PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 116 (130) (2024), 87–96 DOI: https://doi.org/10.2298/PIM2430087W

FRACTIONAL ORDER OPERATIONAL CALCULUS AND EXTENDED HERMITE–APOSTOL TYPE FROBENIUS–EULER POLYNOMIALS

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Abstract. The combined use of integral transforms and special classes of polynomials provides a powerful tool to deal with models based on fractional order derivatives. In this article, the operational representations for the extended Hermite–Apostol type Frobenius–Euler polynomials are introduced via integral transforms. The recurrence relations and some identities involving these polynomials are established. Finally, the quasi-monomial properties for the Hermite–Apostol type Frobenius–Euler polynomials and for their extended forms are derived.

1. Introduction and preliminaries

Various significant properties of the classical and generalized polynomials including the recurrence and explicit relations; functional and differential equations, summation formulae, symmetric and convolution identities, determinant forms etc., are useful and have potential for applications in certain problems of number theory, combinatorics, classical and numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics.

The Appell polynomials constitute an important class of polynomials because of their remarkable applications in numerous fields. The interest in Appell polynomials and their applications in different fields has significantly increased. The recent applications of Appell polynomials are in probability theory and statistics. The presentation of theoretic results provides new examples of applications of Appell polynomials and gives evidence to their central role as orthogonal polynomials.

The Appell class contains the Euler polynomials $E_n(x)$ [**10**] as one of the important member. The Euler polynomials are presented in the Taylor expansion in a neighborhood of the origin of the trigonometric and hyperbolic secant functions. These polynomials are defined by the following generating function

2020 *Mathematics Subject Classification:* 33E20; 33E05; 33B99; 33B20.
Key words and phrases: quasi-monomiality; extended Hermite-Apostol type Frobenius-Euler polynomials; fractional operators; operational rules; integral transforms.
Communicated by Stevan Pilipović.

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

Various generalizations related to the Euler polynomials, namely the Frobenius–Euler polynomials $\mathcal{F}_n(x;u)$ and later the Apostol type Frobenius–Euler polynomials $\mathcal{F}_n(x;\lambda;u)$ were introduced [**13**, **15**]. The Apostol type Frobenius–Euler polynomials are defined by

$$\left(\frac{1-u}{\lambda e^t-u}\right)e^{xt}=\sum_{n=0}^{\infty}\mathcal{F}_n(x;\lambda;u)\frac{t^n}{n!},\quad u,\lambda\in\mathbb{C};\ u\neq 1,$$

which for $\lambda=1$ reduces to the Frobenius–Euler polynomials.

Next, we recall that the 3-variable Hermite polynomials (3VHP) $H_n(x,y,z)$ [**8**], which are defined by the following generating function:

$$e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!},$$

which for $z=0$ reduce to the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [**2**] and for $z=0,\ x=2x$ and $y=-1$ become the classical Hermite polynomials $H_n(x)$ [**1**].

A hybrid class of the 3-variable Hermite–Apostol type Frobenius–Euler polynomials denoted by ${}_H\mathcal{F}_n(x,y,z;\lambda;u)$ is introduced in [**4**] by considering the discrete Apostol type Frobenius–Euler convolution of the 3-variable Hermite polynomials. The Hermite–Apostol type Frobenius–Euler polynomials are defined by the following operational rule

$$(1.1)\qquad {}_H\mathcal{F}_n(x,y,z;\lambda;u)=\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\{\mathcal{F}_n(x;\lambda;u)\}$$

and these polynomials possess the following generating function

$$(1.2)\qquad \left(\frac{1-u}{\lambda e^t-u}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_H\mathcal{F}_n(x,y,z;\lambda;u)\frac{t^n}{n!},\ u,\ \lambda\in\mathbb{C};\ u\neq 1.$$

Fractional order operators have been attracting the attention of mathematicians and engineers from long time ago [**14**, **18**]. Special polynomials and fractional operators have a noteworthy relationship within the realm of mathematics. Fractional order operators introduce a new dimension for ordinary differentiation and integration by extending these operations to non-integer orders, which are often represented by fractional order exponents. Special polynomials, on the other hand, are a class of mathematical functions with unique properties and generated expressions. The combination of special polynomials with fractional order operators leads to the development of more sophisticated and versatile mathematical tools [**5**, **9**]. By applying fractional order operators to special polynomials, researchers can create new families of fractional special polynomials, which can offer enhanced capabilities in solving complex mathematical problems and the modeling of various phenomena.

