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CERTAIN CURVATURE CONDITIONS ON RIEMANNIAN MANIFOLDS ADMITTING THE SPACE MATTER TENSOR

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Abstract. We extend the work applied to Einstein spaces by A. Z. Petrov to some spaces that can be considered as modified forms of this space. For this purpose, the first study which was done on the quasi Einstein manifold continued with the generalized quasi-Einstein and the pseudo generalized quasi-Einstein manifolds in this article and our evidences are supported by several examples in the last section.

1. Introduction

In 1949, the celebrated theorem [**4**] showing the existence of three types of Einstein space with signature $(-,-,-,+)$ and the corresponding three canonical forms were established. During this study, the gravitation fields are classified on the basis of the algebraic structures of the space-matter tensor. A. Z. Petrov proposed and studied a (0,4) tensor field P as follows:

$$P = R + \frac{k}{2}(g \wedge T) - \sigma G. \tag{1.1}$$

Here R, T, σ and k denote respectively the Riemann curvature tensor, the energy-momentum tensor, the energy density and a cosmological constant. $g \wedge T$ is the well-known Kulkarni–Nomizu product of two (0,2) tensors g and T. The (0,4) tensor G is given by

$$G(V_1, V_2, V_3, V_4) = g(V_1, V_4)g(V_2, V_3) - g(V_1, V_3)g(V_2, V_4)$$

for all $V_1, V_2, V_3, V_4 \in \chi(M)$, the Lie-algebra of smooth vector fields on M. The (0,4) type tensor P is called the space-matter tensor (SMT) of M.

Einstein's field equation (EFE) having cosmological constant λ is presented by

$$kT = S + \Big(\lambda - \frac{r}{2}\Big)g, \tag{1.2}$$

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where as usual r denotes the curvature scalar and S stands for the Ricci tensor. Then (1.1) changes into

$$P = R - \Big(\sigma - \lambda + \frac{r}{2}\Big)G + \frac{1}{2}(g \wedge S), \tag{1.3}$$

by virtue of (1.2).

If T is of Codazzi type and σ is constant in our manifold, then the SMT satisfies the second Bianchi identity [**3**] i.e.,

$$(\nabla_X P)(U_1, U_2, U_3, U_4) + (\nabla_{U_1} P)(U_2, X, U_3, U_4) + (\nabla_{U_2} P)(X, U_1, U_3, U_4) = 0.$$

In Section 2 we deal with several characteristics of SMT satisfying different curvature conditions on the Riemannian manifolds.

In 2001 Chaki introduced the notion of generalized quasi Einstein manifolds [**1**]. A Riemannian manifold of dimension greater than three is defined to be generalized quasi-Einstein manifold iff its Ricci tensor S is non-zero and satisfies the relation

$$S(X_1, X_2) = \gamma_1 g(X_1, X_2) + \gamma_2 \vartheta(X_1)\vartheta(X_2) + \gamma_3[\vartheta(X_1)\nu(X_2) + \nu(X_1)\vartheta(X_2)], \tag{1.4}$$

where γ_1, $\gamma_2(\neq 0)$, γ_3 are scalars and ϑ, ν are (non-zero) 1-forms such that $\vartheta(X_1) = g(\varsigma_3, X_1)$, $\nu(X_1) = g(\varsigma_4, X_1)$ for all X_1 and ς_3, ς_4 are the unit vector fields. This class of manifolds is classified by the symbol $G(QE)_n$. The significance of a $G(QE)_n$ rests in the fact that such a four dimensional semi Riemannian manifold is applicable to the study of a general relativistic fluid spacetime admitting heat flux [**5**], where ς_3 is taken as the velocity vector field of the fluid and ς_4 is considered as the heat flux vector field.

Further the notion of $G(QE)_n$ was generalized by Shaikh and Jana [**6**]. A Riemannian manifold of dimension greater than three is called a pseudo generalized quasi-Einstein manifold [**6**] if S is non-zero and it satisfies

$$S(X_1, X_2) = \delta_1 g(X_1, X_2) + \delta_2 H(X_1)H(X_2) + \delta_3 F(X_1)F(X_2) + \delta_4 D(X_1, X_2). \tag{1.5}$$

Here $\delta_1, \delta_2, \delta_3, \delta_4$ are non-zero scalars and H, F are (non-zero) 1-forms such that $H(X) = g(\varsigma_5, X)$, $F(X) = g(\varsigma_6, X)$ for all X and ς_5, ς_6 are unit vector fields; D is a symmetric (0,2) tensor with zero trace such that $D(\varsigma_5, X) = 0$ for all X. Such type of manifold of dimension n is denoted by $P(GQE)_n$. The significance of a $P(GQE)_n$ lies in the fact that such a four dimensional semi Riemannian manifold is applicable to the study of a general relativistic fluid spacetime admitting heat flux and admitting EFE [**6**], where ς_5 is taken as the velocity vector field of the fluid, ς_6 is considered as the heat flux vector field and D is taken as the anisotropic pressure of the fluid.

Sections 3 and 4 deal with generalized quasi Einstein manifolds and pseudo generalized quasi Einstein manifolds with SMT respectively. Some interesting examples are given in the last section.

2. Preliminaries

Here we refer to some basic properties of P under certain curvature conditions. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal frame field on M. Then we have

$$S(X_1, X_2) = g(QX_1, X_2) = \sum_{\alpha=1}^{n} R(e_\alpha, X_1, X_2, e_\alpha),$$
$$r = \sum_{\alpha=1}^{n} S(e_\alpha, e_\alpha) = \sum_{\alpha=1}^{n} g((Qe_\alpha), e_\alpha).$$

Here Q is the symmetric endomorphism related to S.

