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SOME FIXED-CIRCLE RESULTS IN C^*-ALGEBRA VALUED METRIC SPACES

Nilay Değirmen

Abstract. We consider fixed-circle problem in C^*-algebra valued metric spaces and prove some fixed-circle theorems for self-mappings by defining the notion of fixed-circle on such spaces with geometric interpretation. Furthermore, we give some illustrative examples to substantiate the importance of our newly obtained results.

1. Introduction and Preliminaries

The Banach contraction principle [**1**] is a popular and effective tool used for the existence and uniqueness of solutions of many nonlinear problems arising in physics and engineering sciences. Up to now, researchers have generalized the Banach contraction principle in many directions and obtained new results in different metric spaces. With this idea in mind, in 2014, Ma et al. [**2**] introduced the concept of C^*-algebra valued metric space and established some fixed point theorems for self-mappings satisfying the contractive conditions on such spaces. Also, in 2015, Batul and Kamran [**3**] generalized it by weakening the contractive condition introduced by Ma et al. [**2**]. Going in the same direction, many articles on fixed point results in C^*-algebra valued metric spaces, we refer to [**4–9**].

On the other hand, as a geometric approach to the fixed point theory, in [**10**], Özgür and Taş initiated the investigations concerning a fixed-circle problem in metric spaces and examined some fixed-circle theorems for self-mappings on metric spaces with geometric interpretation by giving some necessary examples to validate their own findings. Since this subject has been developed very fast in recent times due to theoretical mathematical studies and some applications in different fields of mathematical sciences such as neural networks, it has attracted considerable interest of many authors. New solutions of fixed-circle problem have been investigated with various aspects and new contractive conditions on both metric spaces and

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some generalized metric spaces. For fixed-circle results using different techniques, we recommend [**11**–**16**].

In the most existing literature, it has been observed that fixed circles have not been defined in C^*-algebra valued metric spaces. This motivates us to analyze solutions of the fixed-circle problem in such spaces with geometric interpretation. So, our purpose in this study is to define the notion of a fixed circle on a C^*-algebra valued metric space and obtain some fixed-circle theorems for self-mappings on C^*-algebra valued metric spaces. Also, we construct some nontrivial illustrative examples of mappings which have or not fixed circles. Since the theory of C^*-algebra is one of the most extensive research areas in operator theory and functional analysis, which is extremely active and having huge applications to theoretical physics and noncommutative geometry, we believe that the results of this article will contribute to studying on fixed-circles with different aspects for C^*-algebra valued metric spaces in the future.

Now, we summarize a number of known definitions and results about C^*-algebras and C^*-algebra valued metric spaces, which will be needed in our subsequent discussions.

We start with the definition of a C^*-algebra and some its properties used in this research.

Definition 1.1. [**17**] A mapping $x \to x^*$ of a complex algebra $\mathbb{A}$ into $\mathbb{A}$ is called an involution on $\mathbb{A}$ if the following properties hold for all $x, y \in \mathbb{A}$ and $\lambda \in \mathbb{C}$:

$$\text{(i)}\ \ (x^*)^* = x, \quad \text{(ii)}\ \ (xy)^* = y^*x^*, \quad \text{(iii)}\ \ (\lambda x + y)^* = \bar{\lambda}x^* + y^*.$$

A complex Banach algebra A with an involution such that $\|x^*x\| = \|x\|^2$ for every x in A, is called a C^*-algebra.

In the rest of the paper, $\mathbb{A}$ will denote a unital C^*-algebra with a unit I.

Let $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. An element $x \in \mathbb{A}$ is called positive if $x \in \mathbb{A}_h$ and $\sigma(x) \subset \mathbb{R}^+$ where $\sigma(x) = \{\lambda \in \mathbb{R} : x - \lambda I \text{ is non-invertible}\}$, the spectrum of x in $\mathbb{A}$. If $x \in \mathbb{A}$ is positive, we write it as $\theta \preceq x$, where θ is the zero element in $\mathbb{A}$. We denote the set of all positive elements of $\mathbb{A}$ by $\mathbb{A}_+$. Also, $\mathbb{A}_h$ becomes a partially ordered set by defining $x \preceq y$ to mean $y - x \in \mathbb{A}_+$ [**17**].

The following statements about involution on $\mathbb{A}$, positive elements of $\mathbb{A}$ and partial order $\preceq$ on $\mathbb{A}_h$ are true:

(i) $\|x^*\| = \|x\|$ for all $x \in \mathbb{A}$.
(ii) If $x \in \mathbb{A}$ is invertible, then x^* is invertible and $(x^*)^{-1} = (x^{-1})^*$.
(iii) If $x, y, z \in \mathbb{A}_h$, then $x \preceq y$ implies $x + z \preceq y + z$.
(iv) If $x, y \in \mathbb{A}_+$ and $\alpha, \beta \in \mathbb{R}^+ \cup \{0\}$, then $\alpha x + \beta y \in \mathbb{A}_+$.
(v) $\mathbb{A}_+ = \{x^*x : x \in \mathbb{A}\}$.
(vi) If $x, y \in \mathbb{A}_h$ and $z \in \mathbb{A}$, then $x \preceq y$ implies $z^*xz \preceq z^*yz$.
(vii) If $\theta \preceq x \preceq y$, then $\|x\| \leqslant \|y\|$ [**17**].

