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ON THE SPECTRUM OF SINGULAR *q*-STURM–LIOUVILLE OPERATORS ON THE WHOLE AXIS

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Abstract. We give some conditions for the self-adjoint operators associated with the q-Sturm–Liouville expression

$$\tau y := -\frac{1}{q} D_{q^{-1}}(p(x)D_q y(x)) + r(x), \quad -\infty < x < \infty.$$

to have a discrete spectrum, and investigate the continuous spectra of these operators. We also prove that the regular symmetric q-Sturm–Liouville operator is semi-bounded from below which is not studied in literature yet.

1. Introduction

Quantum calculus is a generalization of mathematical objects that have the original object as limits when $q \to 1$. Hence q-calculus is a popular subject. It is of great importance for its applications in several mathematical areas such as the calculus of variations, combinatorics, number theory, basic hypergeometric functions, fractal geometry, quantum theory, orthogonal polynomials, statistic physics and theory of relativity etc. For a deeper understanding of q-calculus we refer the reader to [**1**, **8**, **13**, **19**, **20**, **27**, **33**].

The class of self-adjoint operators is one of the important class in the operator theory because they play an important role in quantum mechanics. For a given self-adjoint operator, a basic question is: What is its spectrum? Specially, the self-adjoint differential operator are widely studied by many authors [**11**, **14**, **15**, **17**, **22**, **23**, **28**, **31**, **34**]. The spectrum of a self-adjoint differential operator is real, consists of discrete spectrum and of continuous spectrum. The spectrum of such operators depends on the behavior of the coefficients of the corresponding differential expression.

Recently, in [**35**], Zhang and Ao have studied the finite spectrum of Sturm–Liouville problems with transmission conditions dependent on the spectral parameter. In [**9**], the authors have studied some spectral properties of the Sturm–Liouville

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equation associated with periodic boundary conditions and additional transmission conditions at one interior singular point. Olgar et al. investigated some spectral properties of a discontinuous boundary value problem [**30**]. In [**10**], Bairamov et al. studied eigenvalues, spectral singularities, resolvent operator, spectrum and scattering function of second–order impulsive matrix difference operators. In [**21**], the authors investigated the discreteness and some other properties of the spectrum for the Schrödinger operator. Fulsche and Nursultanov [**16**] studied the spectral properties of Sturm–Liouville operators with measure potentials. In [**5**], the authors investigated the spectrum of Hahn–Sturm–Liouville operators. In [**4**], Allahverdiev and Tuna studied the spectrum of singular Sturm–Liouville operators on unbounded time scales.

In this paper we extend some results for differential operators obtained in [**17**] to the case of q-Sturm–Liouville expression

$$(\tau y)(x) := -\frac{1}{q}D_{q^{-1}}(p(x)D_q y(x)) + r(x)y(x), \quad -\infty < x < \infty, \tag{1.1}$$

where p, r are real-valued functions defined on $\mathbb{R} := (-\infty, \infty)$, continuous at zero ($p(x) \neq 0$, $x \in \mathbb{R}$) and $\frac{1}{p}, r \in L^1_{q,loc}(\mathbb{R})$. We will prove that the regular symmetric q-Sturm–Liouville operator is semi-bounded from below. Using the splitting method [**17**], we will also give some conditions for the self-adjoint operator associated with the singular expression (1.1) to have a discrete spectrum. Finally, we investigate the continuous spectrum of this operator. The same problem has been investigated by the authors on the semi-axis [**3**].

