Big(\frac{x\beta}{\sqrt{\alpha}}\Big)\Big] - \frac{\beta^{\frac{n+3}{2}}}{2}D_F^x\Big[H_{n,F}\Big(x\sqrt{\beta}\Big)\Big].$$

The golden integral representation of the Fibonacci–Hermite polynomials is given by the following theorem.

THEOREM 3.7. *We have*

$$\int_a^b H_{n,F}(x)\,d_F(x) = \frac{H_{n+1,F}(b) - H_{n+1,F}(a)}{2F_{n+1}}.$$

PROOF. By (2.8) and (3.6), we obtain

$$\int_a^b H_{n,F}(x)\,d_F(x) = \frac{1}{2F_{n+1}} \int_a^b D_F^x[H_{n+1,F}(x)]\,d_F(x) = \frac{H_{n+1,F}(b) - H_{n+1,F}(a)}{2F_{n+1}}.$$

Therefore, we complete the proof of this theorem. □

We note that the following series manipulation formulas hold:

$$\sum_{N=0}^{\infty} f(N)\frac{(x_1+x_2)^N}{F_N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x_1^n}{F_n!}\frac{x_2^m}{F_m!} \tag{3.11}$$

$$\sum_{k,l=0}^{\infty} A(l,k) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} A(l,k-l). \tag{3.12}$$

We give the following theorem.

THEOREM 3.8. *The following implicit summation formula*

$$H_{k+l,F}(x) = \sum_{n,m=0}^{k,l} \binom{k}{n}_F \binom{l}{m}_F H_{k+l-n-m,F}(z)(-2)^{n+m}(x-z)^{n+m} \tag{3.13}$$

holds.

PROOF. Upon setting t by $t+u$ in (3.1), we derive

$$e_F^{-(t+u)^2} = e_F^{-2(t+u)z} \sum_{n=0}^{\infty} H_{k+l,F}(z)\frac{t^k}{F_k!}\frac{u^l}{F_l!}.$$

Again replacing z by x in the last equation, and using (3.11), we get

$$e_F^{-(t+u)^2} = e_F^{-2(t+u)x} \sum_{n=0}^{\infty} H_{k+l,F}(x)\frac{t^k}{F_k!}\frac{u^l}{F_l!}.$$

By the last two equations, we obtain

$$\sum_{n=0}^{\infty} H_{k+l,F}(x)\frac{t^k}{F_k!}\frac{u^l}{F_l!} = e_F^{-2(t+u)(x-z)} \sum_{n=0}^{\infty} H_{k+l,F}(z)\frac{t^k}{F_k!}\frac{u^l}{F_l!},$$

which yields

$$\sum_{n=0}^{\infty} H_{k+l,F}(x)\frac{t^k}{F_k!}\frac{u^l}{F_l!} = \sum_{n,m=0}^{\infty} (2z-2x)^{n+m}\frac{t^n}{F_n!}\frac{u^m}{F_m!} \sum_{n=0}^{\infty} H_{k+l,F}(z)\frac{t^k}{F_k!}\frac{u^l}{F_l!}.$$

Utilizing (3.12), we acquire

$$\sum_{n=0}^{\infty} H_{k+l,F}(x)\frac{t^k}{F_k!}\frac{u^l}{F_l!} = \sum_{k,l=0}^{\infty}\sum_{n,m=0}^{k,l} \frac{(-2)^{n+m}(x-z)^{n+m}H_{k+l-n-m,F}(z)}{F_n!F_m!F_{k-n}!F_{l-m}!}t^k u^l,$$

which implies (3.13). □

COROLLARY 3.3. *Letting $k = 0$ in* (3.13)*, the following implicit summation formula holds:*

$$H_{l,F}(x) = \sum_{m=0}^{l} \binom{l}{m}_F H_{l-m,F}(z)(-2)^m(x-z)^m. \tag{3.14}$$

COROLLARY 3.4. *Upon setting $k = 0$ and replacing x by $x + z$ in* (3.14)*, we attain*

$$H_{l,F}(x+z) = \sum_{m=0}^{l} \binom{l}{m}_F H_{l-m,F}(z)(-2)^m x^m.$$

Now, we give the following theorem.

THEOREM 3.9. *The following identity*

$$H_{n,F}\left(\frac{x}{m}\right) = F_n!\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \frac{(2x)^{n-2k}(-1)^k}{F_{n-2k}!F_k!}m^{2k-n} \tag{3.15}$$

holds for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

PROOF. We observe from (3.3) that

$$\begin{aligned}\sum_{n=0}^{\infty} H_{n,F}\left(\frac{x}{m}\right)\frac{(mt)^n}{F_n!} &= e_F^{2mt(\frac{x}{m})}e_F^{-(mt)^2} = e_F^{2tx}e_F^{-m^2(t)^2}\\ &= \sum_{n=0}^{\infty}(2x)^n\frac{t^n}{F_n!}\sum_{n=0}^{\infty}(-m^2)^n\frac{t^{2n}}{F_n!}\\ &= \sum_{n=0}^{\infty}\left(F_n!\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(2x)^{n-2k}(-m^2)^k}{F_{n-2k}!F_k!}\right)\frac{t^n}{F_n!},\end{aligned}$$

which gives (3.15). □

4. Further Remarks

The Fibonacci–Bernstein polynomial of degree n is defined by (cf. [**5**])

$$B_{k,n}^F(x) = \binom{n}{k}_F x^k\,(1-x)^{n-k} \qquad (n, k \in \mathbb{N} \text{ with } 0 < k \leqq n). \tag{4.1}$$

By (4.1), the generating function of the Fibonacci–Bernstein polynomials is given by (cf. [**5**])

$$\sum_{n=k}^{\infty} B_{k,n}^F(x)\frac{t^n}{F_n!} = \frac{(tx)^k}{F_k!}e_F^{(1-x)t}. \tag{4.2}$$

The generating function in (4.2) is obtained by inspired the derivation of the generating function of the classical Bernstein polynomials [**1**].

