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CATEGORICAL CENTERS AND YETTER–DRINFEL'D-MODULES AS 2-CATEGORICAL (BI)LAX STRUCTURES

Bojana Femić and Sebastian Halbig

Abstract. The bicategorical point of view provides a natural setting for many concepts in the representation theory of monoidal categories. We show that centers of twisted bimodule categories correspond to categories of 2-dimensional natural transformations and modifications between the deloopings of the twisting functors. This explains conceptually the lifting of (rigid) dualities to centers of twisted bimodule categories. Inspired by the notion of (pre)bimonoidal functors due to McCurdy and Street and by bilax functors of Aguiar and Mahajan, we study 2-dimensional functors which are simultaneously lax and colax with a compatibility condition. Our approach is build upon a 2-categorical Yang–Baxter operator. We show how this concept, which we call a bilax functor, generalizes many known notions from the theory of Hopf algebras. We propose a 2-category of bilax functors whose 1-cells generalize Yetter–Drinfel'd modules in ordinary categories. We prove that the 2-category of bilax functors from the trivial 2-category is isomorphic to the 2-category of bimonads, and construct a faithful 2-functor from the latter to the 2-category of mixed distributive laws of Power and Watanabe.

1. Introduction

The concept of a center of a monoid was categorified independently by Majid in [**31**] and Joyal and Street in [**21**]. Since then, it has been extensively studied in Hopf algebra and category theory, see for example [**22**] for an overview. One of its striking features comes from the fact that by passing from sets to categories one can replace the qualitative question: 'Do two elements of a monoid commute with another?' with a quantitative one: 'How many suitably coherent (iso)morphisms exist in a given monoidal category between the tensor product of two objects and its opposite?'. Such (iso)morphisms are called *half-braidings*. The center of a

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monoidal category consists of objects of the underlying category with fixed half-braidings together with morphisms of the base category which satisfy a certain compatibility relation.

The aim of the present paper is twofold. In the first part, we study the center construction from the bicategorical perspective. For a monoidal category $\mathcal{C}$ let $\Sigma\mathcal{C}$ denote the induced one-object bicategory, which is called the *delooping* or *suspension* of $\mathcal{C}$. It was observed in [**44**, page 255], see also [**37**, Corollary 4.13], that the center of $\mathcal{C}$ is isomorphic to the category of pseudonatural transformations of the identity 2-functor of $\Sigma\mathcal{C}$ and their modifications. See also the discussion on pages 187–189 of [**6**]. We extend this bicategorical interpretation to center categories of bimodule categories, introduced in [**38**] and also named *actegories* in [**32**]. In order to keep our notation concise, we will focus throughout the introduction on the special case that the bimodule category in question is the regular one. That is $\mathcal{C}$ with tensoring from the left and right as its action.

Given another monoidal $\mathcal{E}$ and two lax monoidal functors $F, G\colon \mathcal{E} \to \mathcal{C}$, we define the twisted center $\mathcal{Z}(F, \mathcal{C}, G)$. We show that it can be identified with the category of 2-dimensional natural transformations between the deloopings of F and G, see Theorem 3.1. Consequently, twisted centres can be interpreted as the hom categories of a 2-category $\mathcal{Z}(\mathcal{E}, \mathcal{C})$, see Theorem 3.2, whose objects are lax functors $\Sigma\mathcal{E} \to \Sigma\mathcal{C}$. While half-braidings need not be isomorphisms in our setting, we prove that if $\mathcal{E}$ and $\mathcal{C}$ are autonomous and we restrict ourselves to strong monoidal functors, their inverses do exist, see Theorem 3.4. In this case, the bicategory $\mathcal{Z}(\mathcal{E}, \mathcal{C})$ is compact, (Theorem 3.5), i.e., all 1-cells have adjoints. This result provides a natural interpretation of certain duality notions between centers of compatible bimodule categories in [**18**]. A special instance of our construction is Shimizu's bicategory $\mathrm{TF}(\mathcal{E}, \mathcal{C})$ of tensor functors, see [**41**], and the result thereof about duals. Furthermore, we will discuss its relation with the centre of of a bicategory developed in [**34**].

In the second part of the paper, we introduce and study 2-categorical functors which are simultaneously lax and colax with a compatibility relation involving a Yang–Baxter operator. We call them *bilax functors.* For monoidal categories such functors were studied under the names of pre-bimonoidal and bimonoidal functors in [**33**], and, in case the domain category is braided, bilax functors in [**1**]. We show that 2-categorical bilax functors generalize bialgebras in braided monoidal categories and bimonads in 2-categories, defined with respect to a Yang–Baxter operator. They furthermore preserve various types of monads, including bimonads and comodule monads as well as relative bimonad modules. The component functors of a bilax functor factor on the endo-hom-categories through an analogue of Hopf bimodules, defined with respect to Yang–Baxter operators.

We record that instead of working with Yang–Baxter operators, one could equally use local braidings, following the footsteps of [**1**]. In this case, the generalization and preservation results somewhat differ from the ones that we obtained and that are listed above.

We establish a 2-category of bilax functors by introducing bilax natural transformations and bilax modifications. Bilax natural transformations are both lax and colax natural transformations satisfying a compatibility condition. As such they generalize the bimonad morphisms from [**13**] and Yetter–Drinfel'd modules from braided monoidal categories. Classically, the Drinfel'd center of the category of modules over a Hopf algebra is monoidally equivalent to its Yetter–Drinfel'd modules. We will discuss in Section Subsection 5.1 how bilax natural transformations and bilax modifications generalize Yetter–Drinfel'd modules, but need not correspond to center categories. Let $\mathrm{Bilax}(1, \mathcal{K})$ be the 2-category of bilax functors whose source is the trivial 2-category 1. We prove that there is a 2-category isomorphism between $\mathrm{Bilax}(1, \mathcal{K})$ and the 2-category of bimonads from [**14**]. Finally, we show that there is a faithful 2-functor $\mathrm{Bilax}(1, \mathcal{K}) \hookrightarrow \mathrm{Dist}(\mathcal{K})$ to the 2-category of mixed distributive laws of [**39**].

The paper is composed as follows. We first give an overview of bicategories, deloopings, module and center categories in Section 2. Section 3 provides a higher categorical interpretation of center categories and studies compactness of the bicategory of center categories. Bilax functors and their properties are investigated in section 4. In the last section a 2-category of bilax functors is introduced and its relations to the 2-categories $\mathrm{Bimnd}(\mathcal{K})$ and $\mathrm{Dist}(\mathcal{K})$ is shown.

2. Preliminaries: Deloopings and weak twisted centers

We assume that the reader is familiar with the notion of a braided monoidal category and the corresponding notation of string diagrams (see e.g. [**20, 22, 45**]), as well as with the definition of a bicategory, for which we recommend [**4, 19**].

In this section we give a short summary of bicategories, deloopings of monoidal categories, module categories, and weak twisted centers. For a more extensive discussion of module categories we refer the reader to [**10**].

Briefly, a *monoidal category* consists of a category $\mathcal{C}$ together with a suitably associative and unital multiplication $\otimes\colon \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ implemented by a functor which is called the *tensor product*.

A 'many object' generalization of monoidal categories is provided by *bicategories* These can be thought of as higher dimensional categories with *hom-categories* between every pair of objects instead of mere sets. The objects of these hom-categories are called 1-*cells* and the morphisms 2-*cells*. Any bicategory $\mathcal{K}$ admits two ways to compose: *horizontal composition* given by the composition functors

$$\circ_{Z,Y,X}\colon \mathcal{K}(Y,Z) \times \mathcal{K}(X,Y) \longrightarrow \mathcal{K}(X,Z), \quad \text{for } X, Y, Z \in \mathcal{O}b\,\mathcal{K} \text{ (objects of } \mathcal{K}),$$

and *vertical composition* induced by the compositions inside the hom-categories. Instead of identity morphisms, every $X \in \mathcal{O}b\,\mathcal{K}$ has a unit 1-cell $\mathrm{id}_X \in \mathcal{K}(X, X)$. In general, the horizontal composition of a bicategory is associative and unital only up to suitable natural isomorphisms. Bicategories where these morphisms are identities are called 2-*categories.* Since every bicategory is biequivalent to a 2-category, we will restrict ourselves without loss of generality to the setting of 2-categories.

As hinted at before, there is an intimate relationship between monoidal categories and bicategories It is provided by considering a monoidal category $\mathcal{C}$ as a bicategory $\Sigma\mathcal{C}$ with one object (which we will usually denote by $*$) and $\mathcal{C}$ as its unique hom-category. Under this identification, the tensor product of $\mathcal{C}$ becomes the horizontal composition of $\Sigma\mathcal{C}$, and the monoidal unit becomes the identity 1-cell id on the unique object of $\Sigma\mathcal{C}$. The resulting canonical isomorphism of categories between the category of monoidal categories with certain structure preserving functors and one-object bicategories plus structure preserving 2-dimensional functors is called *delooping*:

$$\left\{\begin{matrix}\text{monoidal categories with}\\ \text{lax/colax/strong monoidal functors}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{one-object bicategories with}\\ \text{lax/colax/pseudofunctors}\end{matrix}\right\}.$$

Remark 2.1. While horizontal composition in bicategories is contravariant, the tensor product of a monoidal category is usually defined covariantly. Since our focus lies on 2-dimensional categories and in order to avoid confusion, we will also read tensor products contravariantly. Technically, this means, replacing a monoidal category $(\mathcal{C}, \otimes)$ with $(\mathcal{C}, \otimes^{\mathrm{rev}})$.

Bicategories provide a natural interpretation of the representation theory of monoidal categories. All endomorphism categories of a bicategory are monoidal with horizontal composition as tensor product. Similarly, given two objects $A, B \in \mathcal{O}b\,\mathcal{K}$ of a bicategory with endomorphism categories $\mathcal{D} := \mathcal{K}(A, A)$ and $\mathcal{C} := \mathcal{K}(B, B)$, horizontal composition endows $\mathcal{M} := \mathcal{K}(A, B)$ with the structure of a $(\mathcal{C}, \mathcal{D})$-*bimodule category*[1]. That is, there are two functors

$$\triangleright\colon \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M} \quad \text{and} \quad \triangleleft\colon \mathcal{M} \times \mathcal{D} \longrightarrow \mathcal{D}$$

subject to analogous but weakened version of the axioms of bimodules over a monoid.

Conversely, we can associate to any $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{M}$ a two-object bicategory $\Sigma\mathcal{M}$, which we call the *delooping* of $\mathcal{M}$. We write C and D for its objects. The hom-categories of $\Sigma\mathcal{M}$ are

$$\begin{aligned} \Sigma\mathcal{M}(D, D) &= \mathcal{D}, \quad \Sigma\mathcal{M}(D, C) = M,\\ \Sigma\mathcal{M}(C, C) &= \mathcal{C}, \quad \Sigma\mathcal{M}(C, D) = 1, \text{ the trivial category.} \end{aligned}$$

Horizontal composition is given by the tensor products of $\mathcal{C}$ and $\mathcal{D}$ as well as the left and right action of $\mathcal{C}$ and $\mathcal{D}$ on $\mathcal{M}$. The relation between (bi)module categories and bicategories was already observed by Bénabou, [**4**, Section 2.3].

If the categories $\mathcal{C}$ and $\mathcal{D}$ coincide, one can define the center of a bimodule category. The aim of the paper at hand will be the study of these centers and their interaction with the theory of bicategories in a slightly more general version.

Definition 2.1. Let $F\colon \mathcal{E} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{E} \longrightarrow \mathcal{C}$ be lax monoidal functors and $\mathcal{M}$ a (strict) $(\mathcal{C}, \mathcal{D})$-bimodule category over the (strict) monoidal categories $\mathcal{C}$ and $\mathcal{D}$. A *left half-braiding* of an object $M \in \mathcal{M}$ relative to F and G is a natural transformation

$$\sigma_X\colon M \triangleleft F(X) \longrightarrow G(X) \triangleright M, \quad \text{for all } X \in \mathcal{E},$$

[1](Bi)module categories are also known as *actegories*, see [**32**].

such that for all $X, Y \in \mathcal{E}$ the following diagrams commute:

(2.1)
$$\begin{tikzcd}
M \lhd F(Y) \lhd F(X) \arrow[rr, "\sigma_I \lhd \mathrm{id}_{F(X)}"] \arrow[d, "M \lhd F^2"'] & & G(Y) \rhd M \lhd F(X) \arrow[d, "\mathrm{id}_{G(Y)} \lhd \sigma_X"] \\
M \lhd F(Y \otimes X) \arrow[r, "\sigma_{Y\otimes X}"'] & G(Y \otimes X) \rhd M & G(Y) \rhd G(X) \rhd M \arrow[l, "G^2 \rhd M"]
\end{tikzcd}$$

(2.2)
$$\begin{tikzcd}
M \lhd I \cong I \rhd M \arrow[rr, "G^0 \rhd M"] \arrow[dr, "M \lhd F^0"'] & & G(I) \rhd M \arrow[dl, "\sigma_I"] \\
& M \lhd F(I) &
\end{tikzcd}$$

Similarly, a *right half-braiding* on M relative to F and G is a natural transformation

$$\tilde{\sigma}_X : G(X) \rhd M \longrightarrow M \lhd F(X), \quad \text{for all } X \in \mathcal{E},$$

subject to analogous identities.

The *left weak center of* $\mathcal{M}$ *relative to* F *and* G is the category $\mathcal{Z}_l^w(F, \mathcal{M}, G)$. Its objects are pairs (M, σ) consisting of an object $M \in \mathcal{M}$ together with a left half-braiding σ on M relative to F and G. A morphism between objects $(M, \sigma), (N, \tau) \in \mathcal{Z}_l^w(F, \mathcal{M}, G)$ is an arrow $f \in \mathcal{M}(M, N)$ such that

$$(\mathrm{id}_{G(X)} \rhd f)\sigma_X = \tau_X(f \lhd \mathrm{id}_{F(X)}), \quad \text{for all } X \in \mathcal{E}.$$

The full subcategory $\mathcal{Z}_l^s(F, \mathcal{M}, G)$ of $\mathcal{Z}_l^w(F, \mathcal{M}, G)$ whose objects have invertible half-braidings is called the *(strong) left center of* $\mathcal{M}$ *relative to* F *and* G. When the functors are clear from the context, we will call the latter two categories simply *left weak* or *strong twisted centers of* $\mathcal{M}$, respectively.