Let us discuss on Riemannian manifolds admitting different restrictions on the space-matter tensor.

2.1. Vanishing SMT. Let M be of dimension at least three. If P vanishes identically, then (1.3) reduces to

$$R = \Big(\sigma - \lambda + \frac{r}{2}\Big)G - \frac{1}{2}(g \wedge S). \tag{2.1}$$

Contracting (2.1) we get

$$nS + rg + 2\Big(\lambda - \frac{r}{2} - \sigma\Big)(n-1)g = 0.$$

Further contraction of the last relation leads to the following relation

$$r = \frac{2(n-1)}{(n-3)}(\lambda - \sigma). \tag{2.2}$$

Thus we have the following:

LEMMA 2.1. *In a Riemannian manifold with dimension greater than three if SMT vanishes then the scalar curvature takes the form* (2.2).

2.2. Symmetric SMT. If P is symmetric in a Riemannian manifold of dimension greater than three, then we know that

$$\nabla P = 0. \tag{2.3}$$

Covariant differentiation of (1.3) and then use of (2.3) yields

$$\begin{aligned} 2(\nabla_X R)(Y,Z,U,V) &+ g(Z,U)(\nabla_X S)(Y,V) + g(Y,V)(\nabla_X S)(Z,U) \\ &- g(Z,V)(\nabla_X S)(Y,U) - g(Y,U)(\nabla_X S)(Z,V) \\ &- 2[d\sigma(X) + \frac{1}{2}dr(X)]G(Y,Z,U,V) = 0. \end{aligned} \tag{2.4}$$

Contracting (2.4) over V and Y, one obtains

$$n(\nabla_X S)(Z,U) - g(Z,U)\{2(n-1)d\sigma(X) + (n-2)dr(X)\} = 0. \tag{2.5}$$

Putting $Z = e_\alpha = U$ in (2.5) and then taking sum over α, we find

$$2(n-1)d\sigma(X) + (n-3)dr(X) = 0. \tag{2.6}$$

Further contraction of (2.5) over X and U yields

$$2(n-1)d\sigma(Z)+(n-4)dr(Z)=0. \tag{2.7}$$

By the virtue of (2.6) and (2.7), it follows

$$d\sigma(X)=0=dr(X) \tag{2.8}$$

for all $X\in\chi(M)$. Again using (2.8) in (2.5), we have

$$\nabla S=0. \tag{2.9}$$

Finally using (2.8) and (2.9) in (2.4), we obtain

$$\nabla R=0. \tag{2.10}$$

If (2.10) holds, then the relations (2.9) and (2.8) also hold and consequently differentiating (1.3) covariantly we obtain

$$(\nabla_X P)=-d\sigma(X)\circ G, \tag{2.11}$$

using (2.8) and (2.9). Thus it leads to the following:

LEMMA 2.2. *In a Riemannian manifold with dimension greater than three if SMT is symmetric then the relation* (2.11) *holds.*

2.3. Recurrent SMT. In a Riemannian manifold of dimension greater than three admitting EFE we consider that the SMT is recurrent $\nabla P=L\circ P$, where L is the (non-zero) 1-form of recurrence. By using the relations [**2**, (8)–(10)] and by performing some calculations, we find

$$r=\frac{2(n-1)}{n-3}(\lambda-\sigma),\quad \text{since}\quad L\neq 0. \tag{2.12}$$

Further contraction of [**2**, (9)] over X and U yields

$$(n-4)dr(Z)-4(n-1)d\sigma(Z)=2[(n-2)r+2(n-1)(\sigma-\lambda)]L(Z)-2nL(QZ),$$

which gives

$$2nL(QZ)=rL(Z)+dr(Z)+6(n-1)d\sigma(Z)\quad \text{for all}\quad Z\in\chi(M), \tag{2.13}$$

by the virtue of [**2**, (10)]. By using the relation [**2**, (7)] in [**2**, (5)] and after some calculations, we get

$$L(QZ)=\frac{r_0}{2n}L(Z),\quad \text{which yields}\quad S(Z,\rho)=\frac{r_0}{2n}g(Z,\rho), \tag{2.14}$$

where $r_0=(n-1)[2(n-2)(\lambda-\sigma)-(n-4)r]$ and $g(X,\rho)=L(X)$, by virtue of (2.13). Thus we have the following:

LEMMA 2.3. *In a Riemannian manifold with dimension as greater than three, admitting EFE and having recurrent SMT, the Ricci tensor satisfies the relation* (2.14).

2.4. Weakly symmetric SMT. We assume that the SMT in a Riemannian manifold of dimension greater than three admitting EFE, is weakly symmetric [**7**] in nature. Then there exist three 1-forms A_1, A_2 and A_3 (non-zero simultaneously) such that the following relation holds:

$$(\nabla_X P)(Y,Z,U,V) = A_1(X)P(Y,Z,U,V) + A_2(Y)P(X,Z,U,V) + A_2(Z)P(Y,X,U,V) + A_3(U)P(Y,Z,X,V) + A_3(V)P(Y,Z,U,X). \tag{2.15}$$

Here ρ_1, ρ_2, ρ_3 be metrically equivalent to A_1, A_2, A_3 respectively.