Using the concept of a positive element in a C^*-algebra, in 2014, Ma and Jiang [**2**] introduced the notion of a C^*-algebra valued metric space in the following way:

Definition 1.2. [**2**] Let X be a nonempty set. Suppose the mapping $d \colon X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta \Leftrightarrow x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then, d is called C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

It is clear that such spaces generalize the concept of metric spaces. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead of the set of real numbers.

Example 1.1. [**2**] Let E be a Lebesgue measurable set and $L(L^2(E))$ denote the set of bounded linear operators on Hilbert space $L^2(E)$. Define $d\colon L^\infty(E) \times L^\infty(E) \to L(L^2(E))$ by $d(f,g) = \pi_{|f-g|}$ for all $f, g \in L^\infty(E)$, where $\pi_h\colon L^2(E) \to L^2(E)$ is the multiplication operator defined by $\pi_h(\varphi) = h \cdot \varphi$ for all $\varphi \in L^2(E)$. Then, d is a C^*-algebra valued metric and $(L^\infty(E), L(L^2(E)), d)$ is a complete C^*-algebra valued metric space.

Example 1.2. [**18**] Define $d\colon [0,\infty) \times [0,\infty) \to \mathbb{C}^2$ by

$$d(x,y) = (\||x-y| + i|x-y|, |x-y| + 2i|x-y|)$$

for all $x, y \in \mathbb{R}$, the norm on $\mathbb{C}^2$ is defined by $\|(Z_1, Z_2)\| = \max\{|Z_1|, |Z_2|\}$, and partial ordering on $\mathbb{C}^2$ is given by

$$(Z_1, Z_2) \preceq (W_1, W_2) \Longleftrightarrow \begin{array}{l} \mathrm{Re}(Z_1) \leqslant \mathrm{Re}(W_1),\ \mathrm{Im}(Z_1) \leqslant \mathrm{Im}(W_1), \\ \mathrm{Re}(Z_2) \leqslant \mathrm{Re}(W_2),\ \mathrm{Im}(Z_2) \leqslant \mathrm{Im}(W_2). \end{array}$$

Then, d is a C^*-algebra valued metric and $([0,\infty)\mathbb{C}^2, d)$ is a C^*-algebra valued metric space.

Example 1.3. [**18**] Define $d\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(z,w) = |z-w|i$ for all $z, w \in \mathbb{C}$, where the norm on $\mathbb{C}$ is defined by $\|(z,w)\| = \max\{|z|, |w|\}$, and partial ordering on $\mathbb{C}$ is given by

$$z = a + ib \preceq c + id = w \Longleftrightarrow a \leqslant c \text{ and } b \leqslant d.$$

Then, d is a C^*-algebra valued metric and $(\mathbb{C}, \mathbb{C}, d)$ is a C^*-algebra valued metric space.

Based on the idea of the Banach contraction principle [**1**] in classical metric spaces, Ma and Jiang [**2**] established the following main theorems which implies the existence and uniqueness of a fixed point on complete C^*-algebra valued metric spaces.

Theorem 1.1. [**2**] *If $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space and $T\colon X \to X$ is a C^*-algebra valued contractive mapping on X, that is, there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that*

$$d(Tx, Ty) \preceq A^* d(x,y) A \tag{1.1}$$

for all $x, y \in X$, then there exists a unique fixed point in X.

In the following theorem and Theorem 2.6, we denote the set $\{a \in \mathbb{A} : ab = ba \text{ for all } b \in \mathbb{A}\}$ by $\mathbb{A}'$.

THEOREM 1.2. [**2**] *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Suppose the mapping $T\colon X \to X$ satisfies for all $x, y \in X$ such that*

$$d(Tx, Ty) \preceq A(d(Tx, y) + d(Ty, x)) \tag{1.2}$$

where $A \in \mathbb{A}'_+$ and $\|A\| < \frac{1}{2}$. Then, there exists a unique fixed point in X.

Inspired by preceding observation, many authors stated new types of contractive mappings and studied fixed point theorems. Shehwar et al. [**4**] gave the extension of Caristi's fixed point theorem [**19**,**20**] for self-mappings defined on C^*-algebra valued metric spaces, which guarantees the existence of fixed point as follows:

THEOREM 1.3. [**4**] *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space, $\phi\colon X \to \mathbb{A}_+$ be a lower semi continuous map and $T\colon X \to X$ be such that*

$$d(x, Tx) \preceq \phi(x) - \phi(Tx) \tag{1.3}$$

for all $x \in X$. Then, T has at least one fixed point in X.