2. Preliminaries

Now, some preliminary concepts related to quantum analysis and essentials of operator theory are presented for the convenience of the reader. Following the standard notations in [**6, 25**], let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$ and $a \in A$. A *q-difference equation* is an equation that contains q-derivatives of a function defined on A. Let y be a complex-valued function on A. The *q-difference operator* D_q, the *Jackson q-derivative* is defined by $D_q y(x) = \frac{y(qx)-y(x)}{qx-x}$ for all $x \in A \setminus \{0\}$. Note that there is a connection between q-deformed Heisenberg uncertainty relation and the Jackson derivative on q-basic numbers (see [**32**]). In the q-derivative, as $q \to 1$, the q-derivative is reduced to the classical derivative. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x. Since the formulation of the extension problems requires the definition of $D_{q^{-1}}$ in the same manner to be

$$D_{q^{-1}} y(x) := \begin{cases} \frac{y(x)-y(q^{-1}x)}{x-q^{-1}x}, & x \in A \setminus \{0\}, \\ D_q f(0), & x = 0, \end{cases}$$

provided that $D_q f(0)$ exists. A *right-inverse* to D_q, the *Jackson q-integration* is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t \quad (a, b \in A).$$

The *q-integration* for a function is defined in [**18**] by the formulas

$$\int_0^{\infty} f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n),$$

$$\int_{-\infty}^{0} f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n),$$

$$\int_{-\infty}^{\infty} f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n [f(q^n) + f(-q^n)].$$

A function f which is defined on A, $0 \in A$, is said to be q-regular at zero if

$$\lim_{n \to \infty} f(xq^n) = f(0),$$

for every $x \in A$. Through the remainder of the paper, we deal only with functions q-regular at zero. If f and g are q-regular at zero, then we have

$$\int_0^a g(t) D_q f(t)\, d_q t - \int_0^a f(qt) D_q\, g(t) d_q t = f(a)g(a) - f(0)g(0).$$

Let $L_q^2(\mathbb{R}$ be the space of all complex-valued functions defined on $\mathbb{R}$ such that

$$\|f\| := \left(\int_{-\infty}^{\infty} |f(x)|^2 d_q x \right)^{1/2} < \infty.$$

The space $L_q^2(\mathbb{R})$ is a separable Hilbert space with the inner product

$$(f, g) := \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, d_q x, \quad f, g \in L_q^2(\mathbb{R})$$

(see [**7**]).

Definition 2.1. Let D_A denote a subset of the complex Hilbert space H. A linear operator A is said to be Hermitian if, for all $x, y \in D_A$, $(Ax, y) = (x, Ay)$ holds. A Hermitian operator with a domain of definition dense in H is called a symmetric operator. An operator A^* defined on H is called the adjoint of symmetric operator A if for all $x, y \in D_A$, $(x, Ay) = (A^* x, y)$. An operator with a domain of definition dense in H is said to be self-adjoint if $A = A^*$. An operator A is said to be compact if it maps every bounded set into a compact set (see [**29**]).

Definition 2.2. A complex number λ is called a regular point of the linear operator A acting in complex Hilbert space H if it satisfies (R1) and (R2) below

(R1) the inverse $R_\lambda(A) = (A - \lambda I)^{-1}$ where I is the identity operator in H exists,
(R2) $R_\lambda(A)$ is a bounded operator defined on the whole space H.
(R3) $R_\lambda(A)$ is defined on a set which dense in H.

If $R_\lambda(A)$ satisfies (R3), then it is called the *resolvent* of the operator A. All non-regular points λ are called points of the *spectrum* of the operator A.

The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. A $\lambda \in \sigma_p(A)$ is called an eigenvalue of A. The spectrum of the operator A is said to be *purely discrete* if it consists of a denumerable set of eigenvalues with no finite point of accumulation. The *continuous spectrum* $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies (R3) but not (R2). The *residual spectrum* $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists but does not satisfy (R3) (see [**26**]).