Contracting Y and V on (2.15) and using [**2**, (4)], we have

$$H_2(QX) = -\frac{n-1}{2n}[(n-4)r + 2(n-2)(\sigma-\lambda)]H_2(X), \tag{2.16}$$

which gives

$$S(X,\tau_2) = r_1 g(X,\tau_2), \tag{2.17}$$

where $r_1 = \frac{n-1}{n}[(n-2)(\lambda-\sigma) - \frac{n-4}{2}r]$ and $g(X,\tau_2) = H_2(X) = (A_2 - A_3)(X)$ for every X. So we have the following:

LEMMA 2.4. *In a Riemannian manifold with dimension as greater than three, admitting EFE and having weakly symmetric SMT, the Ricci tensor satisfies relation* (2.17)*.*

3. Generalized quasi-Einstein manifold possessing SMT

In the present section we deal with generalized quasi-Einstein manifold [**1**], with SMT having certain curvature restrictions.

From (1.4) it follows that

$$S(\varsigma_3,\varsigma_3) = \gamma_1 + \gamma_2, \quad S(\varsigma_4,\varsigma_4) = \gamma_1, \quad S(\varsigma_3,\varsigma_4) = \gamma_3. \tag{3.1}$$

If possible let $P = 0$ in the considered manifold. Then by the virtue of (1.4), the relation (2.1) reduces to

$$R = \Big[\sigma - \lambda + \frac{r}{2} - \gamma_1\Big]G - \frac{\gamma_2}{2} g \wedge \vartheta' - \frac{\gamma_3}{2} g \wedge \nu', \tag{3.2}$$

where $\vartheta'(X_1,X_2) = \vartheta(X_1)\vartheta(X_2)$ and $\nu'(X_1,X_2) = \vartheta(X_1)\nu(X_2) + \nu(X_1)\vartheta(X_2)$. Taking contraction of (3.2), we have

$$S(X_1,X_2) = \Big[(n-1)\Big(\sigma - \lambda + \frac{r}{2} - \gamma_1\Big) - \frac{1}{2}(\gamma_2+\gamma_3)\Big]g(X_1,X_2) - \frac{n-2}{2}[\gamma_2\vartheta'(X_1,X_2) + \gamma_3\nu'(X_1,X_2)]. \tag{3.3}$$

Replacing X_1 and X_2 by ς_3 in (3.3), we get

$$S(\varsigma_3,\varsigma_3) = (n-1)\Big(\sigma - \lambda + \frac{r}{2} - \gamma_1\Big) - \frac{1}{2}(\gamma_2+\gamma_3) - \frac{n-2}{2}\gamma_2. \tag{3.4}$$

Again replacing both X_1 and X_2 by ς_4 in (3.3), we find

$$S(\varsigma_4,\varsigma_4) = (n-1)\Big(\sigma - \lambda + \frac{r}{2} - \gamma_1\Big) - \frac{1}{2}(\gamma_2+\gamma_3). \tag{3.5}$$

Now using (3.1), (3.4) and (3.5), we get $\gamma_2 = 0$, which is not possible. Hence we state the following:

THEOREM 3.1. *The SMT never vanishes in a* $G(QE)_n$ $(n > 3)$ *admitting EFE, if* $\gamma_2 \neq 0$.

From (2.2) we obtain

$$2(n-1)(\sigma - \lambda) + (n-3)(n\gamma_1 + \gamma_2) = 0, \tag{3.6}$$

by the virtue of (1.4). Hence we have the following:

THEOREM 3.2. σ, λ, γ_1, γ_2 *are connected by the relation* (3.6) *in a* $G(QE)_n$ $(n > 3)$ *admitting EFE and non-vanishing SMT.*

Using (1.4) and (2.8), we get $n\, d\gamma_1(X) + d\gamma_2(X) = 0 \quad \text{for all} \quad X$. Due to the arbitrariness of X, this relation gives

$$n \operatorname{grad} \gamma_1 + \operatorname{grad} \gamma_2 = 0, \tag{3.7}$$

which leads to the following:

THEOREM 3.3. γ_1 *and* γ_2 *are connected by* (3.7) *in a* $G(QE)_n$ $(n > 3)$ *admitting EFE and symmetric SMT.*

Putting $X = \rho$ in [**2**, (10)], we get

$$(n-3)[n\, d\gamma_1(\rho) + d\gamma_2(\rho) - n\gamma_1 - \gamma_2] + 2(n-1)[d\sigma(\rho) - \sigma + \lambda] = 0. \tag{3.8}$$

Hence we get the following:

THEOREM 3.4. ρ, σ, λ, γ_1, γ_2 *are connected by* (3.8) *in a* $G(QE)_n$ $(n > 3)$ *admitting EFE and recurrent SMT.*

In view of (1.4), we find that

$$(n-3)(n\gamma_1 + \gamma_2) + 2(n-1)(\sigma - \lambda) = 0 \tag{3.9}$$

from (2.12). We now get the following:

THEOREM 3.5. *If* γ_1, γ_2 *and* σ *are constants in a* $G(QE)_n$ $(n > 3)$ *admitting EFE and recurrent SMT, then they are connected by* (3.9).