In [**9**], Dattoli and coauthors explored the potential of using integral transforms in a wider context. In their research, they use integral transforms beyond their typical limitations. A comprehensive foundation for increasing the applicability and efficiency of integral transformations in numerous domains is provided by Euler's integral:

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-at} t^{\nu-1} dt, \qquad \min\{\mathrm{Re}(\nu), \mathrm{Re}(a)\} > 0, \tag{1.3}$$

Some of the integral transforms consequences of (1.3) (which can be also treated as operational operators of fractional order) are given as:

$$\begin{aligned} \left(\alpha - \frac{\partial}{\partial x}\right)^{-\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1}\; e^{t\frac{\partial}{\partial x}} f(x)\, dt \\ &= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1}\; f(x+t)\, dt, \end{aligned} \tag{1.4}$$

whereas for the cases involving second order derivatives, it is shown that

$$\left(\alpha - \frac{\partial^2}{\partial x^2}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1}\; e^{t\frac{\partial^2}{\partial x^2}} f(x)\, dt.$$

These transforms of fractional operational calculus can be treated in an efficient way by combining the properties of exponential operators and suitable integral representations.

Many works are devoted to use of fractional operational calculus to investigate extended families of polynomials, see for example [**11**, **17**]. Motivated by this, in this article, the extended Hermite–Apostol type Frobenius–Euler polynomials are introduced and studied by means of generating function and operational definition by using fractional order operators. The recurrence relations and summation formulae for the extended Hermite–Apostol type Frobenius–Euler polynomials are also established.

2. Extended Hermite–Apostol type Frobenius–Euler polynomials

Here, we introduce the extended Hermite–Apostol type Frobenius–Euler polynomials using Euler's integral. For this, we have the following result.

Theorem 2.1. *For the extended Hermite–Apostol type Frobenius–Euler polynomials* ${}_{\nu H}\mathcal{F}_n(x,y,z;\lambda;u;\alpha)$*, the following operational connection holds true:*

$$\left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu} \mathcal{F}_n(x;\lambda;u) = {}_{\nu H}\mathcal{F}_n(x,y,z;\lambda;u;\alpha). \tag{2.1}$$

Proof. Replacing a by $\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)$ in integral (1.4) and then operating the resultant equation on $\mathcal{F}_n(x;\lambda;u)$, we find

$$\begin{aligned} &\left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu} \mathcal{F}_n(x;\lambda;u) \\ &\qquad = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} \exp\left(yt\frac{\partial^2}{\partial x^2} + zt\frac{\partial^3}{\partial x^3}\right) \mathcal{F}_n(x;\lambda;u)\, dt, \end{aligned}$$

which in view of (1.1) gives
(2.2)
$$\left(\alpha-\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathfrak{F}_n(x;\lambda;u)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_H\mathfrak{F}_n(x,yt,zt;\lambda;u)\,dt.$$

The transform on the r.h.s. (2.2) defines a new family of polynomials. Denoting this special family of polynomials by ${}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)$ and naming it as the extended Hermite–Apostol type Frobenius–Euler polynomials, so that we have

$$ {}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_H\mathfrak{F}_n(x,yt,zt;\lambda;u)\,dt. \tag{2.3}$$

In view of (2.2) and (2.3), assertion (2.1) follows. □

Remark 2.1. For $\lambda=1$, the Apostol type Frobenius–Euler polynomials reduce to the Frobenius–Euler polynomials. Therefore, taking $\lambda=1$ in the l.h.s. of (2.1) and denoting the resultant extended Hermite Frobenius–Euler polynomials in the r.h.s. by ${}_{\nu H}\mathfrak{F}_n(x,y,z;u;\alpha)$, we obtain the following operational connection between the extended Hermite Frobenius–Euler polynomials and the Frobenius–Euler polynomials

$$\left(\alpha-\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathfrak{F}_n(x;u)={}_{\nu H}\mathfrak{F}_n(x,y,z;u;\alpha).$$

Remark 2.2. We know that for $\lambda=1$ and $u=-1$, the Apostol type Frobenius–Euler polynomials reduce to the Euler polynomials. Therefore, taking $\lambda=1$ and $u=-1$ in the l.h.s. of (2.1) and denoting the resultant extended Hermite–Frobenius–Euler polynomials in the r.h.s. by ${}_{\nu H}\mathfrak{F}_n(x,y,z;\alpha)$, we obtain the following operational connection between the extended Hermite–Euler polynomials and the Euler polynomials

$$\left(\alpha-\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}E_n(x)={}_{\nu H}E_n(x,y,z;\alpha).$$

Next, we derive the generating function of the extended Hermite–Apostol type Frobenius–Euler polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)$ by proving the following result.