Kadelburg and Radenović [**5**] introduced the extension of Ćirić's fixed point theorem [**21**] for self-mappings defined on C^*-algebra valued metric spaces, which guarantees the existence and uniqueness of a fixed point as follows:

THEOREM 1.4. [**5**] *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and $T\colon X \to X$ be a mapping on X. Suppose that there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that for all $x, y \in X$ there exists $u(x, y) \in \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ such that*

$$d(Tx, Ty) \preceq A^* u(x, y) A. \tag{1.4}$$

Then, T has a unique fixed point in X.

2. Main Results

We begin this section by introducing the concept of a circle on a C^*-algebra valued metric space.

DEFINITION 2.1. Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space, $x_0 \in X$ and $r \in \mathbb{A}_+$. Then, the circle with the centered x_0 and the radius r is defined by

$$C^{C^*}_{x_0,r} = \{x \in X : d(x, x_0) = r\}.$$

EXAMPLE 2.1. Consider the C^*-algebra valued metric space

$$(L^\infty(E), L(L^2(E)), d)$$

given in Example 1.1 for $E = [0, 1]$. Choose the center x_0 as the function f defined by

$$f\colon [0,] \to \mathbb{R}, \quad f(x) = \chi_{[\frac{1}{2},1]}(x) = \begin{cases} 1, & x \in [\frac{1}{2}, 1] \\ 0, & x \notin [\frac{1}{2}, 1] \end{cases}$$

and the radius r as the multiplication operator π_h defined by

$$\pi_h\colon L^2[0, 1] \to L^2[0, 1], \quad \pi_h(\varphi) = h \cdot \varphi$$

for the function

$$h\colon [0,1] \to \mathbb{R}, \quad h(x) = \begin{cases} 1, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ \infty, & x \in \mathbb{Q} \cap [0,1] \end{cases}.$$

Then, it can be easily seen that $f, h \in L^\infty[0,1]$ and $\pi_h \in L(L^2[0,1])$. Thus, we get

$$\begin{aligned} C^{C^*}_{f,\pi_h} &= \{g \in L^\infty[0,1] : d(g,f) = \pi_h\} \\ &= \{g \in L^\infty[0,1] : \pi_{|g-f|} = \pi_h\} \\ &= \{g \in L^\infty[0,1] : |g-f| = h\} \\ &= \{g \in L^\infty[0,1] : |g(x) - f(x)| = h(x) \text{ for each } x \in [0,1]\} \\ &= \left\{g \in L^\infty[0,1] : |g(x) - 0| = 1 \text{ for each } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap \left[0, \tfrac{1}{2}\right)\right\} \\ &\quad \cup \left\{g \in L^\infty[0,1] : |g(x) - 0| = \infty \text{ for each } x \in \mathbb{Q} \cap \left[0, \tfrac{1}{2}\right)\right\} \\ &\quad \cup \left\{g \in L^\infty[0,1] : |g(x) - 1| = 1 \text{ for each } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap \left[\tfrac{1}{2}, 1\right]\right\} \\ &\quad \cup \left\{g \in L^\infty[0,1] : |g(x) - 1| = \infty \text{ for each } x \in \mathbb{Q} \cap \left[\tfrac{1}{2}, 1\right]\right\} \\ &= \left\{g \in L^\infty[0,1] : g(x) = -1 \text{ or } g(x) = 1 \text{ for each } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap \left[0, \tfrac{1}{2}\right)\right\} \\ &\quad \cup \left\{g \in L^\infty[0,1] : g(x) = -\infty \text{ or } g(x) = \infty \text{ for each } x \in \mathbb{Q} \cap \left[0, \tfrac{1}{2}\right)\right\} \\ &\quad \cup \left\{g \in L^\infty[0,1] : g(x) = 0 \text{ or } g(x) = 2 \text{ for each } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap \left[\tfrac{1}{2}, 1\right]\right\} \\ &\quad \cup \left\{g \in L^\infty[0,1] : g(x) = -\infty \text{ or } g(x) = \infty \text{ for each } x \in \mathbb{Q} \cap \left[\tfrac{1}{2}, 1\right]\right\}. \end{aligned}$$

For example, the function g defined by

$$g\colon [0,1] \to \mathbb{R}, \quad g(x) = \begin{cases} 1 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, \frac{1}{7}) \\ -1 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (\frac{1}{7}, \frac{1}{2}) \\ \infty & x \in \mathbb{Q} \cap [0, \frac{1}{2}) \\ 2 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [\frac{1}{2}, \frac{99}{100}) \\ 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [\frac{99}{100}, 1] \\ -\infty & x \in \mathbb{Q} \cap [\frac{1}{2}, 1] \end{cases}$$

is in the circle $C^{C^*}_{f,\pi_h}$.