Using (1.4), the equation (2.14) can be converted into the following relation

$$L(QZ) = r_2 L(Z), \quad \text{i. e.,} \quad S(Z,\rho) = r_2 g(Z,\rho),$$

where $r_2 = \frac{n-1}{n}[(n-2)(\lambda - \sigma) - \frac{n-4}{2}(n\gamma_1 + \gamma_2)]$. This gives the following:

THEOREM 3.6. *If* T *is of Codazzi type in a* $G(QE)_n$ $(n > 3)$ *admitting EFE and recurrent SMT, then* r_2 *is an eigen value of* S *with respect to the eigen vector* ρ*, defined by* $g(X,\rho) = L(X)$*, provided that* σ *is constant.*

Applying (1.4) in (2.16), we find

$$H_2(QX) = r_3 H_2(X), \quad \text{i. e.,} \quad S(X,\tau_2) = r_3 g(X,\tau_2),$$

where $r_3 = \frac{n-1}{n}[(n-2)(\lambda - \sigma) - \frac{n-4}{2}(n\gamma_1 + \gamma_2)]$ and $H_2(X) = g(\tau_2, X) = (A_2 - A_3)(X)$ for all X. Hence we have the following:

THEOREM 3.7. *r_3 is an eigen value of S with respect to the eigen vector τ_2 in a $G(QE)_n$ $(n > 3)$ admitting EFE and weakly symmetric SMT.*

Now applying (2.16), (1.4) and [**2**, (21)], we get

$$\begin{aligned}(n^2-2n-4)[n\,d\gamma_1(X)+d\gamma_2(X)]+2(n-1)(n+2)d\sigma(X)\\ =[(n^2-n-4)(n\gamma_1+\gamma_2)+(n-1)(n+2)(\sigma-\lambda)]H_3(X)-2nH_3(QX),\end{aligned}$$

where $H_3(X)=g(\tau_3,X)=(A_1+A_2+A_3)(X)$ for all X. Now if γ_1, γ_2 and σ are constants, then we have

$$H_3(QX)=\frac{1}{2n}[(n^2-n-4)(n\gamma_1+\gamma_2)+(n-1)(n+2)(\sigma-\lambda)]H_3(X),$$

from the above relation, i.e.,

$$S(X,\tau_3)=\frac{1}{2n}[(n^2-n-4)(n\gamma_1+\gamma_2)+(n+2)(n-1)(\sigma-\lambda)]g(X,\tau_3).$$

Again performing (2.16), (1.4) and [**2**, (21)], we get

$$\begin{aligned}(n^2-4n+4)[(n\,d\gamma_1+d\gamma_2)(X)]+2(n-1)(n-2)d\sigma(X)\\ =2nH_4(QX)+(n-1)[(n-4)(n\gamma_1+\gamma_2)+2(n-2)(\sigma-\lambda)]H_4(X),\end{aligned}$$

where $H_4(X)=g(\tau_4,X)=(A_1-A_2-A_3)(X)$ for all X. If γ_1, γ_2 and σ are constants, then we also get, from the above relation,

$$H_4(QX)=r_4H_4(X),\quad \text{i.e.,}\quad S(X,\tau_4)=r_4g(X,\tau_4),$$

where $r_4=\frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\gamma_1+\gamma_2)]$. So it leads towards the following theorem:

THEOREM 3.8. *If σ is constant in a $G(QE)_n$ $(n>3)$ admitting EFE and weakly symmetric SMT, then r_3 and r_4 are the eigen values of S corresponding to the eigen vectors τ_3 and τ_4 respectively, provided γ_1, γ_2 are constants.*

4. Pseudo generalized quasi-Einstein manifolds possessing SMT

The section studies with pseudo generalized quasi-Einstein manifolds with SMT satisfying some curvature restrictions.

Now from (1.5) we have

$$S(\varsigma_5,\varsigma_5)=\delta_1+\delta_2,\quad S(\varsigma_6,\varsigma_6)=\delta_1+\delta_3+\delta_4D(\varsigma_6,\varsigma_6),\quad S(\varsigma_5,\varsigma_6)=0. \tag{4.1}$$

By virtue of (1.5), (2.1) becomes the following

$$R=\big[\sigma-\lambda+\frac{r}{2}-\delta_1\big]G-\frac{\delta_2}{2}g\wedge H'-\frac{\delta_3}{2}g\wedge F'-\frac{\delta}{2}g\wedge D, \tag{4.2}$$

where $H'(X_1,X_2)=H(X_1)H(X_2)$ and $F'(X_1,X_2)=F(X_1)F(X_2)$ for all X_1, X_2. Contracting (4.2), we have

$$\begin{aligned}S(Z,U)=\Big[(n-1)(\sigma-\lambda+\frac{r}{2}-\delta_1)-\frac{\delta_2+\delta_3}{2}\Big]g(Z,U)\\ -\frac{n-2}{2}[\delta_2H'(Z,U)+\delta_3F'(Z,U)+\delta_4D(Z,U)].\end{aligned} \tag{4.3}$$

Setting $Z = \varsigma_5 = U$ in (4.3), we get

$$S(\varsigma_5, \varsigma_5) = (n-1)(\sigma - \lambda + \frac{r}{2} - \delta_1) - \frac{1}{2}(\delta_2 + \delta_3) - \frac{(n-2)}{2}\delta_2. \tag{4.4}$$

Further setting $Z = \varsigma_6 = U$ in (4.3), we get

$$S(\varsigma_6, \varsigma_6) = (n-1)(\sigma - \lambda + \frac{r}{2} - \delta_1) - \frac{1}{2}(\delta_2 + \delta_3) - \frac{(n-2)}{2}[\delta_3 + \delta_4 D(\varsigma_6, \varsigma_6)]. \tag{4.5}$$

Using (4.1) in (4.4) and (4.5), we have $S(\varsigma_5, \varsigma_5) = \delta_1 + \delta_2 = S(\varsigma_6, \varsigma_6)$.