We define the right weak and strong twisted center categories $\mathcal{Z}_r^w(F, \mathcal{M}, G)$ and $\mathcal{Z}_r^s(F, \mathcal{M}, G)$ in an analogous way.

When $\mathcal{C} = \mathcal{D} = \mathcal{M}$ we set

$$Z_l^w(F, G) \coloneqq \mathcal{Z}_l^w(F, \mathcal{D}, G) \quad \text{and} \quad Z_r^w(F, G) \coloneqq \mathcal{Z}_r^w(F, \mathcal{D}, G).$$

For $\mathcal{C}$ a tensor category and F, G tensor functors these present the left and right version of the twisted center category $Z(F, G)$ studied in [**41**, Section 3].

If $F = G = \mathrm{Id}_{\mathcal{C}}$, we write $\mathcal{Z}_{\mathcal{C}}^l(\mathcal{M}) \coloneqq \mathcal{Z}_l^s(\mathrm{Id}, \mathcal{M}, \mathrm{Id})$ and $\mathcal{Z}_{\mathcal{C}}^r(\mathcal{M}) \coloneqq \mathcal{Z}_r^s(\mathrm{Id}, \mathcal{M}, \mathrm{Id})$. These recover the (left and right) center category from [**16**].

Lemma 2.1. *Suppose* $G\colon \mathcal{E} \longrightarrow \mathcal{C}$ *and* $F\colon \mathcal{E} \longrightarrow \mathcal{D}$ *are lax monoidal functors and* $\mathcal{M}$ *is a* $(\mathcal{C}, \mathcal{D})$*-bimodule category. Then there exists an isomorphism of categories*

$$\Xi\colon \mathcal{Z}_l^s(F, \mathcal{M}, G) \longrightarrow \mathcal{Z}_r^s(F, \mathcal{M}, G), \quad \Xi(M, \sigma) = (M, \sigma^{-1}),$$

which is the identity on morphisms.

Proof. Suppose that σ is an invertible left half-braiding on an object $M \in \mathcal{M}$. We show that σ^{-1} defines a right half-braiding. Precomposing the equation in (2.1) by $(\sigma_Y^{-1} \lhd \mathrm{id}_{F(X)})(\mathrm{id}_{G(X)} \rhd \sigma_X^{-1})$ and postcomposing by σ_{YX}^{-1} yields the desired compatibility of right half-braidings with the lax functor structures. Analogous calculations show that σ^{-1} is compatible with the lax units of F and G and that

Ξ sends any morphism in the strong left center to a morphism of the strong right center. The proof is concluded by constructing Ξ^{-1} in the same spirit as Ξ. That is, by mapping invertible right half-braidings to their inverses. $\square$

The construction of (left) strong twisted center categories can be seen as a result of the following composition of 2-functors:

$$\begin{array}{ccccc} \mathcal{C}\text{-}\mathcal{D}\text{-}\,\mathrm{Bimod} & \xrightarrow{(F,G)} & \mathcal{E}\text{-}\mathcal{E}\text{-}\,\mathrm{Bimod} & \xrightarrow{\mathcal{Z}_{\mathcal{E}}} & \mathcal{Z}(\mathcal{E})\text{-}\,\mathrm{Mod} \\ \mathcal{M} & \longmapsto & {}_G\mathcal{M}_F & \longmapsto & \mathcal{Z}_{\mathcal{E}}({}_G\mathcal{M}_F) \end{array}$$

where (F,G) denotes precomposing the left and right action by F and G, respectively, and $\mathcal{Z}_{\mathcal{E}}$ is defined as in [**11**, Section 3.4]. The term 'twisted' is motivated by this composition. Namely, if F and G are strong monoidal functors, a $\mathcal{C}$-$\mathcal{D}$-bimodule category structure is twisted by them into an $\mathcal{E}$-bimodule structure.

3. Categorical centers as a data in a tricategory

At the core of our investigation in this section are (weak) twisted centers and their interpretation from a higher categorical point of view. We will show that center categories are hom-categories of hom-bicategories of a particular tricategory. Namely, the tricategory of bicategories with a single object.

Throughout this section, we fix a monoidal category $\mathcal{E}$ and a 2-category $\mathcal{K}$.

3.1. Categorical centers as (co)lax natural transformations. For the interpretation of center categories from the perspective of 2-categories, we first recall the definitions of lax and colax functors between bicategories and of lax and colax natural transformations between the latter ones.

Definition 3.1. A *lax functor* $(\mathcal{F},\mathcal{F}^2,\mathcal{F}^0)\colon \mathcal{K}\longrightarrow\mathcal{K}'$ between 2-categories consists of

(1) an assignment $\mathcal{O}b\,\mathcal{K}\ni A\mapsto \mathcal{F}(A)\in\mathcal{O}b\,\mathcal{K}'$,
(2) for all $A,B\in\mathcal{O}b\,\mathcal{K}$ a local functor $\mathcal{F}_{A,B}\colon\mathcal{K}(A,B)\longrightarrow\mathcal{K}'(\mathcal{F}(A),\mathcal{F}(B))$,
(3) a natural transformation
$$\mathcal{F}^2_{g,f}\colon F(g)\circ' F(f)\Rightarrow F(g\circ f),\quad \text{for } (g,f)\in\mathcal{K}(B,C)\times\mathcal{K}(A,B),$$
(4) a natural transformation $\mathcal{F}^0_A\colon\ \mathrm{id}_{\mathcal{F}(A)}\Rightarrow\mathcal{F}(\mathrm{id}_A)$, for $A\in\mathcal{O}b\,\mathcal{K}$,

so that $\mathcal{F}^2$ and $\mathcal{F}^0$ satisfy *associativity* and *unitality* laws.

When the natural transformations $\mathcal{F}^2$ and $\mathcal{F}^0$ are directed in the opposite direction and satisfy *coassociativity* and *counitality* laws, one has a *colax functor*. One speaks of a *pseudofunctor* if $\mathcal{F}^2$ and $\mathcal{F}^0$ are isomorphisms.

Lax transformations can be defined for lax and colax functors. The same holds for colax transformations, so that there are four variations of definitions, depending on the situation.

Definition 3.2. Let $(\mathcal{F},\mathcal{F}^2,\mathcal{F}^0)\colon\mathcal{K}\longrightarrow\mathcal{K}'$ and $(\mathcal{G},\mathcal{G}^2,\mathcal{G}^0)\colon\mathcal{K}\longrightarrow\mathcal{K}'$ be lax functors between 2-categories. A colax natural transformation $\chi\colon\mathcal{F}\Rightarrow\mathcal{G}$ consists of

(1) a 1-cell $\chi_A\colon \mathcal{F}(A) \longrightarrow \mathcal{G}(A)$ for each object $A \in \mathcal{O}b\,\mathcal{K}$, and
(2) for every pair of objects $A, B \in Ob\mathcal{K}$ a collection of 2-cells

$$\{\chi_f\colon \chi_B \circ \mathcal{F}_{A,B}(f) \Rightarrow \mathcal{G}_{A,B}(f) \circ \chi_A \mid f \in \mathcal{K}(A,B)\}$$

natural in f subject to *colax multiplicativity*

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\chi_A"'] & \mathcal{F}(B) \arrow[r, "\mathcal{F}(g)"] \arrow[d, "\chi_B"] \arrow[dl, phantom, "\Swarrow\chi_f"] & \mathcal{F}(C) \arrow[d, "\chi_C"] \arrow[dl, phantom, "\Swarrow\chi_g"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] \arrow[rr, bend right, "\mathcal{G}(gf)"'] & \mathcal{G}(B) \arrow[r, "\mathcal{G}(g)"'] \arrow[d, phantom, "\Downarrow\mathcal{G}^2"] & \mathcal{G}(C) \\
& &
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{F}(B) \arrow[dr, "\mathcal{F}(g)"] \arrow[d, phantom, "\Downarrow\mathcal{F}^2"] & \\
\mathcal{F}(A) \arrow[ur, "\mathcal{F}(f)"] \arrow[rr, "\mathcal{F}(gf)"] \arrow[d, "\chi_A"'] & & \mathcal{F}(C) \arrow[d, "\chi_C"] \arrow[dll, phantom, "\Swarrow\chi_{gf}"] \\
\mathcal{G}(A) \arrow[rr, "\mathcal{G}(gf)"'] & & \mathcal{G}(C)
\end{tikzcd}
\tag{3.1}$$

and *colax unitality*

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[rr, bend left, "\mathrm{id}_{\mathcal{F}(A)}"] \arrow[d, "\chi_A"'] & & \mathcal{F}(A) \arrow[d, "\chi_A"] \arrow[dll, phantom, "\Swarrow\mathrm{id}"] \\
\mathcal{G}(A) \arrow[rr, bend left, "\mathrm{id}_{\mathcal{G}(A)}"] \arrow[rr, bend right, "\mathcal{G}(\mathrm{id}_A)"'] & \Downarrow\mathcal{G}^0 & \mathcal{G}(A)
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{F}(A) \arrow[rr, bend left, "\mathrm{id}_{\mathcal{F}(A)}"] \arrow[rr, bend right, "\mathcal{F}(\mathrm{id}_A)"'] \arrow[d, "\chi_A"'] & \Downarrow\mathcal{F}^0 & \mathcal{F}(A) \arrow[d, "\chi_A"] \arrow[dll, phantom, "\Swarrow\chi_{\mathrm{id}_A}"] \\
\mathcal{G}(A) \arrow[rr, bend right, "\mathcal{G}(\mathrm{id}_A)"'] & & \mathcal{G}(A)
\end{tikzcd}$$

If the 2-cells of χ are invertible, it is called a *pseudonatural transformation.* In case they are identities, one speaks of a *strict natural transformation.*

By reverting the direction of the 2-cells of χ one obtains the notion of a *lax natural transformation* between lax functors.

Definition 3.3. Let $\mathcal{F}, \mathcal{G}\colon \Sigma\mathcal{E} \longrightarrow \mathcal{K}$ be two lax functors. The *left weak center of* $\mathcal{K}$ *relative to* $\mathcal{F}$ *and* $\mathcal{G}$ is $\mathcal{Z}_l^w(\mathcal{F}, \mathcal{K}, \mathcal{G}) \coloneqq \mathcal{Z}_l^w(F, \mathcal{M}, \mathcal{G})$, where $\mathcal{M} = \mathcal{K}(\mathcal{F}(*), \mathcal{G}(*))$ is the bimodule category over $\mathcal{C} = \mathcal{K}(\mathcal{G}(*), \mathcal{G}(*))$, $\mathcal{D} = \mathcal{K}(\mathcal{F}(*), \mathcal{F}(*))$ whose actions are given by composition and $F \coloneqq F_{*,*}\colon \mathcal{E} \longrightarrow \mathcal{D}$ and $G \coloneqq G_{*,*}\colon \mathcal{E} \longrightarrow \mathcal{C}$ are the unique lax monoidal component functors of $\mathcal{F}$ and $\mathcal{G}$.

We may now characterise the objects of a tiwsted center in terms of colax transformations.

Proposition 3.1. *Let $\mathcal{E}$ be a monoidal category, $\mathcal{F}, \mathcal{G}\colon \Sigma\mathcal{E} \longrightarrow \mathcal{K}$ two lax functors. The objects of the left weak twisted center $\mathcal{Z}_l^w(\mathcal{F}, \mathcal{K}, \mathcal{G})$ are canonically in bijection with colax natural transformations $\chi\colon \mathcal{F} \Rightarrow \mathcal{G}\colon \Sigma\mathcal{E} \longrightarrow \mathcal{K}$. Under this identification, the objects of the strong center correspond to pseudonatural transformations.*

Proof. Since the bicategory $\Sigma\mathcal{E}$ has a single object, there is a single 1-cell component $\chi_* : \mathcal{F}(*) \longrightarrow \mathcal{G}(*)$ of $\chi\colon \mathcal{F} \Rightarrow \mathcal{G}$, which corresponds to a distinguished object $M_\chi \in \mathcal{M}$. The 2-cell components of χ are arrows $\chi_X\colon M_\chi \circ \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \circ M_\chi$ in $\mathcal{M}$ natural in $X \in \mathcal{E}$ and the colax multiplicativity and unity translate into the commuting diagrams (2.1) and (2.2). The second claim is immediate. $\square$

Suppose $\mathcal{M}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category and $F\colon \mathcal{E} \longrightarrow \mathcal{D}$ and $\mathcal{G}\colon \mathcal{E} \longrightarrow \mathcal{C}$ are two lax monoidal functors. Let $\iota_\mathcal{C} : \Sigma\mathcal{C} \longrightarrow \Sigma\mathcal{M}$ be the 2-functor given by $\iota_\mathcal{C}(*) = C$ and as identity on the hom-categories $\mathcal{C} \longrightarrow \Sigma\mathcal{M}(C, C) = \mathcal{C}$, and define $\iota_\mathcal{D} : \Sigma\mathcal{D} \longrightarrow \Sigma\mathcal{M}$ similarly. According to the previous theorem, we can interpret the objects of the twisted centre $\mathcal{Z}^w_l(F, \mathcal{M}, G)$ as the colax natural transformations $\chi\colon \iota_\mathcal{C} \circ \Sigma F \Rightarrow \iota_\mathcal{D} \circ \Sigma G\colon \Sigma\mathcal{E} \longrightarrow \Sigma\mathcal{M}$.

Remark 3.1. An analogous statement to the previous proposition for right half-braidings can be obtained by considering lax instead of colax natural transformations between lax functors.

To obtain a bicategorical interpretation of the morphisms in a center category, we need to recall the definition of modifications.