EXAMPLE 2.2. Consider the C^*-algebra valued metric space $([0,\infty), \mathbb{C}^2, d)$ given in Example 1.2. Choose the center $x_0 = 0$ and the radii $r_1 = (2+2i, 2+4i)$, $r_2 = (i,i)$, Then, we get

$$\begin{aligned} C^{C^*}_{0,r_1} &= \{x \in [0,\infty) : d(x,0) = (2+2i, 2+4i)\} \\ &= \{x \in [0,\infty) : (|x| + i|x|, |x| + 2i|x|) = (2+2i, 2+4i)\} \\ &= \{x \in [0,\infty) : |x| = 2\} = \{2\}, \end{aligned}$$

$$\begin{aligned} C^{C^*}_{0,r_2} &= \{x \in [0,\infty) : d(x,0) = (i,i)\} \\ &= \{x \in [0,\infty) : (|x| + i|x|, |x| + 2i|x|) = (i,i)\} = \emptyset \end{aligned}$$

Example 2.3. Consider the C^*-algebra valued metric space $(\mathbb{C}, \mathbb{C}, d)$ given in Example 1.3. Choose the center $x_0 = \frac{-1}{2}$ and the radius $r = i$. Then, we get

$$\begin{aligned} C^{C^*}_{\frac{-1}{2},i} &= \left\{a + ib \in \mathbb{C} : d\left(a + ib, \tfrac{-1}{2}\right) = i\right\} \\ &= \left\{a + ib \in \mathbb{C} : \left|a + \tfrac{1}{2} + ib\right| = 1\right\} \\ &= \left\{a + ib \in \mathbb{C} : \left(a + \tfrac{1}{2}\right)^2 + b^2 = 1\right\}, \end{aligned}$$

which represents the circle with centre $\left(-\frac{1}{2}, 0\right)$ and radius 1.

2.1. The existence of fixed circles. In this part, we introduce the notion of a fixed circle on a C^*-algebra valued metric space. Then, we prove some fixed-circle theorems that guarantee the existence of fixed-circles for a self-mapping satisfying some conditions on C^*-algebra valued metric spaces.

Now, we define the new concept as follows:

Definition 2.2. Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space, $C^{C^*}_{x_0,r}$ be a circle on X and $T: X \to X$ be a self-mapping. If $Tx = x$ for all $x \in C^{C^*}_{x_0,r}$, then the circle $C^{C^*}_{x_0,r}$ is called as the fixed circle of T.

The first solution of fixed-circle problem in C^*-algebra valued metric spaces is given using inequality (1.3) in Theorem 1.3, which is an extension of Caristi's fixed point theorem for mappings defined on C^*-algebra valued metric spaces as follows:

Theorem 2.1. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space, $C^{C^*}_{x_0,r}$ be any circle on X. Define the mapping $\varphi: X \to \mathbb{A}_+$ as*

$$\varphi(x) = d(x, x_0) \tag{2.1}$$

for all $x \in X$. If T is a self-mapping defined on X satisfying the conditions

$$d(x, Tx) \preceq \varphi(x) - \varphi(Tx), \tag{2.2}$$

$$r \preceq d(Tx, x_0) \tag{2.3}$$

for all $x \in C^{C^*}_{x_0,r}$, then the circle $C^{C^*}_{x_0,r}$ is a fixed circle of T.

Proof. Let x be any point in the circle $C^{C^*}_{x_0,r}$. Then, using (2.2), (2.1), (2.3) and the definition of the relation $\preceq$, we get

$$d(x, Tx) \preceq \varphi(x) - \varphi(Tx) = d(x, x_0) - d(Tx, x_0) = r - d(Tx, x_0) \preceq r - r = 0$$

and so $d(x, Tx) = 0$ which means that $Tx = x$. Then, we obtain that $C^{C^*}_{x_0,r}$ is a fixed circle of T. □

Remark 2.1. Theorem 2.1 guarantees the existence of at least one fixed circle of a self-mapping on a C^*-algebra valued metric space, while Theorem 1.3 guarantees the existence of at least one fixed point of a self-mapping on a C^*-algebra valued metric space. We note that if the circle $C^{C^*}_{x_0,r}$ has only one element, Theorem 2.1 is a special case of Theorem 1.3.

Remark 2.2. Inequality (2.2) says that Tx is not in the exterior of the circle $C_{x_0,r}^{C^*}$ for each $x \in C_{x_0,r}^{C^*}$. In the same way, inequality (2.3) says that Tx is not in the interior of the circle $C_{x_0,r}^{C^*}$ for each $x \in C_{x_0,r}^{C^*}$. It follows that $T(C_{x_0,r}^{C^*}) \subset C_{x_0,r}^{C^*}$ under conditions (2.2) and (2.3).