THEOREM 4.1. *The scalar* $\delta_1 + \delta_2$ *is the Ricci curvature in the directions of both the generators* ς_5 *and* ς_6 *in a* $P(GQE)_n$ $(n > 3)$ *admitting EFE and vanishing SMT.*

In view of (1.5), (2.2) reduces to the following equation

$$(n-1)(\sigma - \lambda) + \frac{n-3}{2}(n\delta_1 + \delta_2 + \delta_3) = 0. \tag{4.6}$$

Thus we get:

THEOREM 4.2. σ*,* λ*,* δ_1*,* δ_2 *and* δ_3 *are related by* (4.6) *in a* $P(GQE)_n$ $(n > 3)$ *admitting EFE and vanishing SMT.*

In view of (1.5), (2.8) reduces to $d(n\delta_1 + \delta_2 + \delta_3)(X) = 0$ for each X. Since X is arbitrary, from the above relation, we have

$$n \operatorname{grad} \delta_1 + \operatorname{grad} \delta_2 + \operatorname{grad} \delta_3 = 0, \tag{4.7}$$

which leads to the following:

THEOREM 4.3. δ_1*,* δ_2 *and* δ_3 *are connected by the relation* (4.7) *in a* $P(GQE)_n$ $(n > 3)$ *admitting EFE and symmetric SMT.*

Putting $X = \rho$ in [**2**, (10)], we get

$$\begin{aligned}(n-3)d(n\delta_1 + \delta_2 + \delta_3)(\rho) + 2(n-1)[d\sigma(\rho) - \sigma + \lambda]& \\ = (n-3)(n\delta_1 + \delta_2 + \delta_3).&\end{aligned} \tag{4.8}$$

So we have the following:

THEOREM 4.4. ρ*,* σ*,* λ*,* δ_1*,* δ_2 *and* δ_3 *are connected by the relation* (4.8)*, in a* $P(GQE)_n$ $(n > 3)$ *admitting EFE as well as recurrent SMT.*

In the view of (1.5), (2.12) is reduced to the following form

$$(n-3)(n\delta_1 + \delta_2 + \delta_3) + 2(n-1)(\sigma - \lambda) = 0. \tag{4.9}$$

This calculation directs to

THEOREM 4.5. *If* δ_1*,* δ_2*,* δ_3 *and* σ *are constants in a* $P(GQE)_n$ $(n > 3)$ *admitting EFE and recurrent SMT, then they are related by the relation* (4.9).

By the virtue of (1.5), (2.14) changes into the following form

$$L(QZ) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma) - (n-4)(n\delta_1 + \delta_2 + \delta_3)]L(Z),$$

which implies

$$S(Z,\rho) = \frac{n-1}{n}\Big[(n-2)(\lambda-\sigma) - \frac{n-4}{2}(n\delta_1 + \delta_2 + \delta_3)\Big]g(Z,\rho).$$

This gives the following:

THEOREM 4.6. *If T is of Codazzi type in a $P(GQE)_n$ $(n > 3)$ admitting EFE and recurrent SMT, then $\frac{(n-1)}{2n}[2(n-2)(\lambda-\sigma) - (n-4)(n\delta_1 + \delta_2 + \delta_3)]$ is an eigen value of S with respect to the eigen vector ρ, defined by $g(\rho, X) = L(X)$, for all X, provided that σ is constant.*

In view of (1.5), (2.16) reduces to the following:

$$H_2(QX) = -\frac{n-1}{n}\Big[\frac{n-4}{2}(n\delta_1 + \delta_2 + \delta_3) + (n-2)(\sigma-\lambda)\Big]H_2(X),$$

which implies

$$S(X,\tau_2) = \frac{n-1}{n}\Big[(n-2)(\lambda-\sigma) - \frac{n-4}{2}(n\delta_1 + \delta_2 + \delta_3)\Big]g(X,\tau_2),$$

where $H_2(X) = g(X,\tau_2) = (A_2 - A_3)(X)$ for all X. Hence it directs to the following:

THEOREM 4.7. *S possesses an eigen value $\frac{n-1}{n}[(n-2)(\lambda-\sigma) - \frac{n-4}{2}(n\delta_1+\delta_2+\delta_3)]$ corresponding to the eigen vector τ_2 in a $P(GQE)_n$ $(n > 3)$ equipped with EFE and weakly symmetric SMT.*

In view of (2.16), (1.5) and [**2**, (21)], we get

$$(n^2 - 2n - 4)[n\,d\delta_1(X) + d\delta_2(X) + d\delta_3(X)] + 2(n-1)(n+2)d\sigma(X)$$
$$= [(n^2 - n - 4)(n\delta_1 + \delta_2 + \delta_3) + (n-1)(n+2)(\sigma-\lambda)]H_3(X) - 2nH_3(QX),$$

where $H_3(X) = g(X,\tau_3) = (A + B + E)(X)$ for all X. If δ_1, δ_2, δ_3 and σ are constants, then from the above we have

$$H_3(QX) = \frac{1}{2n}[(n^2 - n - 4)(n\delta_1 + \delta_2 + \delta_3) + (n^2 - 3n + 2)(\sigma-\lambda)]H_3(X),$$

which gives

$$S(X,\tau_3) = \frac{1}{2n}[(n^2 - n - 4)(n\delta_1 + \delta_2 + \delta_3) + (n-1)(n+2)(\sigma-\lambda)]g(X,\tau_3).$$

Again by (2.16), (1.5) and [**2**, (21)], we get

$$(n^2 - 4n + 4)d(n\delta_1 + \delta_2 + \delta_3)(X) + 2(n-1)(n-2)d\sigma(X)$$
$$= 2nH_4(QX) + (n-1)[(n-4)(n\delta_1 + \delta_2 + \delta_3) + 2(n-2)(\sigma-\lambda)]H_4(X),$$

where $H_4(X) = g(\tau_4, X) = (A_1 - A_2 - A_3)(X)$ for all X. If δ_1, δ_2, δ_3 and σ are constants, then we get from the above relation

$$H_4(QX) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma) - (n-4)(n\delta_1 + \delta_2 + \delta_3)]H_4(X),$$

which implies

$$S(X,\tau_4) = \frac{n-1}{2n}[2(n-2)(\lambda-\sigma) - (n-4)(n\delta_1+\delta_2+\delta_3)]g(X,\tau_4).$$