Definition 3.4. A modification $a\colon \chi \Rrightarrow \psi$ between two colax natural transformations $\chi, \psi\colon \mathcal{F} \Rightarrow \mathcal{G}\colon \mathcal{K} \longrightarrow \mathcal{K}'$ consists of a family of 2-cells $a_A\colon \chi(A) \Rightarrow \psi(A)$, indexed by the objects $A \in \mathcal{O}b\,\mathcal{K}$, such that for every 1-cell $f \in \mathcal{K}(A, B)$ we have

$$
\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\chi_A"] \arrow[r, bend left=60, "\psi_A"] \arrow[r, phantom, bend left=30, "a_A \Uparrow"] \arrow[d, "F(f)"'] & \mathcal{G}(A) \arrow[d, "G(f)"] \\
\mathcal{F}(B) \arrow[r, "\chi_B"'] \arrow[ur, phantom, "\chi_f \nearrow"] & \mathcal{G}(B)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\psi_A"] \arrow[d, "F(f)"'] & \mathcal{G}(A) \arrow[d, "G(f)"] \\
\mathcal{F}(B) \arrow[r, "\psi_B"] \arrow[r, bend right=60, "\chi_B"'] \arrow[r, phantom, bend right=30, "a_B \Uparrow"] \arrow[ur, phantom, "\psi_f \nearrow"] & \mathcal{G}(B)
\end{tikzcd}
$$

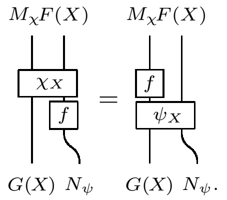
For two lax functors $\mathcal{F}, \mathcal{G}\colon \mathcal{B} \longrightarrow \mathcal{B}'$ among bicategories let $\mathrm{Colax}(\mathcal{F}, \mathcal{G})$ and $\mathrm{Lax}(\mathcal{F}, \mathcal{G})$ denote the categories of colax (respectively lax) natural transformations and their modifications. Similarly, $\mathrm{Pseudo}(F, G)$ denotes the category of pseudonatural transformations and their modifications.

Theorem 3.1. *Suppose $\mathcal{E}$ is a monoidal category and $\mathcal{F}, \mathcal{G}\colon \Sigma\mathcal{E} \longrightarrow \mathcal{K}$ are lax functors. There are canonical isomorphisms of categories:*

$$
\begin{aligned}
\mathcal{Z}^w_l(\mathcal{F}, \mathcal{K}, \mathcal{G}) &\cong \mathrm{Colax}(\mathcal{F}, \mathcal{G}), & \mathcal{Z}^s_l(\mathcal{F}, \mathcal{K}, \mathcal{G}) &\cong \mathrm{Pseudo}(\mathcal{F}, \mathcal{G}) \\
\mathcal{Z}^w_r(\mathcal{F}, \mathcal{K}, \mathcal{G}) &\cong \mathrm{Lax}(\mathcal{F}, \mathcal{G}), & \mathcal{Z}^s_r(\mathcal{F}, \mathcal{K}, \mathcal{G}) &\cong \mathrm{Pseudo}(\mathcal{F}, \mathcal{G}).
\end{aligned}
$$

Proof. We only prove the claim for weak left center categories as the other cases are analogous. Let $\chi, \psi\colon \mathcal{F} \Rightarrow \mathcal{G}$ be colax natural transformations and write $(M_\chi, \chi), (N_\psi, \psi) \in \mathcal{Z}^l_w(\mathcal{F}, \mathcal{K}, \mathcal{G})$ for their corresponding objects in the weak left center. Since $\Sigma\mathcal{E}$ is a delooping of a monoidal category, any modification $a\colon \chi \Rrightarrow \psi$

is defined by a single morphism $f\colon M_\chi \longrightarrow N_\psi$ satisfying for all $X \in \mathcal{E}$ the following identity:



This is precisely the defining equation of a morphism in the weak center and the claim follows. □

3.2. The bicategory of center categories. The bicategorical perspective provides a conceptual understanding why twisted centers can be 'composed' and in particular why 'F-F-twisted' center categories are monoidal. Consider the bicategories $\mathrm{Lax}_{clx}(\Sigma\mathcal{E},\mathcal{K})$ and $\mathrm{Lax}_{lx}(\Sigma\mathcal{E},\mathcal{K})$ of lax functors $\Sigma\mathcal{E} \longrightarrow \mathcal{K}$, (co)lax transformation, and their modifications, see [**19**, Theorem 4.4.11][2]. We may view their hom-categories as twisted centres. Under this identification, horizontal composition provides us with a suitable associative and unital way to 'combine' centers. In particular, endomorphism categories, that is, centeres which are twisted by the same functor from the left and the right, are monoidal.

We write $\mathcal{Z}_l^w(\mathcal{E},\mathcal{K}) := \{\mathcal{Z}_l^w(\mathcal{F},\mathcal{K},\mathcal{G}) \mid \mathcal{F},\mathcal{G}\colon \Sigma\mathcal{E} \longrightarrow \mathcal{K}\}$ for the collection of weak left centers of K relative to $\mathcal{E}$. To formulate our next result, we use the following notation: given two vertically composable 2-cells $x \overset{\alpha}{\Rightarrow} y \overset{\beta}{\Rightarrow} z\colon A \longrightarrow B$ in a 2-category $\mathcal{K}$, we denote their vertical composition in equations by the fraction $\frac{\alpha}{\beta}$.

THEOREM 3.2. *The collection $\mathcal{Z}_l^w(\mathcal{E},\mathcal{K})$ forms a 2-category. For lax functors $\mathcal{F},\mathcal{G},\mathcal{H}\colon \Sigma\mathcal{E} \longrightarrow \mathcal{K}$, the horizontal composition is*

$$\begin{aligned} \circ_{\mathcal{F},\mathcal{G},\mathcal{H}}\colon \mathcal{Z}_l^w(\mathcal{G},\mathcal{K},\mathcal{H}) \times \mathcal{Z}_l^w(\mathcal{F},\mathcal{K},\mathcal{G}) &\longrightarrow \mathcal{Z}_l^w(\mathcal{F},\mathcal{K},\mathcal{H}) \\ ((N,\tau),(M,\sigma)) &\mapsto \left(N \circ M, \frac{\mathrm{id}_N \circ \sigma}{\tau \circ \mathrm{id}_M}\right) \\ (g,f) &\mapsto (g \circ f) \end{aligned}$$

PROOF. The first claim is merely a recapitulation of the fact that we have $\mathcal{Z}_l^w(\mathcal{E},\mathcal{K}) \cong \mathrm{Lax}_{clx}(\Sigma\mathcal{E},\mathcal{K})$ and that $\mathrm{Lax}_{clx}(\Sigma\mathcal{E},\mathcal{K})$ is a bicategory. By Theorem 3.1, the objects and morphisms of any weak center are in correspondence with appropriate colax natural transformations and modifications. Applying this identification, one immediately obtains the above formulas from their respective composition rules. □

Analogous considerations hold for the bicategory of weak right centers, which is $\mathcal{Z}_r^w(\mathcal{E},\mathcal{K}) := \mathrm{Lax}_{lx}(\Sigma\mathcal{E},\mathcal{K})$.

The consequences of the above result are best exemplified by considering the bicategory $\mathcal{Z}_l^w(\mathcal{E},\mathcal{E})$ of all twisted weak left centers $\mathcal{Z}_l^w(F,\mathcal{E},G)$ of $\mathcal{E}$. The weak left analogue $\mathcal{Z}_l^w(\mathcal{E})$ of the Drinfel'd center corresponds to the endo-category on

[2]We note that the latter bicategory was denoted by $\mathrm{Bicat}(\mathcal{B},\mathcal{B}')$ by Bénabou.

the identity functor of $\mathcal{E}$. The previous result recovers its monoidal structure. Moreover, it implies that, if only the left or right action of $\mathcal{E}$ on itself is twisted, the resulting center canonically becomes a right, respectively, left module category over $\mathcal{Z}_l^w(\mathcal{E})$. This gives a theoretical justification to the constructions of [**18**] involving the anti-Drinfel'd center and its opposite.

Let us consider the pseudo-pseudo version of the bicategories $\mathrm{Lax}_{lx}(\mathcal{B},\mathcal{B}')$ and $\mathrm{Lax}_{clx}(\mathcal{B},\mathcal{B}')$. It is a bicategory that we denote by $\mathrm{Ps}_{ps}(\mathcal{B},\mathcal{B}')$ with hom-categories $\mathrm{Ps}(\mathcal{F},\mathcal{G})$ for pseudofunctors $\mathcal{F},\mathcal{G}\colon \mathcal{B}\longrightarrow\mathcal{B}'$. It is called $\mathrm{Hom}(\mathcal{B},\mathcal{B}')$ by Bénabou. In the particular case when $\mathcal{F}=\mathcal{G}=\mathrm{Id}_{\mathcal{B}}$ we have the center category $\mathcal{Z}(\mathcal{B})$ of the bicategory $\mathcal{B}$ introduced in [**34**].

3.3. The tricategory that encompasses strong center categories. We write Bicat for the tricategory of bicategories, pseudofunctors, pseudonatural transformations and modifications.

THEOREM 3.3. *Let* Bicat^* *be the full sub-tricategory of* Bicat *whose objects are bicategories with a single object. By delooping, its objects can be identified with monoidal categories, strong monoidal functors correspond to its 1-cells, and for each pair* $F,G\colon \mathcal{C}\longrightarrow\mathcal{D}$ *of such functors the strong center* $\mathcal{Z}_l^s(F,\mathcal{D},G)\cong\mathcal{Z}_r^s(F,\mathcal{D},G)$ *forms the hom-category.*

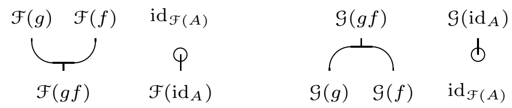
Note that, as explained in [**23**] and [**19**, Chapter 11.3], one cannot replace the 1-cells or 2-cells of Bicat with their (co)lax variants.

By further restricting the tricategory Bicat^* to finite tensor categories (in the sense of [**10**]), one has that every hom-bicategory $\mathrm{Bicat}^*(\mathcal{C},\mathcal{D})$ is precisely the bicategory $\mathrm{TF}(\mathcal{C},\mathcal{D})$ from [**41**, Section 3.2], of tensor functors between finite tensor categories $\mathcal{C}$ and $\mathcal{D}$. As the author works in a context of autonomous categories, he shows that for every hom-category $Z_l^w(F,G)$ of the bicategory $\mathrm{TF}(\mathcal{C},\mathcal{D})$ and every object $(V,\sigma_V)\in Z_l^w(F,G)$ the transformation σ_V is invertible ([**41**, Lemma 3.1]), and (V,σ_V) has a left and a right dual object in $Z_l^w(G,F)$. We will generalize this to 2-categories in Theorem 3.4 and Theorem 3.5.

The idea that half-braidings σ in presence of dual objects are invertible goes back to [**40**], where an analogous result was proved for σ being a monoidal natural transformation $F\Rightarrow G$ between strong monoidal functors F,G, see also [**48**]. The center of a monoidal category was generalized to the center of a monoidal object in a braided monoidal bicategory $\mathcal{B}$ in [**43**], and in [**27**, Theorem 9.5], [**28**] its weak version was considered. The result of [**29**, Corollary 5.6] that for a left autonomous monoidal object in $\mathcal{B}$ weak and strong center coincide, generalizes the case of our Theorem 3.4 when the pseudofunctors $\mathcal{F},\mathcal{G}$ are identities. In [**43**, Proposition 3.1] it was even shown that when the monoidal bicategory $\mathcal{B}$ is moreover left closed (thus also closed), the center of a monoidal object A in $\mathcal{B}$ can be realized as the bicategorical limit of a certain pseudo-cosimplicial diagram CA defined on A. This provides the center of a monoidal object in $\mathcal{B}$, and henceforth the monoidal center, with a Hochschild cohomological construction.

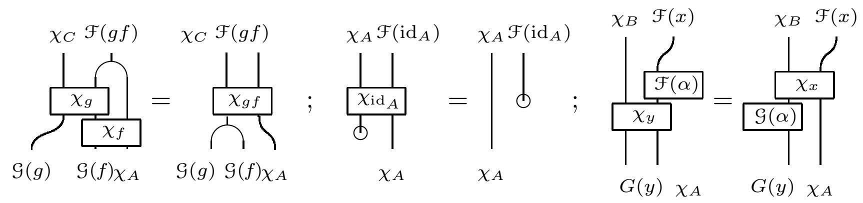
3.4. String diagrams in 2-categories. In Subsection 3.5 and throughout Section 4 and Section 5 we will use string diagrams for 2-categories (again relying on the biequivalence of any bicategory with a 2-category). Our string diagrams are read from top to bottom and (in the context of 2-categories) from right to left. The domains and codomains of the strings stand for 1-cells, while the strings themselves and boxes stand for 2-cells. The 0-cells are to be understood from the context (reading the 1-cells from right to left). Observe that such string diagrams which depict 2-cells in a 2-category $\mathcal{K}$ acting on the same underlying 0-cells $A \in \mathcal{O}b\,\mathcal{K}$ (that is, morphisms in the monoidal categories $\mathcal{K}(A, A)$ for every A) correspond exactly to the string diagrams in the monoidal categories $\mathcal{K}(A, A)$.

Let $\mathcal{F}$ be a lax functor and $\mathcal{G}$ a colax functor. We depict their lax, respectively colax structures by diagrams in the following way:



where g, f are composable 1-cells and A a 0-cell in the domain 2-category. We will often simplify the notation $\circ$ for the composition of 1-cells by concatenation.

Observe that a colax transformation between two colax functors can be interpreted as a distributive law between colax functor structures that is moreover natural in 1-cells. In string diagrams we may write this as follows:

(3.2)


for any 2-cell $\alpha\colon X \Rightarrow y\colon A \longrightarrow B$.

3.5. Adjoints in 2-categories. Dualisability plays a prominent role in the study of monoidal categories and the closely related subject of (extended) topological quantum field theories, see [**3**]. For example, Section 2 of [**9**] and Section 4 of [**18**] discuss and utilize 'duals' of bimodule categories and centers. In the following, we want to provide a 2-categorical perspective on these constructions, thereby giving a theoretical underpinning for some of the ad-hoc constructions of [**18**].