The following example illustrates Theorem 2.1.

Example 2.4. Consider the C^*-algebra valued metric space

$$(L^\infty(E), L(L^2(E)), d)$$

given in Example 1.1 for $E = [0,1]$ and the circle $C_{f,\pi_h}^{C^*}$ given in Example 2.1. Define the function

$$g_0\colon [0,1] \to \mathbb{R}, \quad g_0(x) = \begin{cases} 1, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ \infty, & x \in \mathbb{Q} \cap [0,1] \end{cases}.$$

It is easy to show that $g_0 \in L^\infty[0,1]$. Let us define the self-mapping

$$T\colon L^\infty[0,1] \to L^\infty[0,1], \quad Tg = \begin{cases} g, & g \in C_{f,\pi_h}^{C^*} \\ g_0, & g \notin C_{f,\pi_h}^{C^*} \end{cases}.$$

Then, with a direct computation it can be seen that T satisfies conditions (2.2) and (2.3). That is to say that the circle $C_{f,\pi_h}^{C^*}$ is a fixed circle of T.

Now, we give an example of a self-mapping which satisfies condition (2.2) and does not satisfy condition (2.3).

Example 2.5. Consider the C^*-algebra valued metric space $([0,\infty), \mathbb{C}^2, d)$ given in Example 1.2 and the circle $C_{0,r_1}^{C^*}$ given in Example 2.2. Let us define the self-mapping

$$T\colon [0,\infty) \to [0,\infty) \quad Tx = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

With a simple verification it can be shown that T satisfies condition (2.2) and does not satisfy condition (2.3). Notice that the circle $C_{0,r_1}^{C^*}$ is not a fixed circle of T.

We now state another existence theorem for fixed circles of a self-mapping on a C^*-algebra valued metric space as follows:

Theorem 2.2. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space, $C_{x_0,r}^{C^*}$ be any circle on X and the mapping φ be as in* (2.1). *If T is a self-mapping defined on X satisfying the conditions*

$$d(x, Tx) \preceq \varphi(x) + \varphi(Tx) - 2r, \tag{2.4}$$

$$d(Tx, x_0) \preceq r \tag{2.5}$$

for all $x \in C_{x_0,r}^{C^*}$, then the circle $C_{x_0,r}^{C^*}$ is a fixed circle of T.

PROOF. Let x be any point in the circle $C^{C^*}_{x_0,r}$ Then, using (2.4), (2.1), (2.5) and the definition of the relation $\preceq$, we get

$$\begin{aligned} d(x,Tx) \preceq \varphi(x)+\varphi(Tx)-2r &= d(x,x_0)+d(Tx,x_0)-2r \\ &= d(Tx,x_0)-r \preceq r-r=0 \end{aligned}$$

and so $d(x,Tx)=0$ which implies that $Tx=x$. Then, we deduce that $C^{C^*}_{x_0,r}$ is a fixed circle of T. □

REMARK 2.3. Inequality (2.4) means that Tx is not in the interior of the circle $C^{C^*}_{x_0,r}$ for each $x \in C^{C^*}_{x_0,r}$. In the same fashion, inequality (2.5) means that Tx is not in the exterior of the circle $C^{C^*}_{x_0,r}$ for each $x \in C^{C^*}_{x_0,r}$. These two results show that $T(C^{C^*}_{x_0,r}) \subset C^{C^*}_{x_0,r}$ under conditions (2.4) and (2.5).

We continue with an example which satisfy the requirements of Theorem 2.2.

EXAMPLE 2.6. Consider the C^*-algebra valued metric space $([0,\infty),\mathbb{C}^2,d)$ given in Example 1.2 and the circle $C^{C^*}_{0,r_1}$ given in Example 2.2. Let us define the self-mapping

$$T\colon [0,\infty)\to[0,\infty) \quad Tx=\begin{cases} x, & x\in C^{C^*}_{0,r_1} \\ 5, & x\notin C^{C^*}_{0,r_1} \end{cases}.$$

Clearly, T satisfies conditions (2.4) and (2.5), and we derive that the circle $C^{C^*}_{0,r_1}$ is a fixed circle of T.

Now, we furnish an example in order to evidence that Teorem 2.2 fail out, if condition (2.5) is satisfied but condition (2.4) is not.

EXAMPLE 2.7. Consider the C^*-algebra valued metric space $(\mathbb{C},\mathbb{C},d)$ given in Example 1.3 and the circle $C^{C^*}_{\frac{-1}{2},i}$ given in Example 2.3. Let us define the self-mapping

$$T\colon \mathbb{C}\to\mathbb{C}, \quad Tx=\begin{cases} -\frac{1}{2}, & x\in C^{C^*}_{\frac{-1}{2},i} \\ -\frac{1}{2}+i, & x\notin C^{C^*}_{\frac{-1}{2},i} \end{cases}.$$

It is not hard to prove that T satisfies condition (2.5) and does not satisfy condition (2.4). Notice that the circle $C^{C^*}_{\frac{-1}{2},i}$ is not a fixed circle of T.