Theorem 2.1. [**26**] *The residual spectrum $\sigma_r(A)$ of a self-adjoint linear operator acting on a complex Hilbert space H is empty.*

Theorem 2.2. [**29**] *All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and the same continuous spectrum.*

Definition 2.3. [**29**] The direct sum $A_1 \oplus A_2$ of two operators A_1, A_2 in the spaces H_1, H_2 is an operator in the space $H_1 \oplus H_2$ of all ordered pairs $\{x_1, x_2\}$, $x_1 \in H_1$, $x_2 \in H_2$; its domain of definition is the set of all ordered pairs $\{x_1, x_2\}$, $x_1 \in D_{A_1}$, $x_2 \in D_{A_2}$, and $(A_1 \oplus A_2)\{x_1, x_2\} = \{A_1x_1, A_2x_2\}$. It is easily seen that if A_1 and A_2 are each self-adjoint operators, then their direct sum $A_1 \oplus A_2$ is also a self-adjoint operator.

Definition 2.4. [**29**] A symmetric operator A is said to be semi-bounded from below if there is a number m such that $(Ax, x) \geqslant m\|x\|^2$ for all $x \in D_A$. Similarly, if for all $x \in D_A$, there is a number M such that $(Ax, x) \leqslant M\|x\|^2$, then A is said to be semi-bounded from above.

Theorem 2.3. [**29**] *If a symmetric operator A, with finite deficiency indices (n, n), satisfies the condition $(Ax, x) \geqslant m\|x\|^2$, $x \in D_A$, or the condition $(Ax, x) \leqslant M\|x\|^2$, $x \in D_A$, then the part of the spectrum of every self-adjoint extension of A which lies to the left of m or to the right of M can consist of only a finite number of eigenvalues and the sum of their multiplicities does not exceed n.*

3. Main Results

Let us consider the linear set $D_{\max}$ consisting of all vectors $y \in L_q^2(\mathbb{R})$ such that y and pD_qy are q-regular at zero and $\tau y \in L_q^2(\mathbb{R})$. We define the *maximal operator* $\tau_{\max}$ on $D_{\max}$ by the equality $\tau_{\max}y = \tau y$. The q-Wronskian of $y(.), z(.)$ is defined to be $W_q(y, z)(x) := y(x)D_qz(x) - z(x)D_qy(x)$, $x \in \mathbb{R}$.

For every $y, z \in D_{\max}$ we have q-Green's formula (or q-Lagrange's identity)

$$\int_{-q^{-n}}^{q^{-n}} (\tau y)(x)\,\overline{z(x)}\,d_qx - \int_{-q^n}^{q^n} y(x)\,\overline{(\tau z)(x)}\,d_qx = [y, z](q^{-n}) - [y, z](-q^{-n}), \quad n \in \mathbb{N},$$

where $[y,z](x)$ denotes the q-Lagrange bracket defined by

$$[y,z](x) := p(x)\big[y(x)\overline{D_{q^{-1}}z(x)} - D_{q^{-1}}y(x)\overline{z(x)}\,\big]$$

(see [**2**, **7**]). It is clear that from q-Green's formula limits

$$[y,z](\infty) := \lim_{n\to\infty}[y,z](q^{-n}),\quad [y,z](-\infty) := \lim_{n\to\infty}[y,z](-q^{-n})$$

exist and are finite for all $y,z \in D_{\max}$.

Let $D_{\min}$ be the linear set of all vectors $y \in D_{\max}$ satisfying the conditions

$$[y,z](-\infty) = [y,z](\infty), \tag{3.1}$$

for arbitrary $z \in D_{\max}$. The operator $\tau_{\min}$, that is the restriction of the operator $\tau_{\max}$ to $D_{\min}$ is called the *minimal operator* and the equalities $\tau_{\max} = \tau^*_{\min}$ holds. Further (it follows from (3.1)) $\tau_{\min}$ is closed symmetric operator with deficiency indices $(1,1)$ or $(2,2)$ [**2**, **7**, **12**, **29**].