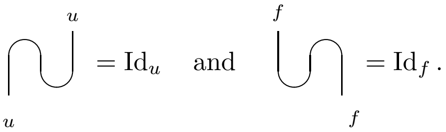
We start by briefly recalling the notion of adjoint 1-cells in 2-categories, see for example [**17, 24**]. Let $f\colon A \longrightarrow B$ be a 1-cell in a 2-category $\mathcal{K}$. A *left adjoint* of f is a 1-cell $u\colon B \longrightarrow A$ together with two 2-cells $\eta\colon \mathrm{id}_A \longrightarrow uf$ and $\varepsilon\colon fu \longrightarrow \mathrm{id}_B$ such that

$$\frac{\mathrm{Id}_f \circ \eta}{\varepsilon \circ \mathrm{Id}_f} = \mathrm{id}_f \quad \text{and} \quad \frac{\eta \circ \mathrm{Id}_u}{\mathrm{Id}_u \circ \varepsilon} = \mathrm{id}_u\,.$$

Similarly, a *right adjoint* of f is a 1-cell $v\colon B \longrightarrow A$ together with two 2-cells $\bar{\eta}\colon \mathrm{id}_B \longrightarrow fv$ and $\bar{\varepsilon}\colon vf \longrightarrow \mathrm{id}_A$ such that

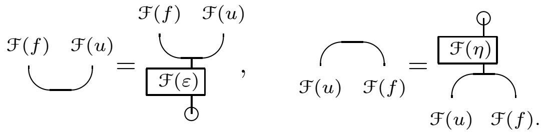
$$\frac{\mathrm{Id}_v \circ \bar{\eta}}{\bar{\varepsilon} \circ \mathrm{Id}_v} = \mathrm{id}_f \quad \text{and} \quad \frac{\bar{\eta} \circ \mathrm{Id}_f}{f \circ \bar{\varepsilon}} = \mathrm{id}_f\,.$$

In string diagrams we will write $\eta =$  and $\varepsilon =$  , and they satisfy the laws:



Pseudofunctors $\mathcal{F}\colon \mathcal{K} \longrightarrow \mathcal{K}'$ preserve (and reflect) adjoints and we have:

(3.3)



In the aforementioned [**18**], certain trace-like morphisms were considered in order to implement pivotal structures on the Drinfel'd centers. This involves a 'lift' of the notion of duals, i.e., adjoints, to the setting of centers of bimodule categories. With our interpretation of $\mathcal{Z}_l^w(\mathcal{E},\mathcal{K}) = \mathrm{Lax}_{clx}(\Sigma\mathcal{E},\mathcal{K})$ in Theorem 3.2 as a bicategory, for a monoidal category $\mathcal{E}$ and a bicategory $\mathcal{K}$, we can now derive a more conceptual version of this construction.

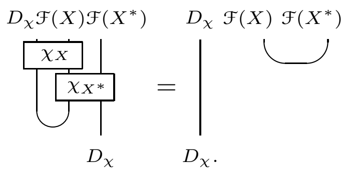
Definition 3.5. We refer to a 2-category $\mathcal{K}$ as *compact* if all 1-cells in $\mathcal{K}$ have left and right adjoints.

The most prominent example of a compact 2-category is given by the delooping $\Sigma\mathcal{E}$ of an autonomous monoidal category $\mathcal{E}$. We will prove at the end of this section that the full sub-bicategory $\mathcal{Z}_l^{w-ps}(\mathcal{E},\mathcal{K})$ whose objects are pseudofunctors is compact.

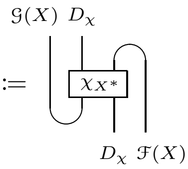
The following result is a 2-categorical interpretation of the fact that the half-braidings over autonomous monoidal categories are automatically invertible, see [**41**, Lemma 3.1].

Theorem 3.4. *Let $\mathcal{K}$ be a compact 2-category and $\mathcal{E}$ an autonomous monoidal category. For any pair of pseudofunctors $\mathcal{F},\mathcal{G}\colon \Sigma\mathcal{E} \longrightarrow \mathcal{K}$, the weak and strong centers coincide. Moreover, the inverse of a left half-braiding is a right half-braiding, that is: $\mathcal{Z}_l^{w-ps}(\mathcal{F},\mathcal{K},\mathcal{G}) \cong \mathcal{Z}_l^{s-ps}(\mathcal{F},\mathcal{K},\mathcal{G}) \cong \mathcal{Z}_r^{s-ps}(\mathcal{F},\mathcal{K},\mathcal{G}) \cong \mathcal{Z}_r^{w-ps}(\mathcal{F},\mathcal{K},\mathcal{G})$.*

Proof. For the first claim it suffices to show that the component 2-cells of any colax natural transformation between pseudofunctors $\chi\colon \mathcal{F} \Rightarrow \mathcal{G}$ are invertible. Hereto, we fix an object $X \in \mathcal{E}$ and write X^* for its left adjoint. Starting from the left-hand side below and applying the left equality in (3.3), the coherence of χ with the functor's multiplicativity, naturality of χ with respect to ε_X and the coherence of χ with respect to the functor's counitality we reach the equality with the right-hand side:



Then the 2-cell:

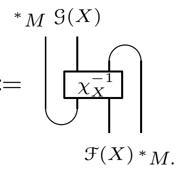
$\gamma_X :=$ 

is clearly a left inverse of $\chi_X : D_\chi \circ \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \circ D_\chi$. The analogous reasoning, but this time using the coherence of χ with the comultiplicativity and unitality of the functors, shows that γ_X is also a right inverse of χ_X. The last statement is proved directly. $\square$

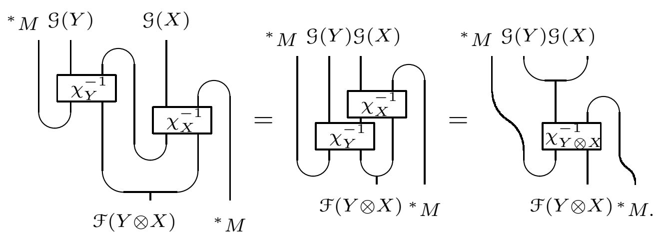
Our next result states that adjoints can be 'lifted' to weak centers.

THEOREM 3.5. *Suppose $\mathcal{E}$ to be an autonomous monoidal category and $\mathcal{K}$ to be a compact 2-category. Then the full sub-bicategory $\mathcal{Z}_l^{w\text{-}ps}(\mathcal{E},\mathcal{K}) \subset \mathcal{Z}_l^{w}(\mathcal{E},\mathcal{K})$ whose objects are pseudofunctors is compact.*

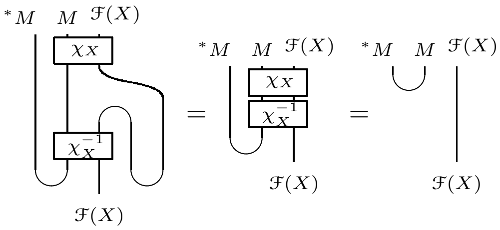
PROOF. Let$(M,\chi) \in \mathcal{Z}_l^{w\text{-}ps}(\mathcal{F},\mathcal{K},\mathcal{G})$ be a 1-cell of $\mathcal{Z}_l^{w\text{-}ps}(\mathcal{E},\mathcal{K})$. We show that it has a right adjoint living in $\mathcal{Z}_l^{w\text{-}ps}(\mathcal{G},\mathcal{K},\mathcal{F})$. The case of left adjoints is analogous. Due to Theorem 3.4, the half-braiding χ, i.e., the 2-cells $\chi_X : M \circ \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \circ M$, is invertible. We utilize this fact, to define a half-braiding on ${}^*M$ as shown in the next diagram

$\sigma_X :=$ 

A direct computation shows that it satisfies the Equations (2.1) and (2.2). For example, we have



To conclude the proof, we show that the unit $\eta\colon \mathrm{id} \longrightarrow M \circ {}^*M$ as well as the counit $\varepsilon\colon {}^*M \circ M \longrightarrow \mathrm{id}$ lift to morphisms in the center category $\mathcal{Z}_l^{w\text{-}ps}(\mathcal{G},\mathcal{K},\mathcal{F})$. For the counit this follows from the computation depicted below.



An analogous argument shows that the unit also becomes a morphism in the center. □

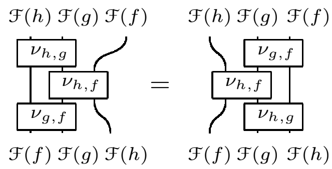
4. Bilax functors

We are interested in functors on bicategories that are both lax and colax but not necessarily pseudofunctors. Likewise, we are interested in natural transformations that are both lax and colax, but not necessarily pseudonatural transformations, as well as in their modifications. In particular, we introduce the notions of a bilax functors, bilax natural transformations and bilax modifications. We formulate them for 2-categories, just to avoid the use of associators and unitors, but the corresponding definitions for bicategories can be formulated in a straightforward fashion. To that end, we fix 2-categories $\mathcal{K}$ and $\mathcal{K}'$. In this section we introduce bilax functors, and leave the remaining two notions for the next section.

4.1. Bilax functors. We reiterate that we will often simplify the notation $\circ$ for the horizontal composition of 1- and 2-cells by concatenation.

Definition 4.1. Let $\mathcal{F}\colon \mathcal{K} \longrightarrow \mathcal{K}'$ be a 2-functor. A *Yang–Baxter operator* for $\mathcal{F}$ consists of a collection of 2-cells $\nu_{g,f}\colon \mathcal{F}(g)\mathcal{F}(f) \Rightarrow \mathcal{F}(f)\mathcal{F}(g)$ in $\mathcal{K}'$, natural in 1-endocells f, g of $\mathcal{K}$, which satisfy the Yang–Baxter equation

(4.1)



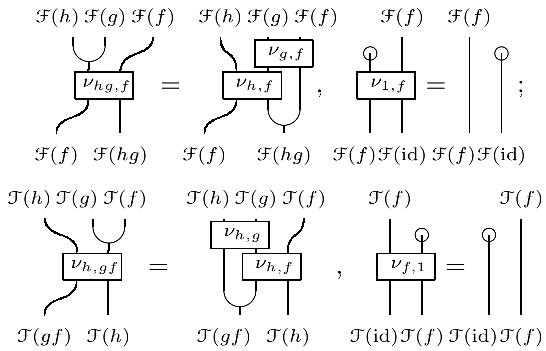
for all 1-endocells f, g, h.

By a *Yang–Baxter operator of* $\mathcal{K}$ we mean a Yang–Baxter operator c for the identity 2-functor $\mathrm{Id}\colon \mathcal{K} \longrightarrow \mathcal{K}$, which additionally satisfies $c_{\mathrm{id}_A,\mathrm{id}_A} = \mathrm{Id}_{\mathrm{id}_A}$ for all $A \in \mathcal{O}b\,\mathcal{K}$. We reserve the notation c for a Yang–Baxter operator on the identity 2-functor on $\mathcal{K}$.

For 2-categories $\mathcal{K} = \Sigma\mathcal{C}$ where $\mathcal{C}$ is a braided monoidal category, a class of Yang–Baxter operators c is given by the braiding(s) of $\mathcal{C}$.

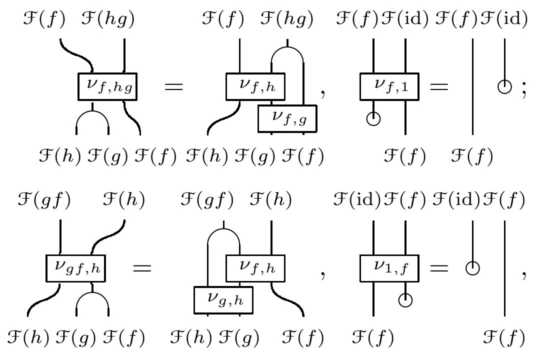
Definition 4.2. Assume that $\mathcal{K}$ possesses a Yang–Baxter operator c. A *bilax functor* is a pair $(\mathcal{F}, \nu)\colon (\mathcal{K}, c) \longrightarrow \mathcal{K}'$ where $\mathcal{F}\colon \mathcal{K} \longrightarrow \mathcal{K}'$ is simultaneously a lax and a colax functor and ν is a Yang–Baxter operator of $\mathcal{F}$, meaning that apart from rule (4.1) two additional groups of rules hold: left and right lax distributive laws

(4.2)



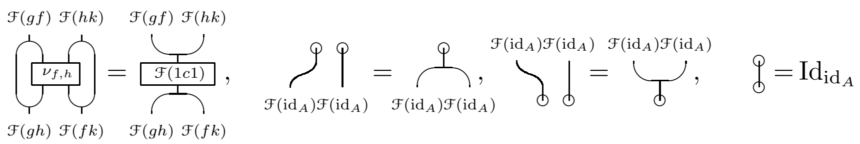
and left and right colax distributive laws

(4.3)



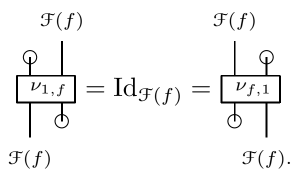
and additionally the *bilaxity condition*

(4.4)



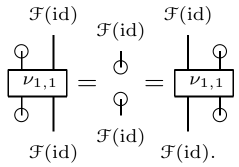
holds for 1-cells $A \xrightarrow{k} B \xrightarrow{h} B \xrightarrow{f} B \xrightarrow{g} C$.

Observe that the unit laws in (4.2) (or the counit laws in (4.3)) together with the fourth rule in (4.4) imply



We record for later use that the unit laws of (4.2) (or the counit laws of (4.3)) imply

(4.5)



The term 'distributive law' in Equations (4.2) and (4.3) is derived from [**12**, Definition 3.1]. Furthermore, monads and lax monoidal functors on monoidal categories can both be interpreted as monoids: the former are monoids in endofunctor categories, while the latter are monoids under Day convolution. In accordance with this suggestive similarity we use the term 'distributive law' also for pseudonatural transformations on lax/colax/bilax functors.