From (4.1) and (4.2), some formulas for the Fibonacci–Bernstein polynomials are presented by (cf. [**2**, **5**])

$$B^F_{k,n}(1-x) = \binom{n}{k}_F (1-x)^k x^{n-k}, \tag{4.3}$$

$$B^F_{k,n}(x) = \frac{F_{n-k+1}}{F_k}\frac{x}{1-x} B^F_{k-1,n}(x), \tag{4.4}$$

$$B^F_{k,n}(x) = (1-x)F_{k-1}B^F_{k,n-1}(x) + xF_{n-k}B^F_{k-1,n-1}(x). \tag{4.5}$$

THEOREM 4.1. *The following identity holds for $x \in [0,1]$ and $k, n \in \mathbb{N}$ with $k \leqq n$:*

$$B^F_{n+k,2n+k}(x) = \frac{F_{2n+k}!F_k!}{F_{n+k}!F_{n+k}!} x^n B^F_{k,n+k}(x). \tag{4.6}$$

PROOF. From (4.1), we calculate that

$$B^F_{n+k,2n+k}(x) = \binom{2n+k}{n+k}_F x^{n+k}(1-x)^n = \frac{F_{2n+k}!F_k!F_n!}{F_n!F_{n+k}!F_{n+k}!} x^n \frac{F_{n+k}!}{F_n!F_k!} x^k (1-x)^n,$$

which gives (4.6). □

Two types of golden hyperbolic sine and cosine functions are defined by (see [**12**])

$$\sinh_F(x) = \frac{e^x_F - e^{-x}_F}{2} \quad \text{and} \quad \cosh_F(x) = \frac{e^x_F + e^{-x}_F}{2},$$
$$\mathrm{SINH}_F\, x = \frac{E^x_F - E^{-x}_F}{2} \quad \text{and} \quad \mathrm{COSH}_F\, x = \frac{E^x_F + E^{-x}_F}{2}.$$

THEOREM 4.2. *We have for $x \in (0,1]$*

$$\sinh_F(t(1-x)) = \frac{1}{2x^k}\sum_{n=0}^{\infty} \frac{(1-(-1)^n)}{\binom{n+k}{k}_F} B^F_{k,n+k}(x)\frac{t^n}{F_n!}, \tag{4.7}$$
$$\cosh_F(t(1-x)) = \frac{1}{2x^k}\sum_{n=0}^{\infty} \frac{(1+(-1)^n)}{\binom{n+k}{k}_F} B^F_{k,n+k}(x)\frac{t^n}{F_n!}.$$

PROOF. Since

$$\begin{aligned}
\sinh_F(t(1-x)) &= \frac{e^{t(1-x)}_F - e^{-t(1-x)}_F}{2} \\
&= \Big(\frac{t^k x^k e^{t(1-x)}_F - t^k x^k e^{-t(1-x)}_F}{F_k!}\Big)\frac{F_k!}{2t^k x^k} \\
&= \frac{F_k!}{2t^k x^k}\bigg(\sum_{n=k}^{\infty} B^F_{k,n}(x)\frac{t^n}{F_n!} - \sum_{n=k}^{\infty}(-1)^{n-k}B^F_{k,n}(x)\frac{t^n}{F_n!}\bigg) \\
&= \frac{F_k!}{2t^k x^k}\sum_{n=k}^{\infty}(1-(-1)^{n-k})B^F_{k,n}(x)\frac{t^n}{F_n!}
\end{aligned}$$

$$= \frac{F_k!}{2x^k}\sum_{n=0}^{\infty}(1-(-1)^n)B^F_{k,n+k}(x)\frac{t^n}{F_{n+k}!}$$
$$= \frac{1}{2x^k}\sum_{n=0}^{\infty}\frac{(1-(-1)^n)}{\binom{n+k}{k}_F}B^F_{k,n+k}(x)\frac{t^n}{F_n!},$$

we get the desired result (4.7). The other can be shown similarly. □

As a last result, we give a correlation between Fibonacci–Hermite polynomials and Fibonacci–Bernstein polynomials.

THEOREM 4.3. *The following correlation is valid for $x \in [0,1)$*

$$H_{n,F}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{F_{2k}!2^{n-2k}(-1)^k}{F_k!(1-x)^{2k}}B^F_{2k,n}(1-x).$$

PROOF. The proof of this theorem follows from (3.2) and (4.3) as

$$H_{n,F}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{F_{2k}!2^{n-2k}(-1)^k}{F_k!(1-x)^{2k}}\binom{n}{2k}_F(1-x)^{2k}x^{n-2k}$$
$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{F_{2k}!2^{n-2k}(-1)^k}{F_k!(1-x)^{2k}}B^F_{2k,n}(1-x).$$ □

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