THEOREM 3.1. *If $p(x) > 0$ ($x \in [-a,a]$), $0 < a < \infty$), then the regular operator $\tau_{\min}$ acting in $L^2_q(-a,a)$ is semi-bounded from below. Further, the negative part of the spectrum of every self-adjoint extension of $\tau_{\min}$ consists of not more than a finite number of negative eigenvalues of finite multiplicity.*

PROOF. For $y \in D_{\min}$ we have

$$y(-a) = (pD_{q^{-1}}y)(-a) = 0,\ y(a) = (pD_{q^{-1}}y)(a) = 0.$$

By q-integration by parts, we get

$$\begin{aligned}(\tau_{\min}y,y) &= \int_{-a}^{a}\tau y\overline{y}\,d_qx = \int_{-a}^{a}\Big[-\frac{1}{q}D_{q^{-1}}(pD_qy) + ry\Big]\overline{y}\,d_qx\\ &= \int_{-a}^{a}\Big[-\frac{1}{q}D_{q^{-1}}(pD_qy)\overline{y} + r(x)|y|^2\Big]d_qx\\ &= \int_{-a}^{a}\big[p|D_qy|^2 + r(x)|y|^2\big]\,d_qx.\end{aligned}$$

We set

$$v(x,\xi) = \begin{cases}1, & \xi \leqslant x\\ 0, & \xi > x,\end{cases} \qquad \text{and} \qquad H(\xi,\eta) = -\int_{-a}^{a} r(x)v(x,\xi)v\,(x,\eta)\,d_qx.$$

For $y \in D_{\min}$ we have

$$y(x) = \int_{-a}^{a}\frac{v(x,\xi)(pD_qy)(\xi)}{p(\xi)}d_q\xi.$$

Hence we get

$$\begin{aligned}(\tau_{\min}y,y) = \int_{-a}^{a}&\frac{|(pD_qy)(\xi)|^2}{p(\xi)}\,d_q\xi\\ &- \int_{-a}^{a}\int_{-a}^{a}\frac{H(\xi,\eta)(pD_qy)(\xi)pD_q\overline{y}(\eta)}{p(\xi)p(\eta)}\,d_q\xi\,d_q\eta.\end{aligned} \tag{3.2}$$

Let $L^2_{q,p}(-a,a)$ be the Hilbert space of all complex-valued functions defined on $[-a,a]$ with the inner product

$$(f_1,f_2)_1=\int_{-a}^{a} f_1(x)\overline{f_2(x)}\frac{1}{p(x)}\,d_qx.$$

In $L^2_{q,p}(-a,a)$ we consider the integral operator K with the symmetric kernel $H(\xi,\eta)$:

$$Kf=\int_{-a}^{a}\frac{H(\xi,\eta)}{p(\eta)}f(\eta)d_q\eta \quad \text{where} \quad \int_{-a}^{a}\int_{-a}^{a}\frac{|H(\xi,\eta)|^2}{p(\xi)p(\eta)}\,d_q\xi\,d_q\eta<\infty.$$

Then K is a compact operator in the space $L^2_{q,p}(-a,a)$ [**6**].

Let $\varphi_1,\varphi_2,\varphi_3, dots$ be a complete orthonormal system of eigenfunctions of the operator K and $\lambda_1,\lambda_2,\lambda_3,\dots$ be the corresponding eigenvalues. Then we get

$$(Kf,f)_1=\sum_{k=1}^{\infty}\lambda_k|(f,\varphi_k)_1|^2.$$

As $k\to\infty$, we have $\lambda_k\to 0$. Then there is a certain number N such that $\lambda_k<1$ for $k>N$. For $(f,\varphi_k)_1=0,\ k=1,2,\dots,N$, we have

$$(Kf,f)_1=\sum_{k=N+1}^{\infty}\lambda_k|(f,\varphi_k)_1|^2\leqslant\sum_{k=N+1}^{\infty}|(f,\varphi_k)_1|^2,$$

that is,

$$(Kf,f)_1\leqslant(f,f)_1. \tag{3.3}$$

Let $\mathcal{D}$ denote the manifold of all functions $y\in D_{\min}$ which satisfy the conditions

$$(pD_qy,\varphi_k)_1=0,\ \ k=1,2,\dots,N,;\ y\in D_{\min}.$$

By (3.3), we have, for $y\in\mathcal{D}$,

$$\int_{-a}^{a}\int_{-a}^{a}\frac{H(\xi,\eta)(pD_qy)(\xi)(pD_q\overline{y}(\eta))}{p(\xi)p(\eta)}d_q\xi\,d_q\eta$$
$$\leqslant(KpD_qy,pD_qy)_1\leqslant(pD_qy,pD_qy)_1=\int_{-a}^{a}\frac{|(pD_qy)(\xi)|^2}{p(\xi)}d_q\xi.$$