Let $(\mathcal{K}, c)$ and $(\mathcal{K}', d)$ be 2-categories with their respective Yang–Baxter operators, and assume that a bilax functor $(\mathcal{F}, \nu)$ is given acting between them. If $\nu_{f,g} = d_{\mathcal{F}(f),\mathcal{F}(g)}$ for all composable 1-endocells f, g in $\mathcal{K}$, we will say that $\mathcal{F}$ is a bilax functor with a *compatible Yang–Baxter operator*, and we will simply write $\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}', d)$. In the case that the relation between ν and d is not known, we will write $(\mathcal{F}, \nu)\colon (\mathcal{K}, c) \longrightarrow \mathcal{K}'$, as in Definition 4.2.

Given a bilax functor $(\mathcal{F}, \nu)$ the functors on endo-hom-categories

$$\mathcal{F}_{A,A}\colon \mathcal{K}(A, A) \longrightarrow \mathcal{K}'(\mathcal{F}(A), \mathcal{F}(A)) \tag{4.6}$$

are pre-bimonoidal in the sense of [**33**, Section 2], where we rely on the strictification theorem for monoidal categories. If instead of the Yang–Baxter operators one works with braidings on the endo-hom categories, then the functors $\mathcal{F}_{A,A}$ are bilax as in [**1**, Section 19.9.1]. This inspired the terminology in Definition 4.2. While the analogues of the coherence conditions (4.2) and (4.3) do not appear explicitly in [**33**, Section 2], we incorporate them in accordance with the discussion in [**1**, Section 19.9.1] in our definition. Accordingly, we recover a 'braided bialgebra' from [**45**, Definition 5.1], as we will show further below. In the situation $\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}', d)$ (i.e., that ν is compatible with d), the functors $\mathcal{F}_{A,A}$ are bimonoidal in the sense of [**33**, Section 2]. The other way around, we clearly have:

Example 4.1. Let $(\mathcal{C}, \Phi_{\mathcal{C}})$ and $(\mathcal{D}, \Phi_{\mathcal{D}})$ be braided monoidal categories. We identify their braidings with Yang–Baxter operators c and d on their delooping categories $\Sigma\mathcal{C}$ and $\Sigma\mathcal{D}$, respectively. Any bilax functor $\mathcal{F}\colon (\Sigma\mathcal{C}, c) \longrightarrow (\Sigma\mathcal{C}, d)$ with a compatible Yang–Baxter operator is a bimonoidal functor in the sense of [**33**, Section 2].

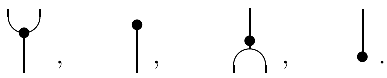
Example 4.2. Let $(\mathcal{C}, \Phi)$ be a braided monoidal category and 1 the trivial 2-category (it has a single 0-cell and only identity higher cells). Any bilax functor $\mathcal{F}\colon 1 \longrightarrow \Sigma\mathcal{C}$ with invertible ν which coincides with Φ can be identified with a bialgebra in $\mathcal{C}$. To see this, note that $\mathcal{F}$ determines and is determined by a 1-cell in $\Sigma\mathcal{C}$, i.e., an object B of $\mathcal{C}$, and a (co)multiplication and (co)unit on that 1-cell, which are subject to (co)associativity and (co)unitality due to $\mathcal{F}$ being lax and colax. On the other hand, c of the trivial 2-category is trivial: it is the identity 2-cell on the identity 1-cell on $*$. Then the equations (4.4) recover bialgebra axioms on $\mathcal{F}(\mathrm{id}_*)$. (The rest of axioms of the bilax functor are automatically fulfilled by the braiding, and do not contribute any additional information.)

If moreover $\mathcal{F}$ is a pseudofunctor, then it is trivial: the obtained bialgebra is isomorphic to the monoidal unit of $\mathcal{C}$. This can be proved considering the lax unitality structure of $\mathcal{F}$.

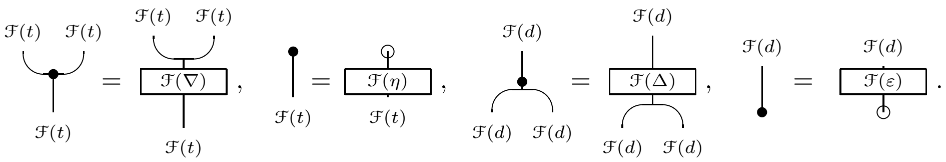
The following result is straightforwardly proved, see also [**33**, Proposition 3.9].

Lemma 4.1. *Let $\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}', d)$ and $\mathcal{G}\colon (\mathcal{K}', d) \longrightarrow (\mathcal{K}'', e)$ be compatible bilax functors. Then $\mathcal{G}\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}'', e)$ is a compatible bilax functor with a Yang–Baxter operator $\nu_{g,f} := \nu^{\mathcal{G}}_{\mathcal{F}(g),\mathcal{F}(f)}$, where $\nu^{\mathcal{G}}$ is a Yang–Baxter operator of $\mathcal{G}$ and g, f are 1-endocells of $\mathcal{K}$.*

4.2. Bimonads. The theory of monads and comonads in the context of 2-categories was introduced by Street in [**42**]. Recall that a monad in $\mathcal{K}$, is a 1-endocell $t \in \mathcal{K}(A, A)$ endowed with 2-cells $\mu\colon tt \longrightarrow t$ and $\eta\colon \mathrm{id}_A \longrightarrow t$ subject to associativity and unitality conditions. Dually, that is, swapping the direction of the structure 2-cells, one obtains the notion of a comonad in $\mathcal{K}$: a 1-endocell endowed with a coassociative and counital comultiplication. In order to differentiate notations for (co)lax functors and (co)monad structures, we will represent the multiplication and unit of a monad as well as the comultiplication and counit of a comonad by:



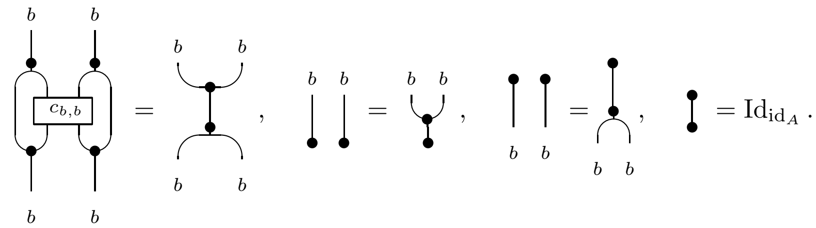
One shows in a straightforward manner that lax functors preserve monads and colax functors preserve comonads. Specifically, for a monad t in $\mathcal{K}$ and a lax functor $\mathcal{F}\colon \mathcal{K} \longrightarrow \mathcal{K}'$, and a comonad d in $\mathcal{K}$ and a colax functor $\mathcal{G}$ we have the following monad and comonad structures:



We are going to introduce bimonads in 2-categories with respect to Yang–Baxter operators. Observe that their 1-categorical analogue is different from the bimonads of [**35**] and [**36**] in ordinary categories, but they are a particular instance of τ-bimonads and bimonads in 2-categories from [**14**]. Whereas bimonads in [**36**] are opmonoidal monads on monoidal categories, bimonads in [**35**] are monads and comonads on a not necessarily monoidal category with compatibility conditions that involve a distributive law λ. The bimonads in [**14**, Definition 7.1] generalize the latter to 2-categories and are equipped with an analogous 2-cell i.e., distributive law λ. The τ-bimonads that we also introduced in [**14**], are a particular case of bimonads, where the 2-cell λ is given in terms of a 2-cell τ which is a distributive law both on the left and on the right, both with respect to monads and comonads.

In the special case of a Yang–Baxter operator coming from a braiding one gets examples à la [**35**].

Definition 4.3. Let $(\mathcal{K}, c)$ be a 2-category with a Yang–Baxter operator c and suppose that $b \in \mathcal{K}$ is a 1-endocell endowed with a monad and comonad structure. We call b a *c-bimonad* if $c_{b,b}$ is a distributive law both on the left and on the right, both with respect to monads and comonads, and the following compatibilities hold:

(4.7) 

Obviously, for any object $A \in (\mathcal{K}, c)$ one has that id_A is a c-bimonad. The following observation is inspired by Bénabou:

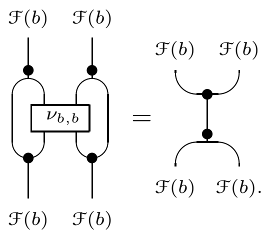
Lemma 4.2. *There exists a bijection between c-bimonads in $(\mathcal{K}, c)$ and compatible bilax functors $1 \longrightarrow (\mathcal{K}, c)$. It is given by mapping any c-bimonad $b\colon A \longrightarrow A$ in $(\mathcal{K}, c)$ to the bilax functor $\mathcal{T}\colon 1 \longrightarrow \mathcal{K}$ with $\mathcal{T}(\mathrm{id}_*) = b$, whose bilax structure 2-cells agree with the respective 2-cells of the c-bimonad b.*

Proof. As observed by Bénabou, a lax functor from 1 to $\mathcal{K}$ defines and is defined by a monad in $\mathcal{K}$. Likewise, the colaxity of $\mathcal{T}\colon 1 \longrightarrow \mathcal{K}$ is equivalent to $\mathcal{T}(\mathrm{id}_*)$ being a comonad. Equations(4.2) and (4.3) correspond to the four distributive laws of $c_{b,b}$ with $b = \mathcal{T}(\mathrm{id}_*)$ and the bilaxity conditions given in (4.4) correspond to conditions (4.7). Such a bilax functor is also a τ-bimonad in the sense of [**14**, Definition 7.1] (it is a monad and a comonad with a monad-morphic and a comonad-morphic distributive law on both sides, in the sense of (4.2) and (4.3)). $\square$

Similarly, we clearly have:

Lemma 4.3. *Any bilax functor $(\mathcal{T}, \nu)\colon 1 \longrightarrow \mathcal{K}$ determines a ν-bimonad $\mathcal{T}(\mathrm{id}_*) = b$.*

Theorem 4.1. *Let $(\mathcal{F}, \nu)\colon (\mathcal{K}, c) \longrightarrow \mathcal{K}'$ be a bilax functor and $b \in \mathcal{K}$ a c-bimonad. Then $\mathcal{F}(b)$ is a ν-bimonad in $\mathcal{K}'$. That is, it satisfies the last three axioms of* (4.7) *and a variant of the first axiom shown below:*



Moreover, $\mathcal{F}(b)$ is a τ-bimonad in the sense of [**14**].
Similarly, for $\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}', d)$ a compatible bilax functor $\mathcal{F}(b)$ is a d-bimonad.

PROOF. The claim follows by the naturality of the (co)lax structure of $\mathcal{F}$ with respect to the 2-cells from the (co)monad structures of b. (For the three compatibilities of the (co)unit structures for $\mathcal{F}(b)$, alternatively, apply $\mathcal{F}(\eta_b)$ and $\mathcal{F}(\varepsilon_b)$ at suitable places in the last three equations in (4.4) and use the naturality of the (co)lax structure of $\mathcal{F}$.) The penultimate statement follows from (4.2) and (4.3). $\square$

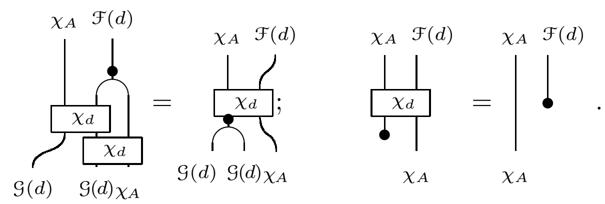
EXAMPLE 4.3. Given a bilax functor $(\mathcal{F},\nu)\colon (\mathcal{K},c) \longrightarrow \mathcal{K}'$, or a compatible bilax functor $\mathcal{F}\colon (\mathcal{K},c) \longrightarrow (\mathcal{K}',d)$, we have for all $A \in \mathcal{O}b\,\mathcal{K}$ that $\mathcal{F}(\mathrm{id}_A)$ is a ν-bimonad, respectively d-bimonad in $\mathcal{K}'$.

Given that the notion of a Yang–Baxter operator is more general than that of a braiding, a ν-bimonad generalizes the notion of a bialgebra. Moreover, we have that the notion of a bilax functor $\mathcal{F}\colon \mathcal{K} \longrightarrow \mathcal{K}'$ recovers that of a 'braided bialgebra' from [**45**, Definition 5.1], in the case $\mathcal{K} = \Sigma\mathcal{C}$, where $\mathcal{C}$ is a monoidal category with a single object. (Another direction of generalization of the notion of a braided bialgebra of Takeuchi can be found in [**2**] in the notion of a weak braided bialgebra in a monoidal category.)

EXAMPLE 4.4. Let $(\mathcal{K},c)$ and $(\mathcal{K}',d)$ be two 2-categories with their respective Yang–Baxter operators, and let $b\colon A_0 \longrightarrow A_0$ be a d-bimonad in $\mathcal{K}'$. We define $\mathcal{F}_b(A) = A_0, \mathcal{F}_b(x) = b, \mathcal{F}_b(\alpha) = \mathrm{Id}_b$ for all objects A, 1-cells x and 2-cells α in $\mathcal{K}$. Then $\mathcal{F}_b$ is a bilax functor with $\nu_{f,g} = d_{b,b}$ for all 1-endocells f,g in $\mathcal{K}$. (Alternatively, instead of the Yang–Baxter operator d in $\mathcal{K}'$ one could require a Yang–Baxter operator ν on $\mathcal{F}_b$, take a ν-bimonad b and obtain a bilax functor $(\mathcal{F}_b,\nu)$.)

The following claim is directly proved:

LEMMA 4.4. *For a comonad $d\colon A \longrightarrow A$ and a colax transformation between two colax functors $\chi\colon \mathcal{F} \Rightarrow \mathcal{G}$ it holds:*



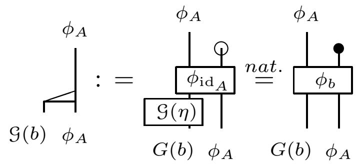
Monads and comonads in $\mathcal{K}$ can act on other 1-cells of $\mathcal{K}$. That is, for example a left module over a bimonad $b\colon A \longrightarrow A$ is a 1-cell $x\colon A' \longrightarrow A$ endowed with a left action $\triangleright\colon bx \Rightarrow x$ of b (in [**13**, Definition 2.3] the axioms are expressed in string diagrams). Note that in category theory, what we call a b-(co)module is also referred to as a b-(co)algebra, see for example [**25**].