Hence we have the following:

THEOREM 4.8. *In a $P(GQE)_n$ $(n > 3)$ satisfying EFE and weakly symmetric SMT, then $\frac{1}{2n}[(n^2-n-4)(n\delta_1+\delta_2+\delta_3)+(n-1)(n+2)(\sigma-\lambda)]$ and $\frac{n-1}{2n}[2(n-2)(\lambda-\sigma)-(n-4)(n\delta_1+\delta_2+\delta_3)]$ are eigen values of S attached with the eigen vectors τ_3 and τ_4, respectively, provided that σ, δ_1, δ_2 and δ_3 are constants.*

5. Some illustrative examples

Let us deal with various examples of Riemannian manifolds satisfying EFE and SMT satisfying certain curvature restrictions.

EXAMPLE 5.1. Let M^4 be $\mathbb{R}^4$ with coordinates (x^i), $i = 1, \dots, 4$ and be endowed with the Riemannian metric

$$ds^2 = g_{ij}dx^i dx^j, \tag{5.1}$$

where $g_{11} = e^{2x^2} = g_{33}$, $g_{22} = 1$, $g_{44} = 4$ and $g_{ij} = 0$, otherwise. Thus only non-zero components of R, S and the curvature scalar are given by

$$\begin{aligned} R_{1212} &= -e^{2x^2} = R_{2323}, \quad R_{1313} = -e^{4x^2}; \\ S_{11} &= 2e^{2x^2} = S_{33}, \quad S_{22} = 2; \quad r = 6. \end{aligned} \tag{5.2}$$

To verify that M^4 is a $G(QE)_4$ we consider the 1-forms ϑ, ν and the scalars γ_1, γ_2, γ_3 as follows:

$$\vartheta_i = \begin{cases} 4 & \text{for } i = 4 \\ 0 & \text{otherwise,} \end{cases} \qquad \nu_i = \begin{cases} 1 & \text{for } i = 4 \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

$$\gamma_1 = 2; \quad \gamma_2 = -2; \quad \gamma_3 = 3. \tag{5.4}$$

Accordingly (1.4) is reduced to

$$S_{ii} = \gamma_1 g_{ii} + \gamma_2\vartheta_i\vartheta_i + 2\gamma_3\vartheta_i\nu_i, \quad \text{for all possible } i. \tag{5.5}$$

By virtue of (5.1)–(5.4), it follows that r.h.s. of (5.5) $= 2g_{ii} - 2\vartheta_i\vartheta_i + 6\vartheta_i\nu_i = 2e^{2x^2} =$ l.h.s. of (5.5) for $i = 1$. Putting similar arguments it can be shown that the relation (5.5) holds for the remaining values of i. So M^4 with considered g is a $G(QE)_4$.

Now, considering σ as a constant, we calculate the (non-vanishing) components of SMT.

$$\begin{aligned} &P_{1221} = e^{2x^2}(\lambda-\sigma) = P_{2332}, && P_{1331} = e^{4x^2}(\lambda-\sigma), \\ &P_{1441} = 4e^{2x^2}(-2+\lambda-\sigma) = -P_{3434}, && P_{2442} = 4(-2+\lambda-\sigma) \end{aligned}$$

and its covariant derivatives $P_{hijk,l} = 0$ for all h, i, j, k, l. Also it can be calculated that $R_{hijk,l} = 0$ for all $h, i, j, k, l = 1, 2, 3, 4$. Therefore here M^4 is symmetric also. Hence we have the theorem as follows:

THEOREM 5.1. *The smooth manifold* (M^4, g) *admitting EFE and symmetric SMT and equipped with the Riemannian metric given in* (5.1) *is a* $G(QE)_4$ *with non-vanishing curvature scalar. It is also a symmetric manifold.*

EXAMPLE 5.2. Let $M^4 = \mathbb{R}^4$ with coordinates(x^i), $i = 1, \dots, 4$ be endowed with the Riemannian metric

$$ds^2 = e^{x^1}(dx^1)^2 + e^{2x^1}(dx^2)^2 + e^{2x^1}(dx^3)^2 + e^{2x^1}(dx^4)^2. \tag{5.6}$$

To verify that the manifold under consideration is a $G(QE)_4$, we take the 1-forms ϑ, ν and the scalars γ_1, γ_2, γ_3 as follows:

$$\vartheta_i = \begin{cases} -1 & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad \nu_i = \begin{cases} \frac{e^{x^1}-1}{2} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{5.7}$$

$$\gamma_1 = \frac{5e^{-x^1}}{2}; \quad \gamma_2 = -e^{-x^1}; \quad \gamma_3 = e^{-x^1}. \tag{5.8}$$

Thus (1.4) is reduced to the following equations

$$S_{ii} = \gamma_1 g_{ii} + \gamma_2 \vartheta_i \vartheta_i + 2\gamma_3 \vartheta_i \nu_i, \quad i \in \{1, 2, 3, 4\}. \tag{5.9}$$

By virtue of (5.6), (5.7), (5.8), it can be proved that (5.9) is true. Therefore M^4 with the considered metric g in (5.6) is a $G(QE)_4$. We take $\sigma = \lambda$ (a constant) and consider L as follows:

$$L\Big(\frac{\partial}{\partial x^i}\Big) = L_i = \begin{cases} -1, & i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

Calculating the non-zero components of SMT and its covariant derivatives, the validity of the following relations with the 1-form given by (5.10) can be checked easily:

$$P_{1j1j,1} = L_1 P_{1j1j}, \quad P_{jkjk,1} = L_1 P_{jkjk},$$

where j and k run from 2 to 4 and $j \neq k$. Thus we have

THEOREM 5.2. *The smooth manifold* (M^4, g) *admitting EFE and recurrent SMT and equipped with the Riemannian metric given in* (5.6) *is a* $G(QE)_4$ *with non-vanishing curvature scalar and such that* $\sigma = \lambda$.