From the equality (3.2), we obtain $(\tau_{\min}y,y)\geqslant 0$.

On the other hand, the dimension of the manifold $D_{\min}$ modulo $\mathcal{D}$ is N, and consequently, the operator $\tau_{\min}$ is semi bounded from below on the whole manifold $D_{\min}$. By Theorem 2.3, we get the desired result. □

Let H' denotes the set of all functions f from $L^2_q(\mathbb{R})$ which vanish outside a finite interval $[\alpha,\beta]\subset(-\infty,\infty)$ and $D'_{\min}=H'\cap D_{\min}$. Further, let $\tau'_{\min}$ denote the restriction of the operator $\tau_{\min}$ to $D'_{\min}$. Then $\tau_{\min}$ is the closure of the operator $\tau''_{\min}$, i.e., $\widetilde{\tau'_{\min}}=\tau_{\min}$ [**29**].

Now we restrict $D'_{\min}$ by imposing the additional conditions

$$y(-c)=(pD_{q^{-1}}y)(-c)=0,\ \ y(c)=(pD_{q^{-1}}y)(c)=0,$$

where c is a fixed point of the interval $\mathbb{R}$. By this restriction, we obtain the manifold $D''_{\min}$. The restriction $\tau''_{\min}$ of the operator $\tau'_{\min}$ to $D''_{\min}$ is called the splitting of the operator $\tau'_{\min}$ at the points $-c$ and c of the interval $\mathbb{R}$. It is clear that

$$\tau''_{\min} = \tau'_1 \oplus \tau'_2 \oplus \tau'_3, \tag{3.4}$$

i.e., the operator $\tau''_{\min}$ is the direct sum of three operators τ'_1, τ'_2 and τ'_3 in the spaces $L^2_q(-\infty,-c)$, $L^2_q(-c,c)$ and $L^2_q(c,\infty)$, where τ'_1, τ'_2 and τ'_3 are generated in these spaces from the q-Sturm–Liouville expression τ in the same way as $\tau'_{\min}$ was.

If $\tau_1 = \widetilde{\tau'_1}$, $\tau_2 = \widetilde{\tau'_2}$ and $\tau_3 = \widetilde{\tau'_3}$ are the closures of the operators τ'_1, τ'_2 and τ'_3, then (3.4) implies that $\widetilde{\tau''_{\min}} = \tau_1 \oplus \tau_2 \oplus \tau_3$. If we extend the symmetric operators τ_1, τ_2 and τ_3 into self-adjoint operators $\tau_{1,s}$, $\tau_{2,s}$ and $\tau_{3,s}$ in the spaces $L^2_q(-\infty,-c)$, $L^2_q(-c,c)$ and $L^2_q(c,\infty)$ respectively, then the direct sum $A = \tau_{1,s} \oplus \tau_{2,s} \oplus \tau_{3,s}$ will be a self-adjoint extension of the symmetric operator $\widetilde{\tau''_{\min}}$. The spectrum of the operator A is the set-theoretic sum of the spectra of $\tau_{1,s}$, $\tau_{2,s}$ and $\tau_{3,s}$. Since the deficiency indices of the operator $\widetilde{\tau''_{\min}}$ are finite, by Theorem 2.2, all its self-adjoint extensions have one and the same continuous spectrum. Both the operator A and also each self-adjoint extension τ_s of the operator $\tau_{\min}$ are such extensions. Hence, the continuous parts of spectrum of the two operators A and τ_s coincide.