THEOREM 4.2. *Let $\mathcal{F},\mathcal{G}$ be two bilax functors and $b\colon A \longrightarrow A$ a c-bimonad in $\mathcal{K}$ (indeed, a monad and comonad satisfying the two last identities in* (4.7)*).*

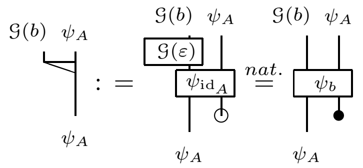
For a colax transformation $\phi\colon \mathcal{F} \Rightarrow \mathcal{G}$ of colax functors, (4.8) *defines a left $\mathcal{G}(b)$-comodule structure on $\phi(A)$.*

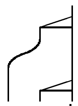
Dually, for a lax transformation $\psi\colon \mathcal{F} \Rightarrow \mathcal{G}$ of lax functors, (4.9) *defines a left $\mathcal{G}(b)$-module structure on $\psi(A)$.*

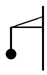
(4.8)



(4.9)



Proof. We indicate the steps of the proof for the comodule structure. Starting from  apply first the second rule in (4.4), then the first rule in (3.2), naturality of  , naturality of χ, third rule in (4.7) for b, and lastly naturality of χ.

For the counitality, starting from  apply first the fourth rule in (4.7), the second rule in (3.2), and lastly the fourth rule in (4.4). □

For $\mathcal{F} = \mathcal{G}$ acting on the trivial 2-category $\mathcal{K} = 1$, and ϕ and ψ being endo-transformations, Theorem 4.2 recovers [**13**, Proposition 2.4], proved for 2-(co)monads and their distributive laws. The latter is a 2-categorical formulation for a fact possibly used in (braided) monoidal categories by different authors, but we are not aware of an exact reference.

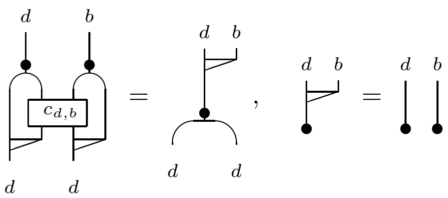
It is important to note that modules/comodules over a c-bimonad in $\mathcal{K}$ do not form a monoidal category (unless the Yang–Baxter operator c is a half-braiding).

4.3. Module comonads, comodule monads and relative bimonad modules. In [**25**] the notion of a wreath was introduced as monad in the free completion 2-category $\mathrm{EM}^M(\mathcal{K})$ of the 2-category $\mathrm{Mnd}(\mathcal{K})$ of monads in $\mathcal{K}$ under the Eilenberg–Moore construction. Dually, cowreaths are comonads in $\mathrm{EM}^C(\mathcal{K})$, where the latter is the analogous completion of the 2-category $\mathrm{Comnd}(\mathcal{K})$ of comonads. In [**13**] the first author introduced the 2-category $\mathrm{bEM}(\mathcal{K})$ so that there are forgetful 2-functors $\mathrm{bEM}(\mathcal{K}) \to \mathrm{EM}^M(\mathcal{K})$ and $\mathrm{bEM}(\mathcal{K}) \to \mathrm{EM}^C(\mathcal{K})$. Moreover, in *loc. cit.* biwreaths were introduced as bimonads in $\mathrm{bEM}(\mathcal{K})$. Biwreaths as a notion integrate both wreaths and cowreaths as well as their mixed versions: mixed wreaths and mixed cowreaths. In particular, a biwreath also behaves like a '(co)module (co)monad' with respect to monad-morphic or comonad-morphic distributive law in $\mathcal{K}$, where the highlighted notions in the 2-categorical setting were introduced in [**13**].

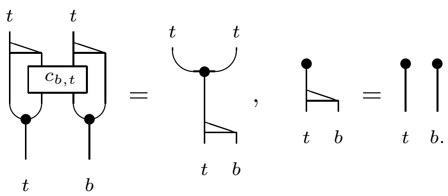
In the present paper, similarly to the above-mentioned idea (see diagrams (67) and (65) of *loc. cit.*), but now with respect to Yang–Baxter operators, we will consider the notions that we introduce in the definition below.

For the sake of examples that we will study further below, we record that in [**13**] the 2-category Bimnd($\mathcal{K}$) of bimonads in $\mathcal{K}$ (with respect to distributive laws) was defined, so that there are inclusion and projection 2-functors $E_B\colon$ Bimnd($\mathcal{K}$) $\to$ bEM($\mathcal{K}$) and $\pi\colon$ bEM($\mathcal{K}$) $\to$ Bimnd($\mathcal{K}$) which are identities on 0- and 1-cells. In [**14**] we have considered a variation of the 2-category Bimnd($\mathcal{K}$) We will recall it in Example 5.1.

Definition 4.4. Let $b\colon A \to A$ be a c-bimonad in a 2-category $(\mathcal{K}, c)$ with a Yang–Baxter operator. Let $d\colon A \to A$ be a comonad and a right b-module, and $t\colon A \to A$ a monad and a right b-module. We say that d is a (right) *module comonad* if the two equations

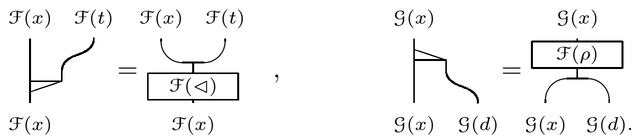


hold. We say that t is a (right) *comodule monad* if the two equations below hold:



The left-hand side versions of these notions can be clearly deduced.

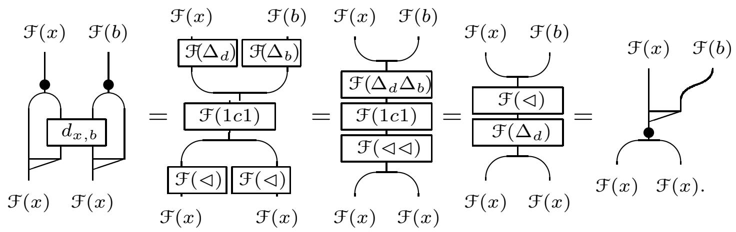
We continue with some simple yoga of (co)lax and bilax functors. Consider 1-cells: $A \underset{d}{\overset{t}{\rightrightarrows}} A \xrightarrow{x} B$ in $\mathcal{K}$, where t is a monad, d is a comonad and x is a right t-module (via a 2-cell $\triangleleft$) and a right d comodule (via a 2-cell ρ). Recall that, as we have already used before, lax functors preserve monads and colax functors preserve comonads. Given a lax functor $\mathcal{F}$ and a colax functor $\mathcal{G}$ one has that $\mathcal{F}(x)$ is a right $\mathcal{F}(t)$-module and a right $\mathcal{G}(d)$-comodule with structure 2-cells:

(4.10) 

The analogous claims hold on left sides.

Theorem 4.3. *Bilax functors $\mathcal{F}\colon (\mathcal{K}, c) \to (\mathcal{K}', d)$ with compatible Yang–Baxter operators preserve module comonads and comodule monads.*

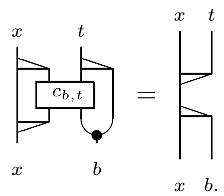
PROOF. The arguments for showing that $\mathcal{F}$ preserves comodule monads are analogous to proving that it maps module comonads to module comonads. Therefore, we will restrict ourselves to the latter. Let $b\colon A \longrightarrow A$ be a bimonad in $(\mathcal{K}, c)$ and $x\colon A \longrightarrow A$ a b-module comonad. The bilaxity of $\mathcal{F}$ implies that $\mathcal{F}(b)$ is a d-bimonad with an action on the comonad $\mathcal{F}(x)$. Hence, we only have to show that the first two compatibility conditions in Definition 4.4 hold. Using the functoriality of $\mathcal{F}$ we have:



The compatibility of the action of $\mathcal{F}(b)$ with the counit of $\mathcal{F}(x)$ is an immediate consequence of $\mathcal{F}$ being lax. $\square$

For the next property we introduce the following notion:

DEFINITION 4.5. Let $b\colon A \longrightarrow A$ be a c-bimonad in a 2-category $(\mathcal{K}, c)$ with a Yang–Baxter operator and let $t\colon A \longrightarrow A$ be a right b-comodule monad. A right t-module and a right b-comodule $x\colon A \longrightarrow B$ is a right *relative t-b-module* if the following relation holds:



Morphisms of right relative t-b-modules are right t-linear and right b-colinear 2-cells in $\mathcal{K}$. Analogously, one defines a left *relative t-b-module.* If $t = b$ we call x a *relative bimonad module.*

The above notions correspond to those of relative Hopf modules [**7**] and Hopf modules [**30**] in braided monoidal categories, which in turn are categorifications of Hopf modules introduced in [**26**]. Obviously, b itself is a relative bimonad module.

EXAMPLE 4.5. Let $\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}', d)$ be a bilax functor with compatible Yang–Baxter operator. For any 1-cell $x\colon A \longrightarrow B$ the 1-cell $\mathcal{F}(x)$ is a left relative bimonad module over $\mathcal{F}(\mathrm{id}_B)$ and a right relative bimonad module over $\mathcal{F}(\mathrm{id}_A)$. This follows from the first equation in (4.4) by setting $\mathrm{id}_B, \mathrm{id}_B, \mathrm{id}_B, x$, respectively $x, \mathrm{id}_A, \mathrm{id}_A, \mathrm{id}_A$ for the 1-cells g, f, h, k.

Analogously to Theorem 4.3 we get the following result:

THEOREM 4.4. *Bilax functors $\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}', d)$ with compatible Yang–Baxter operators preserve relative bimonad modules.*

Hopf bimodules, for Hopf algebras over a field, appeared in the construction of bicovariant differential calculi over a Hopf algebra in [**47**]. They were generalized in [**5**, Section 4.2] to the context of a braided monoidal category $\mathcal{C}$. For a bialgebra B in $\mathcal{C}$ a Hopf bimodule is a B-bimodule M in $\mathcal{C}$ which is moreover a B-bicomodule in the monoidal category of B-bimodules ${}_B\mathcal{C}_B$ (for the structures on B itself the regular (co)action on B and the diagonal action on tensor product of comodules are used). This means that both left and right B-comodule structures of M are left and right B-bimodule morphisms, meaning that there are four conditions to be fulfilled. Together with simultaneously B-linear and B-bicolinear morphisms Hopf bimodules make a category denoted by ${}^B_B\mathcal{C}^B_B$. We mark that the name 'Hopf bimodules' is somewhat misleading, as the Hopf structure on the bialgebra B is not necessary here.

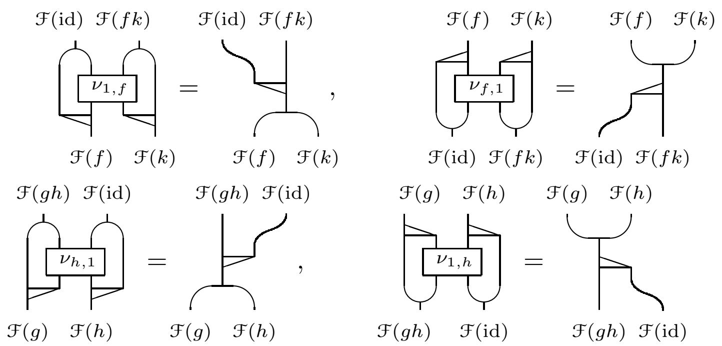
Substituting a braided monoidal category $\mathcal{C}$ and a bialgebra B in it with a monoidal category with a Yang–Baxter operator c and a c-bimonad in it, we can consider the analogous category of Hopf bimodules ${}^B_B(\mathcal{C},c)^B_B$, where $\mathcal{C}$ and B now have these new meanings.

Corollary 4.1. *Let $\mathcal{F}\colon (\mathcal{K},c) \longrightarrow (\mathcal{K}',d)$ be a bilax functor with compatible Yang–Baxter operator. Functors* (4.6) *for every $A \in \mathcal{O}b\,\mathcal{K}$ factor through the category of Hopf bimodules over the d-bimonads $\mathcal{F}(\mathrm{id}_A)$ in $(\mathcal{K}'(\mathcal{F}(A),\mathcal{F}(A)),d)$.*

Proof. For any 1-endocell $x\colon A \longrightarrow A$ we should check if $\mathcal{F}(x)$ satisfies the four relations. Two of them are satisfied by Example 4.5, which means that the left coaction is left linear and that the right coaction is right linear. The other two, meaning the two mixed versions of compatibilities, one gets by setting $x = f$, respectively $x = h$, and the resting three 1-cells to be identities, in the first equation in (4.4).

To check the claim on morphisms, observe that for any 2-cell $\alpha\colon X \longrightarrow y$ in $\mathcal{K}$, i.e., morphism in $\mathcal{K}(A,A)$, $\mathcal{F}(\alpha)$ is $\mathcal{F}(\mathrm{id}_A)$-(co)linear by the naturality of the (co)lax structure of $\mathcal{F}$. $\square$

We record here some direct consequences for a bilax functor $(\mathcal{F},\nu)$ from the first equation in (4.4). For simplicity, we may consider $\mathcal{F}\colon (\mathcal{K},c) \longrightarrow (\mathcal{K}',d)$ to have a compatible Yang–Baxter operator. By (4.10) note that $\mathcal{F}(x)$ is a bi(co)module over $\mathcal{F}(\mathrm{id})$ for any 1-cell x (acting among suitable 0-cells). Then we may write:



for 1-cells $A \xrightarrow{k} B \xrightarrow{h} B \xrightarrow{f} B \xrightarrow{g} C$.

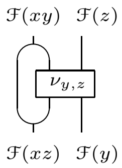
5. 2-category of bilax functors

In this section we introduce the rest of the ingredients to construct a 2-category of bilax functors.