The following theorem presents a new solution for fixed-circle problem obtained by the help of the inequality (1.3) in Theorem 1.3.

THEOREM 2.3. *Let $(X,\mathbb{A},d)$ be a C^*-algebra valued metric space, $C^{C^*}_{x_0,r}$ be any circle on X and the mapping φ be as in* (2.1). *If T is a self-mapping defined on X satisfying the conditions*

$$d(x,Tx)\preceq\varphi(x)-\varphi(Tx), \tag{2.6}$$

$$r\preceq A^*d(x,Tx)A+d(Tx,x_0) \tag{2.7}$$

for all $x\in C^{C^*}_{x_0,r}$ and some $A\in\mathbb{A}$ with $\|A\|<1$, then the circle $C^{C^*}_{x_0,r}$ is a fixed circle of T.

Proof. Let x be any point in the circle $C_{x_0,r}^{C^*}$. Let $x \neq Tx$. Then, using (2.6), (2.1) and (2.7) we get

$$\begin{aligned} 0 &\preceq d(x,Tx) \preceq \varphi(x) - \varphi(Tx) = d(x,x_0) - d(Tx,x_0) = r - d(Tx,x_0) \\ &\preceq A^* d(x,Tx)A + d(Tx,x_0) - d(Tx,x_0) = A^* d(x,Tx)A \end{aligned}$$

and so

$$\begin{aligned} 0 \leqslant \|d(x,Tx)\| &\leqslant \|A^* d(x,Tx)\| \\ &\leqslant \|A^*\| \|d(x,Tx)\| \|A\| = \|A\|^2 \|d(x,Tx)\| < \|d(x,Tx)\|. \end{aligned}$$

This yields a contradiction with our assumption. Therefore, we conclude that $x = Tx$ for all $x \in C_{x_0,r}^{C^*}$ and hence $C_{x_0,r}^{C^*}$ is a fixed circle of T. □

The following example supports Theorem 2.3.

Example 2.8. Consider the C^*-algebra valued metric space $([0,\infty),\mathbb{C}^2,d)$ given in Example 1.2 and the circle $C_{0,r_1}^{C^*}$ given in Example 2.2. Let us define the self-mapping

$$T\colon [0,\infty) \to [0,\infty), \quad Tx = \begin{cases} x, & x \in C_{0,r_1}^{C^*} \\ 5, & x \notin C_{0,r_1}^{C^*} \end{cases}.$$

Then, with a direct computation it can be seen that the self-mapping T satisfies conditions (2.6) and (2.7) for $A = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \in \mathbb{C}^2$ with $\|A\| = \frac{1}{\sqrt{2}} < 1$. Note that the circle $C_{0,r_1}^{C^*}$ is a fixed circle of T.

Now, we give an example of a self-mapping which satisfies condition (2.6) and does not satisfy condition (2.7).

Example 2.9. Consider the C^*-algebra valued metric space $(\mathbb{C},\mathbb{C},d)$ given in Example 1.3, the circle $C_{\frac{-1}{2},i}^{C^*}$ given in Example 2.3 and the self-mapping $T\colon \mathbb{C} \to \mathbb{C}$ given in Example 2.7. It can be easily seen that T satisfies condition (2.6). But since there does not exist $A \in \mathbb{C}$ with $\|A\| < 1$ such that $i \preceq A^*|x + \frac{1}{2}|iA$ for $x = \frac{1}{2}$, condition (2.7) does not hold. Obviously, the circle $C_{\frac{-1}{2},i}^{C^*}$ is not a fixed one of T.

We finish this section by giving our last existence theorem for fixed circles of a self-mapping on a C^*-algebra valued metric space as follows.

Theorem 2.4. *Let $(X,\mathbb{A},d)$ be a C^*-algebra valued metric space, $C_{x_0,r}^{C^*}$ be any circle on X and the mapping φ be as in* (2.1). *T is a self-mapping defined on X satisfying the condition*

$$A^* d(x,Tx)A \preceq \varphi(x) - \varphi(Tx) \tag{2.8}$$

for all $x \in X$ where $A \in \mathbb{A}$ is an invertible element and $\|A^{-1}\| < 1$ if and only if T fixes the circle $C_{x_0,r}^{C^*}$ and $T = I_X$.

