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GENERALIZED ABSOLUTE MATRIX SUMMABILITY FACTORS

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Abstract. We generalize a theorem dealing with absolute summability factors of an infinite series to absolute matrix summability under weaker conditions by using an almost increasing sequence.

1. Introduction

Let (p_n) be a sequence of positive numbers such that $P_n = \sum_{v=0}^{n} p_v \to \infty$ as $n \to \infty$, $(P_{-i} = p_{-i} = 0,\ i \geqslant 1)$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where $A_n(s) = \sum_{v=0}^{n} a_{nv} s_v$, $n = 0, 1, \dots$ Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A; \delta|_k$, $k \geqslant 1$ and $\delta \geqslant 0$, if (see [**11**])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A; \delta|_k$ summability reduces to $|A, p_n|_k$ summability [**18**]. If we take $\varphi_n = n$ for all values of n, then $\varphi - |A; \delta|_k$ summability reduces to $|A; \delta|_k$ summability [**10**]. Also, if we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability [**2**]. Furthermore, if we take $\delta = 0$, $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $\varphi - |A; \delta|_k$ summability reduces to $|C, 1|_k$ summability [**4**].

2. Known result

In [**3**], Bor has proved the following theorem dealing with absolute Riesz summability of an infinite series.

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Theorem 2.1. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty.$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$|\lambda_m| X_m = O(1) \quad as \quad m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty \tag{2.2}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty,$$

where $\Delta^2 \lambda_n = \Delta(\Delta\lambda_n)$, $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $t_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v$, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.

3. Main result

Some different works on absolute matrix summability methods have been done [**5**–**7**, **12**–**15**, **17**]. The purpose of this article is to generalize Theorem 2.1 by using an almost increasing sequence instead of a positive monotonic non-decreasing sequence. Before giving general theorem, let us mention the definition of almost increasing sequence and some further notations. A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and M such that $Kc_n \leqslant b_n \leqslant Mc_n$ [**1**].

Let $A = (a_{nv})$ be a normal matrix, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \dots, \tag{3.1}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots, \tag{3.2}$$

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v, \quad \bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{3.3}$$

Theorem 3.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \dots, \tag{3.4}$$

$$a_{n-1,v} \geqslant a_{nv}, \quad for \quad n \geqslant v + 1, \tag{3.5}$$

$$a_{nn} = O\Big(\frac{p_n}{P_n}\Big), \tag{3.6}$$

$$|\hat{a}_{n,v+1}| = O(v|\Delta_v(\hat{a}_{nv})|), \tag{3.7}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| = O(\varphi_v^{\delta k-1}) \quad as \quad m \to \infty, \tag{3.8}$$

where $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$. *Let* (X_n) *be an almost increasing sequence and* $\varphi_n p_n = O(P_n)$. *If conditions* (2.1), (2.2) *of Theorem* 2.1 *and*

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1} |t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{3.9}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |A;\delta|_k$, $k \geqslant 1$ *and* $0 \leqslant \delta < 1/k$.

Lemma 3.1. [**9**] *If* (X_n) *is an almost increasing sequence, then under the conditions* (2.1) *and* (2.2), *we have*

$$nX_n \mid \Delta\lambda_n \mid = O(1) \quad as \quad n \to \infty, \tag{3.10}$$

$$\sum_{n=1}^{\infty} X_n \mid \Delta\lambda_n \mid < \infty. \tag{3.11}$$

Proof of Theorem 3.1. Let (T_n) denotes A-transform of the series $\sum a_n \lambda_n$. Then, by (3.3), we have $\bar{\Delta} T_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v$. Applying Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta} T_n &= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \frac{n+1}{n} a_{nn} \lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v(\hat{a}_{nv}) \lambda_v t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{t_v}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

For the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that $\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \mid T_{n,i} \mid^k < \infty$, for $i = 1,2,3,4$. First, using (3.6), we get

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |T_{n,1}|^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} a_{nn}^k |\lambda_n|^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \Big(\frac{p_n}{P_n}\Big)^k |\lambda_n|^{k-1} |\lambda_n| |t_n|^k.
\end{aligned}$$

Now, using condition (2.1) and the fact that (X_n) is an almost increasing sequence, we obtain that $|\lambda_n|^{k-1} = O(1)$. Also, using Abel's transformation, we have

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |T_{n,1}|^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k-1} |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \varphi_v^{\delta k-1} |t_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \varphi_n^{\delta k-1} |t_n|^k.
\end{aligned}$$

Thus, we get

$$\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |T_{n,1}|^k = O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,$$

by (3.9), (3.11), (2.1).

Applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k}' = 1$, we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}.$$

Here, we have

$$\begin{aligned} \Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv} - \hat{a}_{n,v+1} \\ &= \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} \\ &= a_{nv} - a_{n-1,v}, \end{aligned}$$

by (3.2) and (3.1). Thus (3.5), (3.1) and (3.4) imply

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \leqslant a_{nn}. \tag{3.12}$$

Hence, using (3.12), (3.6) and (3.8), we get

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \\ &= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| \\ &= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} |\lambda_v| |t_v|^k. \end{aligned}$$

Then, as in $T_{n,1}$, we get $\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,2}|^k = O(1) \quad \text{as} \quad m \to \infty$. Now, using condition (3.7) and Hölder's inequality, we get

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} v |\Delta_v(\hat{a}_{nv})| |\Delta\lambda_v| |t_v| \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} (v|\Delta\lambda_v|)^k |t_v|^k |\Delta_v(\hat{a}_{nv})| \right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k} \sum_{v=1}^{n-1} (v|\Delta\lambda_v|)^k |t_v|^k |\Delta_v(\hat{a}_{nv})| \\ &= O(1) \sum_{v=1}^{m} (v|\Delta\lambda_v|)^k |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| \end{aligned}$$

by using (3.12), (3.6).

Now, again using the fact that (X_n) is an almost increasing sequence and the condition (3.10), we have $(v|\Delta\lambda_v|)^{k-1} = O(1)$, and also using (3.8), we obtain

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,3}|^k = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} v|\Delta\lambda_v||t_v|^k.$$

Here, using Abel's transformation, we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,3}|^k = O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \varphi_r^{\delta k-1} |t_r|^k + O(1) m|\Delta\lambda_m| \sum_{v=1}^{m} \varphi_v^{\delta k-1} |t_v|^k.$$

Then, we get

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,3}|^k &= O(1) \sum_{v=1}^{m-1} vX_v|\Delta^2\lambda_v| + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1) m|\Delta\lambda_m| X_m \\ &= O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by (3.9), (2.2), (3.11), (3.10).

Finally, again using Hölder's inequality, and conditions (3.7), (3.12) and (3.6), we get

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,4}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_{v+1}||t_v| \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k |t_v|^k \right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1} \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k |t_v|^k.\end{aligned}$$

Then,

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,4}|^k &= O(1) \sum_{v=1}^{m} |\lambda_{v+1}||t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| \\ &= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} |\lambda_{v+1}||t_v|^k\end{aligned}$$

by using (3.8).

Then, as in $T_{n,1}$, we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,4}|^k = O(1) \quad \text{as} \quad m \to \infty.$$

Hence, the proof of Theorem 3.1 is completed. □

4. Conclusions

In the special case, when we take (X_n) as a positive monotonic non-decreasing sequence, $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability (see [**16**]). If we take (X_n) as a positive monotonic non-decreasing sequence, $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 2.1. Also, if we take (X_n) as a positive monotonic non-decreasing sequence, $\delta = 0$, $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a theorem about $|C, 1|_k$ summability of an infinite series (see [**8**]).

References

1. N. K. Bari, S. B. Stečkin, *Best approximations and differential properties of two conjugate functions*, Tr. Mosk. Mat. O.-va **5** (1956), 483–522 (in Russian).
2. H. Bor, *On two summability methods*, Math. Proc. Camb. Philos. Soc. **97**(1) (1985), 147–149.
3. ———, *On absolute summability factors*, Proc. Am. Math. Soc. **118**(1) (1993), 71–75.
4. T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. Lond. Math. Soc. (3) **7** (1957), 113–141.
5. A. Karakaş, *On absolute matrix summability factors of infinite series*, J. Class. Anal. **13**(2) (2018), 133–139.
6. B. Kartal, *On an extension of absolute summability*, Konuralp J. Math. **7**(2) (2019), 433–437.
7. ———, *Factors for generalized matrix summability*, Erc. Unv. J. Inst. Sci and Tech. **37**(3) (2021), 462–467.
8. S. M. Mazhar, *On $|C, 1|_k$ summability factors of infinite series*, Indian J. Math. **14** (1972), 45–48.
9. ———, *Absolute summability factors of infinite series*, Kyungpook Math. J. **39**(1) (1999), 67–73.
10. H. S. Özarslan, *A note on generalized absolute Riesz summability*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat. **60**(1) (2014), 51–56.
11. ———, *A new study on generalised absolute matrix summability methods*, Maejo Int. J. Sci. Technol. **12**(3) (2018), 199–205.
12. ———, *A new factor theorem for absolute matrix summability*, Quaest. Math. **42**(6) (2019), 803–809.
13. ———, *An application of absolute matrix summability using almost increasing and δ-quasi-monotone sequences*, Kyungpook Math. J. **59**(2) (2019), 233–240.
14. ———, *Generalized almost increasing sequences*, Lobachevskii J. Math. **42**(1) (2021), 167–172.
15. ———, *A new theorem on generalized absolute matrix summability*, Azerb. J. Math. **13**(1) (2023), 3–13.
16. H. S. Özarslan, H. N. Öğdük, *On absolute matrix summability methods*, Math. Commun. **12**(2) (2007), 213–220.
17. H. S. Özarslan, B. Kartal, *Absolute matrix summability via almost increasing sequence*, Quaest. Math. **43**(10) (2020), 1477–1485.
18. W. T. Sulaiman, *Inclusion theorems for absolute matrix summability methods of an infinite series. IV*, Indian J. Pure Appl. Math. **34**(11) (2003), 1547–1557.

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