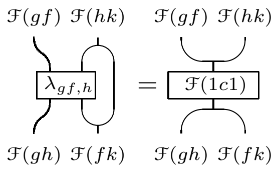
5.1. Bilax natural transformations. Among bilax functors we introduce bilax natural transformations. Recall that for a lax transformation ψ and a colax transformation ϕ, both acting between (bilax) functors $\mathcal{F} \Rightarrow \mathcal{F}'$, for every 1-cell $f\colon A \longrightarrow B$ there are 2-cells

$$\psi_f\colon \mathcal{F}'_{A,B}(f) \circ \psi(A) \Rightarrow \psi(B) \circ \mathcal{F}_{A,B}(f),$$
$$\phi_f\colon \phi(B) \circ \mathcal{F}_{A,B}(f) \Rightarrow \mathcal{F}'_{A,B}(f) \circ \phi(A)$$

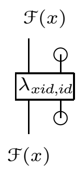
natural in f. For the sake of the following definition we introduce the notation:

$\lambda_{xy,z} \coloneqq$ 

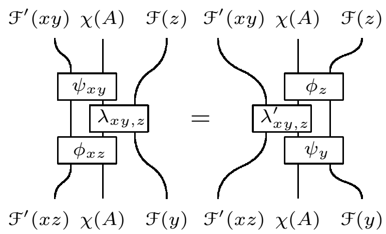
for a bilax functor $\mathcal{F}$ and 1-cells $A \xrightarrow{z} A \xrightarrow{y} A \xrightarrow{x} B$. Then the first equation in (4.4) can be expressed also as:



Observe that by (4.5) and the rules of the (co)lax structures of $\mathcal{F}$, one has:

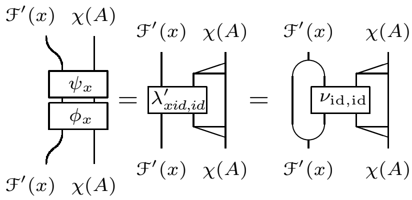
(5.1)  $= \mathrm{Id}_{\mathcal{F}(x)}\,.$

Definition 5.1. A bilax natural transformation $\chi\colon \mathcal{F} \Rightarrow \mathcal{F}'$ between bilax functors is a pair (ψ, ϕ) consisting of a lax natural transformation ψ of lax functors and a colax natural transformation ϕ of colax functors, which agree on the 1-cell components, i.e., $\psi(A) = \phi(A) \coloneqq \chi(A)$ for every $A \in \mathcal{O}b\,\mathcal{K}$, and whose 2-cell components are related through the relation:

(5.2) 

for composable 1-cells: $A \xrightarrow{z} A \xrightarrow{y} A \xrightarrow{x} B$. We will denote it shortly as a triple (χ, ψ, ϕ).

In particular, if $y = z = \mathrm{id}_A$ and one applies the unity of the lax structure of $\mathcal{F}$ on the top right in (5.2), and the counity of the colax structure of $\mathcal{F}$ on the bottom right, one obtains (by (5.1) and Theorem 4.2):

(5.3) 

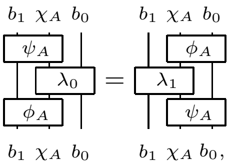
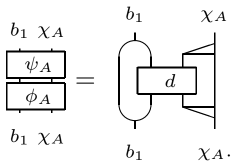
which will be called the *Yetter–Drinfel'd condition on the bilax natural transformation* χ. Observe that the left module and comodule structures of $\chi(A)$ above are over $\mathcal{F}'(\mathrm{id}_A)$.

Example 5.1. Let $\mathcal{K}$ and $\mathcal{K}'$ be 2-categories with Yang–Baxter operators c and d, respectively. We fix two d-bimonads $b_0 \colon A_0 \longrightarrow A_0$ and $b_1 \colon A_1 \longrightarrow A_1$ in $\mathcal{K}'$ and define bilax functors $\mathcal{F}_0, \mathcal{F}_1 \colon \mathcal{K} \longrightarrow \mathcal{K}'$ by $\mathcal{F}_0 \coloneqq \mathcal{F}_{b_0}$ and $\mathcal{F}_1 \coloneqq \mathcal{F}_{b_1}$ with $\nu_i = d$ for $i = 1, 2$, as in Example 4.4.

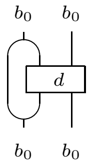
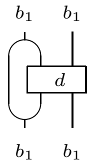
A bilax natural transformation (χ, ψ, ϕ) between bilax functors $\chi \colon \mathcal{F}_0 \Rightarrow \mathcal{F}_1$ consists of a family of 2-cells

$$\psi_A \colon b_1\chi(A) \Rightarrow \chi(A)b_0 \quad \text{and} \quad \phi_A \colon \chi(A)b_0 \longrightarrow b_1\chi(A),$$

indexed by $A \in \mathcal{O}b\,\mathcal{K}$, where ψ_A (respectively ϕ_A) are distributive laws with respect to the monad (resp. comonad) structures of b_0, b_1 (by (3.2) and its vertical dual), and it holds:

(5.4)  and consequently 

Note that λ_0 and λ_1 have the form:

$\lambda_0 =$  and $\lambda_1 =$ 

and that the third equation in (3.2) is now trivial. For every $A \in \mathcal{O}b\,\mathcal{K}$, the triple $(\chi(A), \psi_A, \phi_A)$ is a 1-cell in the 2-category $\mathrm{Bimnd}(\mathcal{K}')$ from [**14**, Section 7], which we mentioned at the beginning of Subsection 4.3. The 0-cells of $\mathrm{Bimnd}(\mathcal{K})$ are bimonads in $\mathcal{K}$ defined via a distributive law 2-cell λ, 1-cells are triples (F, ψ, ϕ) where (F, ψ) is a 1-cell in $\mathrm{Mnd}(\mathcal{K})$ and (F, ϕ) is a 1-cell in $\mathrm{Comnd}(\mathcal{K})$ with a compatibility condition between ψ, ϕ and λ (as in (5.4) on the left), and a 2-cell is a single 2-cell ζ in $\mathcal{K}$ which is simultaneously a 2-cell in $\mathrm{Mnd}(\mathcal{K})$ and in $\mathrm{Comnd}(\mathcal{K})$.

EXAMPLE 5.2. If $\mathcal{K} = \Sigma\mathcal{C}$ in the above Example is induced by a braided monoidal category $\mathcal{C}$, the above bilax natural transformation $\chi\colon \mathcal{F}_0 \Rightarrow \mathcal{F}_1\colon \Sigma\mathcal{C} \longrightarrow \mathcal{K}'$ is precisely a single 1-cell in $\mathrm{Bimnd}(\Sigma\mathcal{C})$.

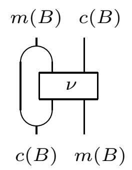
EXAMPLE 5.3. By Lemma 4.3 actually *any* two bilax functors $(\mathcal{T}_0, \nu), (\mathcal{T}_1, \nu')\colon 1 \longrightarrow \mathcal{K}$ determine two bimonads in $\mathcal{K}$: $\mathcal{T}_0$ yields a ν-bimonad $b_0 := \mathcal{T}(\mathrm{id}_*)$ on $A = \mathcal{T}(*)$ and $\mathcal{T}_1$ a ν'-bimonad $b_1 := \mathcal{T}_1(\mathrm{id}_*)$ on $A' = \mathcal{T}'(*)$. Then analogously as in Example 5.2, any bilax natural transformation $\chi\colon \mathcal{T}_0 \Rightarrow \mathcal{T}_1\colon 1 \longrightarrow \mathcal{K}'$ is precisely a single 1-cell in $\mathrm{Bimnd}(\mathcal{K})$.

EXAMPLE 5.4. Consider a bilax transformation $\chi\colon (\mathcal{T}, \nu) \Rightarrow (\mathcal{T}', \nu')\colon 1 \longrightarrow \mathcal{K}$ from the trivial 2-category to a 2-category $\mathcal{K}$. Let $B := \mathcal{T}(\mathrm{id}_*)$ be the ν-bimonad on $\mathcal{A} = \mathcal{T}(*)$ and $B' := \mathcal{T}'(\mathrm{id}_*)$ the ν'-bimonad on $\mathcal{T}'(*)$ as in Lemma 4.3. Let $m(B)$ denote the monad part of B and $c(B)$ the comonad part of B, and set $\chi(*) = X$. We find that $\psi\colon m(B')X \Rightarrow Xm(B)$ is a distributive law with respect to monads, and that $\phi\colon Xc(B) \Rightarrow c(B')X$ is a distributive law with respect to comonads.

It is a nice exercise to prove the following lemma that we will use to pursue with this example.

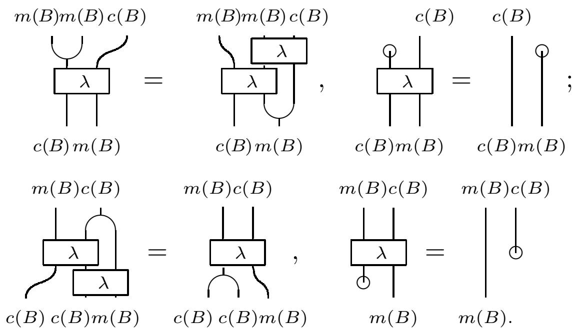
LEMMA 5.1. *Let B be both a monad (to which we refer to as $m(B)$) and a comonad (we refer to it as $c(B)$) and suppose that $\nu\colon BB \Rightarrow BB$ is a distributive law, both on left and right side, with respect to monad $m(B)$ and with respect to comonad $c(B)$ (this means four distributive laws). Then*

$\lambda\colon =$

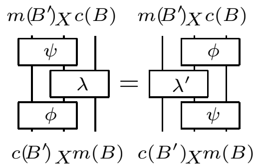


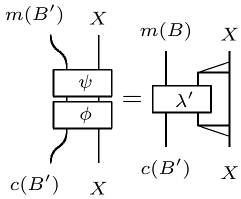
is a distributive law on the left both with respect to monads and comonads, that is:

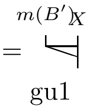
(5.5)

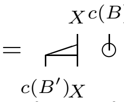


Continuing with the Example, we have that similarly ν' induces λ'. Moreover, the compatibilities

(5.6) 

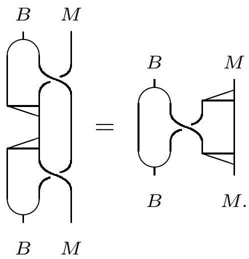
and 

hold. Then $(X, \psi, \phi)\colon (\mathcal{A}, m(B), c(B), \lambda) \to (\mathcal{A}', m(B'), c(B'), \lambda')$ is a 1-cell in the 2-category of mixed distributive laws $\mathrm{Dist}(\mathcal{K})$ of [**39**, Definition 6.2]. (In the specific case when X is a left $m(B')$-module and left $c(B')$-comodule, and ψ and ϕ are given by $\psi =$ 

gu1 and $\phi =$ 

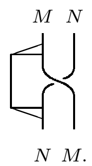
, the two expressions in (5.6) are equivalent and one recovers a particular form of λ-bialgebras by Turin and Plotkin, [**46**, Section 7.2].)

Suppose that B is a bialgebra in a braided monoidal category $\mathcal{C}$. A (left) Yetter–Drinfel'd module over B is an object M together with a (left) action $B \otimes M \to M$ and a (left) coaction $M \to B \otimes M$ of B subject to the compatibility condition:

(5.7) 

The category of (left) Yetter–Drinfel'd modules over B in $\mathcal{C}$ and left B-linear and B-colinear morphisms we denote by ${}^B_B\mathcal{YD}(\mathcal{C})$.

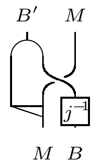
Remark 5.1. Observe that the antipode, i.e., a Hopf algebra structure on a bialgebra in the context of Yetter–Drinfel'd modules, is used in the following two instances. One is to construct the inverse for the braiding of the respective category. Another one is to formulate an equivalent condition to (5.7). Thus, the category of Yetter–Drinfel'd modules over a bialgebra is monoidal and even it has a pre-braiding (non-invertible), given by:

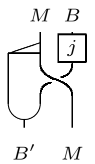


Example 5.5. Consider two braided monoidal categories $\mathcal{C}$ and $\mathcal{D}$ and two bialgebras B_0, B_1 in $\mathcal{D}$. These give rise to two bilax (and bimonoidal) functors

$F_{B_0}, F_{B_1}\colon \Sigma\mathcal{C} \longrightarrow \Sigma\mathcal{D}$ as in Example 4.4. The bilax natural transformation $\chi\colon \mathcal{F}_0 \Rightarrow \mathcal{F}_1\colon \mathcal{K} \longrightarrow \mathcal{K}'$ from Example 5.1 corresponds to a generalized notion of a Yetter–Drinfel'd module over B_1.

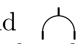
THEOREM 5.1. *Any Yetter–Drinfel'd module M over a bialgebra B' in a braided monoidal category $(\mathcal{C}, \Phi)$ comes from a bilax natural transformation of Example 5.5 where ψ and ϕ are given by*

(5.8) $\psi =$ 

, $\phi =$ 

,

for any bialgebra isomorphism $j\colon B \longrightarrow B'$. (More precisely, from a bilax endo-transformation with $j = \mathrm{id}_B$ in (5.8).)

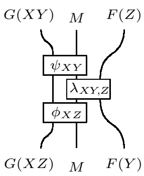
PROOF. The notation in these two diagrams is the usual one for braided monoidal categories, concretely  and  stand for the (co)multiplication of B'. That the given ψ and ϕ are desired distributive laws (i.e., (co)lax natural transformations) it was proved at the beginning of [**13**, Section 5.1], though for $B = B'$ and trivial j. The algebra (resp. coalgebra) morphism property of j^{-1} (resp. j) makes (5.8) the desired distributive laws for nontrivial j. The first claim now follows from [**14**, Proposition 7.5], whose conditions are fulfilled since $\mathcal{C}$ is braided. Set $\nu_{F,X} = (M \otimes j^{-1})\Phi_{F',X}$ and $\nu_{X,F} = \Phi_{X,F'}(M \otimes j)$. The second claim follows from [**14**, Corollary 7.6]. $\square$

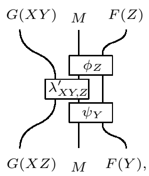
We record that another direction of generalization of Yetter–Drinfel'd modules was carried out in [**8**, Section 8] in the enriched setting. There a (left-right) Yetter–Drinfel'd $\mathcal{C}$-module is defined as a certain $\mathcal{V}$-functor $F : \mathcal{C} \longrightarrow \mathcal{V}$ for a comonoidal $\mathcal{V}$-category $\mathcal{C}$, where $\mathcal{V}$ is at least a braided monoidal closed category.