Proof. Suppose that T is a self-mapping defined on X satisfying condition (2.8). Let x be any point in X. Let $x \neq Tx$. Then, using the (2.8), (2.1) and (iii) given in Definition 1.2, we get

$$A^* d(x,Tx)A \preceq \varphi(x) - \varphi(Tx) = d(x,x_0) - d(Tx,x_0)$$

$$\preceq d(x,Tx)+d(Tx,x_0)-d(Tx,x_0)=d(x,Tx)$$

and so

$$d(x,Tx)\preceq (A^*)^{-1}d(x,Tx)A^{-1}=(A^{-1})^*d(x,Tx)A^{-1}.$$

It follows that

$$\begin{aligned}\|d(x,Tx)\| &\leqslant \|(A^{-1})^*d(x,Tx)A^{-1}\|\\ &=\|(A^{-1})^*\|\|d(x,Tx)\|\|A^{-1}\|\\ &=\|A^{-1}\|^2\|d(x,Tx)\|\\ &<\|d(x,Tx)\|.\end{aligned}$$

But it is impossible. Hence, we can write $x=Tx$ for all $x\in X$ and $T=I_X$.

Conversely, suppose that T fixes the circle $C^{C^*}_{x_0,r}$ and $T=I_X$. Then, since $Tx=x$ for all $x\in X$, condition (2.8) holds for any invertible element $A\in\mathbb{A}$ with $\|A^{-1}\|<1$. This completes the proof. □

Remark 2.4. Theorem 2.4 says that if a self-mapping fixes a circle by satisfying conditions (2.2) and (2.3) (or conditions (2.4) and (2.5)), but does not satisfy condition (2.8), then the self-mapping is different from the identity map.

2.2. The uniqueness of fixed circles. In this part, we discuss the uniqueness of fixed circles in the existence theorems proved in subsection 2.1.

Before giving our uniqueness theorems, we emphasize that the fixed circles $C^{C^*}_{x_0,r}$ in Theorems 2.1, 2.2 and 2.3 are not unique with the following result.

Proposition 2.1. *Let $(X,\mathbb{A},d)$ be a C^*-algebra valued metric space and $C^{C^*}_{x_1,r_1}$, $C^{C^*}_{x_2,r_2},\dots,C^{C^*}_{x_n,r_n}$ be any given circles. Then, there exists at least one self-mapping T of X such that T fixes all the circles $C^{C^*}_{x_1,r_1},C^{C^*}_{x_2,r_2},\dots,C^{C^*}_{x_n,r_n}$.*

Proof. Let the self-mapping $T\colon X\to X$ be defined as

$$Tx=\begin{cases}x, & x\in\bigcup_{i=1}^{n}C^{C^*}_{x_i,r_i}\\ \alpha, & \text{otherwise.}\end{cases}$$

where $\alpha\in X$ is a constant satisfying $d(\alpha,x_i)\neq r_i$ and the mapping $\varphi_i\colon X\to[0,\infty)$ be defined as $\varphi_i(x)=d(x,x_i)$ for $i=1,2,\dots,n$. Then it is not hard to verify that conditions (2.2) and (2.3) in Theorem 2.1 are satisfied for the circles $C^{C^*}_{x_1,r_1},C^{C^*}_{x_2,r_2},\dots,C^{C^*}_{x_n,r_n}$. Therefore, all the circles $C^{C^*}_{x_1,r_1},C^{C^*}_{x_2,r_2},\dots,C^{C^*}_{x_n,r_n}$ are fixed circles of the self-mapping T by Theorem 2.1. □

Firstly, we focus on the uniqueness of fixed circles in Theorem 2.1 using inequality (1.1) in Theorem 1.1 in the following theorem.

Theorem 2.5. *Let $(X,\mathbb{A},d)$ be a C^*-algebra valued metric space, $C^{C^*}_{x_0,r}$ be any circle on X and T be a self-mapping satisfying conditions* (2.2) *and* (2.3) *given in Theorem* 2.1. *If T satisfies the contraction condition*

$$d(Tx,Ty)\preceq A^*d(x,y)A \tag{2.9}$$

for all $x \in C^{C^*}_{x_0,r}$*,* $y \in X - C^{C^*}_{x_0,r}$ *and some* $A \in \mathbb{A}$ *with* $\|A\| < 1$*, then the circle* $C^{C^*}_{x_0,r}$ *is unique fixed circle of* T*.*

Proof. Assume that $C^{C^*}_{x_1,\delta}$ is another fixed circle of T. Let a and b be any points in $C^{C^*}_{x_0,r}$ and $C^{C^*}_{x_1,\delta}$, respectively. Then, we get by (2.9)

$$d(a,b) = d(Ta,Tb) \preceq A^* d(a,b) A$$

and so,

$$\|d(a,b)\| \leqslant \|A^* d(a,b) A\| \leqslant \|A\|^2 \|d(a,b)\| < \|d(a,b)\|.$$

But this is impossible. Hence, the self-mapping T fixes only circle $C^{C^*}_{x_0,r}$. □

Now, we determine the uniqueness condition for the fixed circles in Theorem 2.2 using condition (1.2) in Theorem 1.2.