Therefore, we have the following theorem.

THEOREM 3.2. *The continuous parts of the spectrum of every self-adjoint extension of the operator $\tau_{\min}$ is the set-theoretic sum of the continuous parts of the spectra of $\tau_{1,s}$, $\tau_{2,s}$ and $\tau_{3,s}$, where $\tau_{1,s}$, $\tau_{2,s}$ and $\tau_{3,s}$ have been obtained by the splitting of the operator $\tau_{\min}$.*

THEOREM 3.3. *If*

$$\lim_{x\to\pm\infty} r(x) = +\infty, \tag{3.5}$$

$$p(x) > 0, \quad x \in \mathbb{R} \tag{3.6}$$

then every self-adjoint extension τ_s of the singular operator $\tau_{\min}$ has a purely discrete spectrum.

PROOF. Let $N > 0$ be an arbitrary number. From (3.5), one can choose numbers $-c$ and c such that $r(x)| > N$ for $x \in \mathbb{R} \setminus (-c,c)$. By condition (3.6), via q-integration by parts, we obtain ($y \in D_{\tau'_1}$)

$$\begin{aligned}(\tau'_1 y, y) &= \int_{-\infty}^{-c} \tau y \overline{y}\, d_q x = \int_{-\infty}^{-c} \left[-\frac{1}{q} D_{q^{-1}}(pD_q y) + ry\right] \overline{y}\, d_q x \\ &= \int_{-\infty}^{-c} \left[-\frac{1}{q} D_{q^{-1}}(pD_q y)\overline{y} + r(x)|y|^2\right] d_q x \\ &= \int_{-\infty}^{-c} p|D_q y|^2 + r(x)|y|^2 d_q x > N \int_{-\infty}^{-c} |y|^2 d_q x = N(y,y).\end{aligned}$$

Hence the operator τ'_1 is bounded from below and its closure τ_1 is also bounded from below by the number N. Therefore, by Theorem 2.3, the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of the self-adjoint extension $\tau_{1,s}$ of τ_1. Similarly, by condition (3.6), via q-integration by parts, we obtain ($y \in D_{\tau_3'}$)

$$\begin{aligned}(\tau_3' y, y) &= \int_c^\infty \tau y \overline{y}\, d_q x = \int_c^\infty \left[-\frac{1}{q} D_{q^{-1}}(pD_q y) + ry\right] \overline{y}\, d_q x \\ &= \int_c^\infty \left[-\frac{1}{q} D_{q^{-1}}(pD_q y)\overline{y} + r(x)|y|^2\right] d_q x \\ &= \int_c^\infty p|D_q y|^2 + r(x)|y|^2 d_q x > N \int_c^\infty |y|^2 d_q x = N(y,y).\end{aligned}$$

Hence the operator τ_3' is bounded from below and its closure τ_3 is also bounded from below by the number N. Therefore, by Theorem 2.3, the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of the self-adjoint extension $\tau_{3,s}$ of τ_3. On the other hand, since the operator τ_2 is regular, the spectrum of any self-adjoint extension $\tau_{2,s}$ of τ_2 is purely discrete. Hence the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of $A = \tau_{1,s} \oplus \tau_{2,s} \oplus \tau_{3,s}$. By Theorem 3.2, every self-adjoint extension τ_s of the operator $\tau_{\min}$ has this property. Since the number N is arbitrary, the spectrum of the operator τ_s has no continuous part at all. □

THEOREM 3.4. *Let* $\lim_{x\to\pm\infty} r(x) = M$ *and* $p(x) > 0$ $(x \in \mathbb{R})$. *Then the interval* $(-\infty, M)$ *contains no point of the continuous spectrum of any, self-adjoint extension* τ_s *of the singular operator* $\tau_{\min}$*; on the contrary, any* τ_s *can only have at most point-eigenvalues on this interval and these can have a point of accumulation only at* $\lambda = M$