EXAMPLE 5.3. Let M^4 be $\mathbb{R}^4$ with coordinates (x^i), $i = 1, \dots, 4$ and be endowed with the Riemannian metric

$$ds^2 = g_{ij} dx^i dx^j, \tag{5.11}$$

where $g_{11} = e^{2x^1} = g_{22} = g_{33} = g_{44}$ and $g_{ij} = 0$, otherwise. Let us consider the 1-forms ϑ, ν and the scalars γ_1, γ_2, γ_3 as follows:

$$\vartheta\Big(\frac{\partial}{\partial x^i}\Big) = \vartheta_i = \begin{cases} -1, & i = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad \nu\Big(\frac{\partial}{\partial x^i}\Big) = \nu_i = \begin{cases} \frac{e^{2x^1}-1}{7}, & i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_1 = 2e^{-2x^1}; \quad \gamma_2 = -2e^{-2x^1}; \quad \gamma_3 = 7e^{-2x^1}.$$

With these considerations it can be easily shown that M^4 is a $G(QE)_4$. Let us take $\sigma = \lambda - 2e^{-2x^1} + ve^{-4x^1}$, where v is an arbitrary(non-zero) constant and consider the 1-forms as follows:

$$A\Big(\frac{\partial}{\partial x^i}\Big) = A_i = \begin{cases} \frac{4e^{2x^1}-4v}{-2e^{2x^1}+v}, & i = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad B\Big(\frac{\partial}{\partial x^i}\Big) = B_i = \begin{cases} \frac{2}{-2+ve^{-2x^1}}, & i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$E\Big(\frac{\partial}{\partial x^i}\Big) = E_i = \begin{cases} \frac{2}{-2+ve^{-2x^1}}, & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can check easily find the non-zero components of SMT and its covariant derivatives and also check the validity of the following relations with the above 1-forms:

$$P_{1j1j,1} = A_1 P_{1j1j} + B_1 P_{1j1j} + B_j P_{111j} + E_1 P_{1j1j} + E_j P_{1j11},$$
$$P_{jkjk,1} = A_1 P_{jkjk} + B_j P_{1kjk} + B_k P_{j1jk} + E_j P_{jk1k} + E_k P_{jkj1},$$

where j and k run from 2 to 4 and $j \neq k$. Hence we have

THEOREM 5.3. *The smooth manifold* (M^4, g) *admitting EFE and weakly symmetric SMT and equipped with the Riemannian metric given in* (5.11) *is a* $G(QE)_4$ *with non-zero curvature scalar and such that* $\sigma = \lambda - 2e^{-2x^1} + ve^{-4x^1}$, *where* v *is an arbitrary non-zero constant.*

EXAMPLE 5.4. Let M^4 be $\mathbb{R}^4$ with coordinates (x^i), $i = 1, \ldots, 4$ and be furnished with the Riemannian metric

$$ds^2 = g_{ij} dx^i dx^j, \tag{5.12}$$

where $g_{11} = 1 + \sin x^2 = g_{22} = g_{33} = g_{44}$, $g_{ij} = 0$, otherwise and $\sin x^2 \neq -1$. Let us consider the 1-forms H, F, a symmetric (0,2) tensor D and the scalars δ_1, δ_2, δ_3, δ_4 as follows:

$$H_i = \begin{cases} \big[\frac{3+\sin x^2}{3}\big]^{\frac{1}{2}} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad D_{ij} = \begin{cases} \frac{2}{7} & \text{for } i = 1 = j \\ -\frac{2}{14} & \text{for } i = 2 = j \\ -\frac{2}{28} & \text{for } i = 3 = j \\ -\frac{2}{28} & \text{for } i = 4 = j \\ 0 & \text{otherwise,} \end{cases}$$

$$F_i = \begin{cases} \big[\frac{3+\sin x^2}{3}\big]^{\frac{1}{2}} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_1 = \frac{1}{4}; \quad \delta_2 = \delta_3 = -\frac{3}{4}; \quad \delta_4 = \frac{7}{2}$$

With these considerations it can be easily shown that (M^4, g) is a $P(GQE)_4$. Now taking σ as any arbitrary function of x^1 only, it can be easily proved that $P_{hijk,l} = 0$ for all h, i, j, k, l. Also it can be shown that $R_{hijk,l} = 0$ for all h, i, j, k, l. Thus our manifold M^4 is symmetric. Hence we have

THEOREM 5.4. *The smooth manifold* (M^4, g) *admitting EFE and symmetric SMT and equipped with the Riemannian metric given in (5.12) is a* $P(GQE)_4$ *with non-zero curvature scalar. It is also a symmetric manifold.*