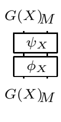
EXAMPLE 5.6. Let (χ, ψ, ϕ) be a bilax natural transformation between bilax functors with compatible Yang Baxter operators $\chi\colon \mathcal{F} \Rightarrow \mathcal{F}'\colon \Sigma\mathcal{C} \longrightarrow \Sigma\mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are braided monoidal categories with braidings $\Phi_{\mathcal{C}}$ and $\Phi_{\mathcal{D}}$, respectively. Then $F\colon = \mathcal{F}_{*,*}$ and $G\colon = \mathcal{F}'_{*,*}$ are bimonoidal functors $\mathcal{C} \longrightarrow \mathcal{D}$ as in Example 4.1, $\chi(*) = M$ is an object in $\mathcal{D}$ and there are morphisms

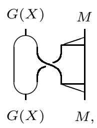
$$\psi_X\colon G(X) \otimes M \longrightarrow M \otimes F(X) \quad \text{and} \quad \phi_X\colon M \otimes F(X) \longrightarrow G(X) \otimes M$$

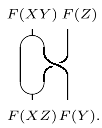
natural in $X \in \mathcal{C}$, where ψ is a distributive law for the monoidal functor structures, and ϕ is a distributive law for the comonoidal functor structures, so that the left identity below holds, and consequently the one next to it:



$=$ 



$=$ 

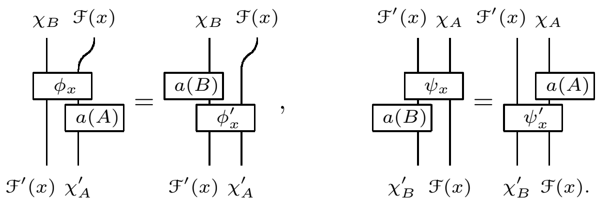
with $\lambda_{XY,Z}\colon =$ 

For bialgebras B in $\mathcal{C}$ by Theorem 4.1 $F(B), G(B)$ are bialgebras in $\mathcal{D}$, and if ψ_B, ϕ_B are of the form as in (5.8), we recover classical Yetter–Drinfel'd modules in $\mathcal{C}$.

The bilax natural transformations, i.e., identities (5.2) and (5.3), offer the following point of view. Given any monoidal category $\mathcal{D}$, when one considers the center category $\mathcal{Z}_l^w(\mathcal{D})$, one is given a family of colax transformations ϕ. In particular, when $\mathcal{D} = {}_H\mathcal{C}$ the category of modules over a bialgebra or a Hopf algebra H in a braided monoidal category $\mathcal{C}$, one is able to construct lax transformations ψ (as in (5.8)) so that the given ϕ and this ψ obey (5.3) – since ϕ is H-linear, being a morphism in ${}_H\mathcal{C}$. Similarly, considering $\mathcal{Z}_r^w({}^H\mathcal{C})$ one is given ψ's and one constructs ϕ's, so that they together obey (5.3). As in the proof of the above Theorem (that is, as proved in [**14**, Proposition 7.5]), in this setting the bilax condition (5.2) follows.

5.2. Bilax modifications. We finally introduce:

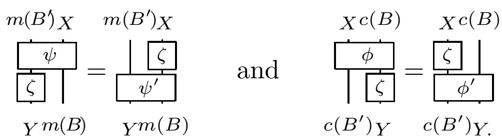
Definition 5.2. Let $\chi, \chi' \colon (\mathcal{F}, \nu) \Rightarrow (\mathcal{F}', \nu') \colon (\mathcal{K}, c) \longrightarrow \mathcal{K}'$ be bilax natural transformations. A *bilax modification* $a \colon \chi \Rrightarrow \chi'$ is a collection of 2-cells $(a(A))_{A \in \mathcal{O}b\,(\mathcal{K})}$ satisfying equations:



Equivalently, a bilax modification is a modification both of lax and colax natural transformations: $a \colon \psi \Rrightarrow \psi'$ and $a \colon \phi \Rrightarrow \phi'$, where (ψ, ϕ) constitute χ and (ψ', ϕ') constitute χ'.

Example 5.7. Pursuing Example 5.3 a bilax modification between bilax natural transformations of bilax functors $1 \longrightarrow \mathcal{K}$ is precisely a 2-cell in Bimnd($\mathcal{K}$).

Example 5.8. Recall Example 5.4 where bilax natural transformations are 1-cells in the 2-category of mixed distributive laws Dist($\mathcal{K}$) of [**39**, Definition 6.2]. In this setting a bilax modification of bilax natural transformations is a 2-cell $\zeta \colon X \Rightarrow Y$ in $\mathcal{K}$ that satisfies:



As such it is a 2-cell in the 2-category Dist($\mathcal{K}$) of [**39**, Definition 6.2].

Example 5.9. In the setting of Example 5.5, where bilax natural transformations are generalized Yetter–Drinfel'd modules, a bilax modification of bilax natural transformations is a morphism f in $\mathcal{D}$ satisfying:

By (4.8) and (4.9) this means that f is both a morphism of left B'-modules and left B'-comodules. This is a morphism of generalized Yetter–Drinfel'd modules from Example 5.5.

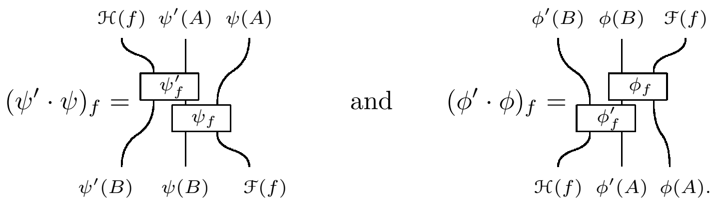
Now we may formulate:

THEOREM 5.2. *The category of Yetter–Drinfel'd modules* ${}^B_B\mathcal{YD}(\mathcal{D})$ *over a bialgebra* B *in a braided monoidal category* $(\mathcal{D}, \Phi_{\mathcal{D}})$ *is a full subcategory of the category* $\mathrm{Bilax}(\mathcal{F}_B)$ *of bilax endo-transformations on a bilax functor* $\mathcal{F}_B\colon (\Sigma\mathcal{C}, \Phi_{\mathcal{C}}) \longrightarrow (\Sigma\mathcal{D}, \Phi_{\mathcal{D}})$ *with compatible Yang–Baxter operator as in Example* 5.5 *and bilax modifications.*

Similarly one has:

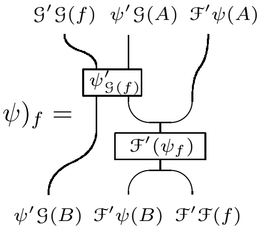
THEOREM 5.3. *The category* ${}^B_B\mathcal{YD}(\mathcal{D})$ *is isomorphic to the category* $\mathrm{Bilax}(\mathcal{T}_B)$ *of bilax endo-transformations on a bilax functor* $\mathcal{T}_B\colon 1 \longrightarrow (\Sigma\mathcal{D}, \Phi_{\mathcal{D}})$ *with compatible Yang–Baxter operator as in Lemma* 4.2 *and bilax modifications.*

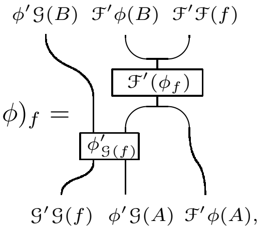
5.3. 2-category of bilax functors. We finish this section by concluding that bilax functors (with compatible Yang–Baxter operator) $\mathcal{K} \longrightarrow \mathcal{K}'$, bilax natural transformations and bilax modifications form a 2-category $\mathrm{Bilax}(\mathcal{K}, \mathcal{K}')$. (Strictly speaking, we should differentiate a 2-category whose objects are bilax functors $(\mathcal{F}, \nu)\colon (\mathcal{K}, c) \longrightarrow \mathcal{K}'$, and another one with objects compatible bilax functors of the form $\mathcal{F}\colon (\mathcal{K}, c) \longrightarrow (\mathcal{K}', d)$. Though for both of them the following results hold, so we will abstract this small difference in that we will use the same notation for both 2-categories, but the reader should be aware of this subtlety.) The composition of bilax transformations $(\chi, \psi, \phi)\colon \mathcal{F} \Rightarrow \mathcal{G}$ and $(\chi', \psi', \phi')\colon \mathcal{G} \Rightarrow \mathcal{H}$ is easily seen to be induced by the vertical compositions of the (co)lax transformations $\psi' \cdot \psi, \phi' \cdot \phi$, namely by:



Bilax modifications compose both horizontally and vertically, in the obvious and natural way.

We comment for the record that although the lax and colax natural transformations compose horizontally by:

$(\psi' \circ \psi)_f =$ 

and $(\phi' \circ \phi)_f =$ 

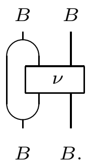
the horizontal composition of lax and colax natural transformations does not induce a bilax transformation. Namely, in order for this to work, the (co)lax structures should be identities.

We finally compare the 2-category $\mathrm{Bilax}(\mathcal{K}, \mathcal{K}')$, more precisely its special case $\mathrm{Bilax}(1, \mathcal{K})$, with two other existing 2-categories in the literature, namely $\mathrm{Dist}(\mathcal{K})$ and $\mathrm{Bimnd}(\mathcal{K})$ mentioned before. From Lemma 4.3, Example 5.3 and Example 5.7 we clearly have:

THEOREM 5.4. *There is a 2-category isomorphism* $\mathrm{Bilax}(1, \mathcal{K}) \cong \mathrm{Bimnd}(\mathcal{K})$.

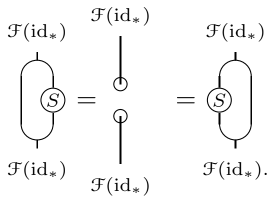
From Lemma 4.3, Example 5.4 and Example 5.8, it can be appreciated that on the level of 1- and 2-cells there is a faithful assignment $\mathrm{Bilax}(1, \mathcal{K}) \hookrightarrow \mathrm{Dist}(\mathcal{K})$. Since the 0-cells of $\mathrm{Dist}(\mathcal{K})$ are given by tuples $(\mathcal{A}, T, D, \lambda)$, where T is a monad and D a comonad on a 0-cell $\mathcal{A}$ in $\mathcal{K}$, and $\lambda\colon TD \Rightarrow DT$ is a distributive law with respect to monad and comonad as in (5.5), we clearly have:

THEOREM 5.5. *There is a faithful 2-functor* $\mathrm{Bilax}(1, \mathcal{K}) \hookrightarrow \mathrm{Dist}(\mathcal{K})$*, which is defined on 0-cells by* $(\mathcal{F}, \nu) \overset{\cong}{\mapsto} (\mathcal{F}(*) = \mathcal{A}, \mathcal{F}(\mathrm{id}_*) = B, \nu) \mapsto (\mathcal{A}, m(B), c(B), \lambda)$*, with* λ *being*

$\lambda(\nu)\colon =$ 

Observe that Theorem 5.3 is a consequence of the above 2-category isomorphism $\mathrm{Bilax}(1, \mathcal{K}) \cong \mathrm{Bimnd}(\mathcal{K})$.

We finish with the following remark. Although bilax natural transformations generalize Yetter–Drinfel'd modules, they do it only in the case of bilax functors from the trivial 2-category, and so that this generalization is in the form of (mixed) distributive laws (ψ and ϕ), which are a priori not braidings (of the category of Yetter–Drinfel'd modules). Passing from these distributive laws to modules and comodules over the bimonad (see [**13**, Proposition 2.4]) and to a sub-2-category $\mathrm{Bilax}^*(1, \mathcal{K})$ of $\mathrm{Bilax}(1, \mathcal{K})$ in which distributive laws are of a specific form (as in [**14**, Proposition 7.5]), one gets to 2-categorical Yetter–Drinfel'd modules, due to [**14**, Proposition 7.5]. Then one can prove that the endo-hom-categories of $\mathrm{Bilax}^*(1, \mathcal{K})$ (i.e., the categories of Yetter–Drinfel'd modules in this setting) are isomorphic to (certain subcategories of) the monoidal centers (of the categories of (co)modules over the bimonads), by taking the left-left version of [**15**, Proposition 7.1]. (One can also get a 2-category isomorphism including bimonad isomorphisms analogous to j in Theorem 5.1.) In view of this one could call the sub-2-category $\mathrm{Bilax}^*(1, \mathcal{K})$ a *bilax center* 2*-category of* $\mathcal{K}$. The *strong center of* $\mathcal{K}$ would correspond to a sub-2-category of the latter where not only are bilax transformations moreover pseudotransformations, but also bilax functors posses an *antipode* 2*-cell* $S : \mathcal{F}(\mathrm{id}_*) \Rightarrow \mathcal{F}(\mathrm{id}_*)$ in $\mathcal{K}$ satisfying the axiom



In the case of the center categories considered in the first part of the paper for lax functors $\mathcal{F}, \mathcal{G} : \Sigma\mathcal{E} \longrightarrow \mathcal{K}$, one has monads $\mathcal{F}(I_{\mathcal{E}})$ and $\mathcal{G}(I_{\mathcal{E}})$, but no module structure on objects χ_* of $\mathcal{K}$ for transformations χ (since monads do not have counit 2-cells). For those centers no relation to some kind of a Yetter–Drinfel'd module is established.

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Mathematical Institute
of Serbian Academy of Sciences and Arts
Belgrade
Serbia
bfemic@mi.sanu.ac.rs

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Research Group Algebraic Lie Theory
Philipps-Universität Marburg
Marburg
Germany
Sebastian.Halbig@uni-marburg.de