PROOF. If we decompose the operator at the points $-c$ and c such that

$$r(x) > M - \varepsilon \text{ for } x \in \mathbb{R} \setminus (-c, c),$$

then we obtain $(\tau_2' y, y) > (M - \varepsilon)(y, y)$. Hence, the part of the spectrum of τ_1 lying in the interval $(-\infty, M - \varepsilon)$ can consist only of a finite number of eigenvalues of finite multiplicity. Likewise, we obtain $(\tau_3' y, y) > (M - \varepsilon)(y, y)$. Consequently, the part of the spectrum of τ_3 lying in the interval $(-\infty, M - \varepsilon)$ can consist only of a finite number of eigenvalues of finite multiplicity. On the other hand, by Theorem 3.1, the operator τ_2 is regular and bounded below. Hence its spectrum is purely discrete; and any point of accumulation of the spectrum $\tau_{2,s}$ can only be at $\lambda = +\infty$. Thus, from Theorem 3.2, we get the desired result. □

Now, we need following lemma.

LEMMA 3.1. *If the interval* $[\lambda_0 - \delta, \lambda_0 + \delta]$ *contains no point of the spectrum of a self-adjoint operator* A *except perhaps for a finite number of eigenvalues each of finite multiplicity, and if* Q *is a bounded Hermitian operator satisfying the condition* $\|Q\| < \delta$, *then the point* λ_0 *does not lie in the continuous part of the spectrum of the operator* $A + Q$.

PROOF. See [**29**]. □

THEOREM 3.5. *Let* $p(x) \equiv 1$ *and* $\lim_{x\to\pm\infty}|r(x)| = M$. *Then any interval, of length greater than* $2M$, *of the positive half-axis contains of the continuous spectrum of any self-adjoint extension* τ_s *of the singular operator* $\tau_{\min}$.

PROOF. Suppose, contrary to our claim, that an interval $[\lambda_0-\delta, \lambda_0+\delta]$ of the half-axis $\lambda > 0$ contains no point of the continuous spectrum of τ_s, $\delta > M$. Then, the operator may be decomposed, this interval would contain no point of the continuous spectrum of any self-adjoint extension of $\tau_{\min}$. If we choose the points $-c$ and c such that $|r(x)| \leqslant M+\varepsilon < \delta$ for $|x| > c$, then, by Lemma 3.1, λ_0 can not belong to the continuous spectrum of the self-adjoint extension of the minimal operator generated by the expression $-\frac{1}{q}D_{q^{-1}}D_q$ and the same boundary conditions. But this is contradiction because the continuous spectrum of last operator covers the whole of the positive half-axis. $\square$

In particular, for $M = 0$ we have the following corollary.

COROLLARY 3.1. *Let* $p(x) \equiv 1$ *and* $\overline{\lim}_{x\to\pm\infty}|r(x)| = 0$. *Then the whole positive half-axis is covered by the continuous spectrum of any self-adjoint extension* τ_s *of the singular operator* $\tau_{\min}$.

COROLLARY 3.2. *Let* $p(x) \equiv 1$ *and*

$$\overline{\lim_{x\to\pm\infty}}|r(x)| = \rho < \infty, \quad \underline{\lim_{x\to\pm\infty}}|r(x)| = \sigma > -\infty.$$

Then any interval, of length greater than $(\rho-\sigma)$, *of the half-axis* $\lambda > \frac{1}{2}(\rho+\sigma)$ *contains the continuous spectrum of any self-adjoint extension* τ_s *of the singular operator* $\tau_{\min}$.

PROOF. For, if $r_1(x) = r(x) - \frac{1}{2}(\rho+\sigma)$, then $\overline{\lim}_{x\to\pm\infty}|r_1(x)| = \frac{1}{2}(\rho-\sigma)$, and the result follows by replacing $r(x)$ by $r_1(x)$, i.e., by applying Theorem 3.5 to the operator $\tau_s - \frac{1}{2}(\rho+\sigma)I$. $\square$

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