EXAMPLE 5.5. Let M^4 be $\mathbb{R}^4$ with coordinates (x^i), $i = 1, \ldots, 4$ and be furnished with the Riemannian metric

$$ds^2 = g_{ij} dx^i dx^j, \tag{5.13}$$

where $g_{11} = e^{2x^1}$, $g_{22} = e^{x^1}$, $g_{33} = g_{44} = 1$, $g_{ij} = 0$, otherwise. Also let us consider the 1-forms H, F, a symmetric (0,2) tensor D and the scalars δ_1, δ_2, δ_3, δ_4 as follows:

$$H_i = \begin{cases} \left[\frac{e^{2x^1}+e^{x^1}-1}{3}\right]^{\frac{1}{2}} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad F_i = \begin{cases} \left[\frac{e^{2x^1}+e^{x^1}-1}{3}\right]^{\frac{1}{2}} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$D_{ij} = \begin{cases} \frac{3(e^{-x^1}-e^{-2x^1})}{10} & \text{for } i = 1 = j \\ -\frac{3e^{-x^1}}{10} & \text{for } i = 2 = j \\ -\frac{3e^{-2x^1}}{20} & \text{for } i = 3 = j \\ -\frac{3e^{-2x^1}}{20} & \text{for } i = 4 = j \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_1 = \frac{e^{-2x^1}}{4}; \quad \delta_2 = \delta_3 = -\frac{3e^{-2x^1}}{4}; \quad \delta_4 = \frac{5}{3}.$$

With these it can be shown that our (M^4, g) is a $P(GQE)_4$. Let us take $\sigma = \lambda$, and take the 1-form L as follows:

$$L\Big(\frac{\partial}{\partial x^i}\Big) = L_i = \begin{cases} -2, & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

at any point of M. Calculating the non-zero components of SMT and its covariant derivatives, it can be verified easily the validity of the following relations with the above 1-forms: $P_{1j1j,1} = L_1 P_{1j1j}$, $P_{jkjk,1} = L_1 P_{jkjk}$, where j and k run from 2 to 4 and $j \neq k$. So the considered manifold is recurrent. Hence we have

THEOREM 5.5. *The smooth manifold (M^4, g) admitting EFE and recurrent SMT such that $\sigma = \lambda$ and equipped with the Riemannian metric given in* (5.13) *is a $P(GQE)_4$ with non-zero curvature scalar. It is also a recurrent manifold.*

EXAMPLE 5.6. Let M^4 be $\mathbb{R}^4$ with coordinates (x^i), $i = 1, \ldots, 4$ and be furnished with the Riemannian metric

$$ds^2 = g_{ij} dx^i dx^j, \tag{5.14}$$

where $g_{11} = x^2 e^{x^1}$, $g_{22} = g_{33} = g_{44} = 1$, $g_{ij} = 0$, otherwise and $x^2 > 0$. Let us suppose the 1-forms H, F and the scalars δ_1, δ_2, δ_3, δ_4 as follows:

$$H\Big(\frac{\partial}{\partial x^i}\Big) = H_i = \begin{cases} \left[\frac{3x^2e^{x^1}+5}{8}\right]^{\frac{1}{2}} & \text{for } i=1 \\ 0 & \text{otherwise,} \end{cases} \qquad F\Big(\frac{\partial}{\partial x^i}\Big) = F_i = \begin{cases} \left[\frac{3x^2e^{x^1}+5}{8}\right]^{\frac{1}{2}} & \text{for } i=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$D\Big(\frac{\partial^2}{\partial x^i \partial x^j}\Big) = D_{ij} = \begin{cases} \frac{5}{4(x^2)^2} & \text{for } i=1=j \\ -\frac{3}{4(x^2)^2} & \text{for } i=2=j \\ -\frac{1}{4(x^2)^2} & \text{for } i=3=j \\ -\frac{1}{4(x^2)^2} & \text{for } i=4=j \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_1 = \frac{1}{8(x^2)^2}; \quad \delta_2 = \delta_3 = -\frac{1}{2(x^2)^2}; \quad \delta_4 = \frac{1}{2}.$$

With these choices, our (M^4, g) is a $P(GQE)_4$. Lastly let us take $\sigma = \lambda$, and the 1-forms as

$$A(\frac{\partial}{\partial x^i}) = A_i = \begin{cases} -\frac{2}{x^2} & \text{for } i = 2 \\ 0 & \text{otherwise,} \end{cases} \qquad B(\frac{\partial}{\partial x^i}) = B_i = \begin{cases} \frac{1}{7x^2} & \text{for } i = 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$E(\frac{\partial}{\partial x^i}) = E_i = \begin{cases} -\frac{1}{7x^2} & \text{for } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

One can now easily calculate the non-zero components of SMT and its covariant derivatives. Also,the validity of the following relations can be verified with the above 1-forms:

$$P_{1j1j,2} = A_2 P_{1j1j} + B_1 P_{2j1j} + B_j P_{121j} + E_1 P_{1j2j} + E_j P_{1j12},$$
$$P_{jkjk,2} = A_2 P_{jkjk} + B_j P_{2kjk} + B_k P_{j2jk} + E_j P_{jk2k} + E_k P_{jkj2}.$$

Hence we have

THEOREM 5.6. *The smooth manifold* (M^4, g) *admitting EFE and weakly symmetric SMT such that* $\sigma = \lambda$ *and equipped with the Riemannian metric given in* (5.14) *is a* $P(GQE)_4$ *with non-zero curvature scalar.*

6. Conclusion

We have obtained several interesting results on the basis of purely geometric view point. For example, there is no mathematical constraint which can force SMT to vanish identically in case of generalized quasi-Einstein manifolds. We expect that the results obtained will be useful in studying physical behaviours of different cosmological models.

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