Theorem 2.6. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued metric space,* $C^{C^*}_{x_0,r}$ *be any circle on* X *and* T *be a self-mapping satisfying conditions* (2.4) *and* (2.5) *given in Theorem* 2.2*. If* T *satisfies the contraction condition*

$$d(Tx,Ty) \preceq A(d(Tx,y) + d(Ty,x)) \tag{2.10}$$

for all $x \in C^{C^*}_{x_0,r}$*,* $y \in X - C^{C^*}_{x_0,r}$ *and some* $A \in \mathbb{A}'_+$ *with* $\|A\| < \frac{1}{2}$*, then the circle* $C^{C^*}_{x_0,r}$ *is a unique fixed circle of* T*.*

Proof. Assume that $C^{C^*}_{x_1,\delta}$ is another fixed circle of T. Let a and b be any points in $C^{C^*}_{x_0,r}$ and $C^{C^*}_{x_1,\delta}$, respectively. Then, we get by (2.10)

$$d(a,b) = d(Ta,Tb) \preceq A(d(Ta,b) + d(Tb,a)),$$

so that

$$\begin{aligned} \|d(a,b)\| &\leqslant \|A(d(Ta,b) + d(Tb,a))\| \\ &\leqslant \|A\| \|d(a,b) + d(b,a)\| = 2\|A\| \|d(a,b)\| < \|d(a,b)\| \end{aligned}$$

which is a contradiction which means that $a = b$. This shows that the self-mapping T fixes only circle $C^{C^*}_{x_0,r}$. □

Finally, we state our last uniqueness theorem for the fixed circles in Theorem 2.3 using condition (1.4) in Theorem 1.4.

Theorem 2.7. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued metric space,* $C^{C^*}_{x_0,r}$ *be any circle on* X *and* T *be a self-mapping satisfying conditions* (2.6) *and* (2.7) *given in Theorem* 2.3*. If* T *satisfies the contraction condition that there exists* $u(x,y) \in \{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ *such that*

$$d(Tx,Ty) \preceq A^* u(x,y) A \tag{2.11}$$

for all $x \in C^{C^*}_{x_0,r}$*,* $y \in X - C^{C^*}_{x_0,r}$ *and some* $A \in \mathbb{A}$ *with* $\|A\| < 1$*, then the circle* $C^{C^*}_{x_0,r}$ *is a unique fixed circle of* T*.*

Proof. Assume that $C^{C^*}_{x_1,\delta}$ is another fixed circle of T. Let a and b be any points in $C^{C^*}_{x_0,r}$ and $C^{C^*}_{x_1,\delta}$, respectively. Then, we get by (2.11)

$$d(a,b) = d(Ta,Tb) \preceq A^*u(a,b)A,$$

so that

$$\begin{aligned}\|d(a,b)\| &\leqslant \|A^*u(a,b)A\| \leqslant \|A\|^2\|u(a,b)\| < \|u(a,b)\| \\ &\leqslant \max\{\|d(a,b)\|,\|d(a,Tb)\|,\|d(b,Tb)\|,\|d(a,Tb)\|,\|d(b,Ta)\|\} \\ &= \|A\|^2\max\{\|d(a,b)\|,0\} = \|d(a,b)\|\end{aligned}$$

which is a contradiction. Hence $a = b$. This implies that the self-mapping T fixes only circle $C^{C^*}_{x_0,r}$. □

Remark 2.5. (1) The uniqueness result in Theorem 2.1 can be also stated using the contraction conditions (2.10) given in Theorem 2.6 or (2.11) given in Theorem 2.7 instead of the contraction condition (2.9).

(2) The uniqueness result in Theorem 2.2 can be also stated using the contraction conditions (2.9) given in Theorem 2.5 or (2.11) given in Theorem 2.7 instead of the contraction condition (2.10).

(3) The uniqueness result in Theorem 2.3 can be also stated using the contraction conditions (2.9) given in Theorem 2.5 or (2.10) given in Theorem 2.6 instead of the contraction condition (2.11).

3. Conclusion and Future Works

We discuss the existence and uniqueness of the fixed-circle for self-mappings satisfying some special conditions on C^*-algebra valued metric spaces. We also furnish some examples to show effectiveness of our theoretical results. Using similar approaches and new contractive conditions, new fixed-circle results on C^*-algebra valued metric spaces can be studied. Since with this article we develop a new and different perspective instead of the classical fixed point theory in C^*-algebra valued metric spaces, we hope that our results will support researchers for future works and applications to other related areas.

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Department of Mathematics
Ondokuz Mayıs University
Samsun
Türkiye
`nilay.sager@omu.edu.tr`

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