Theorem 2.2. *For the extended Hermite–Apostol type Frobenius–Euler polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)$, the following generating function holds true*

$$\frac{(1-u)\ \exp(xw)}{(\lambda e^w-u)\ (\alpha-(yw^2+zw^3))^\nu}=\sum_{n=0}^\infty {}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\frac{w^n}{n!}. \tag{2.4}$$

Proof. Multiplying both sides of (2.3) by $\frac{w^n}{n!}$, then summing it over n and making use of (1.2) in the r.h.s. of the resultant equation, we find

$$\sum_{n=0}^\infty {}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\frac{w^n}{n!}=\frac{(1-u)\ \exp(xw)}{(\lambda e^w-u)\ \Gamma(\nu)}\int_0^\infty e^{-\left(\alpha-(yw^2+zw^3)\right)t}\ t^{\nu-1}dt,$$

which in view of integral (1.3) yields assertion (2.4). □

REMARK 2.3. We know that for $\lambda = 1$, the Apostol type Frobenius–Euler polynomials reduce to the Frobenius–Euler polynomials. Therefore, by taking $\lambda = 1$ in generating equation (2.4), we obtain the following generating function for extended Hermite–Frobenius–Euler polynomials [**3**, **4**]

$$\frac{(1-u)\ \exp(xw)}{(e^w-u)\ (\alpha-(yw^2+zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}\mathfrak{F}_n(x,y,z;u;\alpha)\frac{w^n}{n!}.$$

REMARK 2.4. For $\lambda = 1$ and $u = -1$, the Apostol type Frobenius–Euler polynomials reduce to the Euler polynomials. Therefore, by taking $\lambda = 1$ and $u = -1$ in generating equation (2.4), we obtain the following generating function for extended Hermite–Euler polynomials [**3**, **4**]

$$\frac{2\exp(xw)}{(e^w+1)\ (\alpha-(yw^2+zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}E_n(x,y,z;\alpha)\frac{w^n}{n!}.$$

Next, we derive the recurrence relations for the extended Hermite–Apostol type Frobenius–Euler polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)$ by taking into consideration their generating relation. A recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms. Differentiating generating function (2.4), with respect to x, y, z and α, we find the following recurrence relations for the extended Hermite–Apostol type Frobenius–Euler polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)$

$$\begin{aligned}
&\frac{\partial}{\partial x}\big({}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\big) = n\ {}_{\nu H}\mathfrak{F}_{n-1}(x,y,z;\lambda;u;\alpha),\\
&\frac{\partial}{\partial y}\big({}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\big) = \nu\ n(n-1)_{\nu+1 H}\mathfrak{F}_{n-2}(x,y,z;\lambda;u;\alpha),\\
&\frac{\partial}{\partial z}\big({}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\big) = \nu\ n(n-1)(n-2)_{\nu+1 H}\mathfrak{F}_{n-3}(x,y,z;\lambda;u;\alpha),\\
(2.5)\quad &\frac{\partial}{\partial \alpha}\big({}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\big) = -\nu_{\nu+1 H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha).
\end{aligned}$$

In view of the above relations, it follows that

$$\begin{aligned}
\frac{\partial}{\partial y}\big({}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\big) &= -\frac{\partial^3}{\partial x^2\partial\alpha}{}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha),\\
\frac{\partial}{\partial z}\big({}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)\big) &= -\frac{\partial^4}{\partial x^3\partial\alpha}\ {}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha).
\end{aligned}$$

Several identities involving Frobenius–Euler polynomials are known. The operational formalism developed in the previous section can be used to obtain the identities for the extended Hermite–Frobenius–Euler polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;u;\alpha)$. To achieve this, we perform the following operation

($\mathcal{O}$) operating $\big(\alpha - (y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3})\big)^{-\nu}$ on both sides of a given relation.

Consider the following identities for the Frobenius–Euler polynomials $\mathcal{F}_n^{(\alpha)}(x;u)$ from [**12**]

$$u\mathcal{F}_n(x;u^{-1})+\mathcal{F}_n(x;u)=(1+u)\sum_{k=0}^{n}\binom{n}{k}\mathcal{F}_{n-k}(u^{-1})\mathcal{F}_k(x;u),$$

$$\begin{aligned}&\frac{1}{n+1}\mathcal{F}_k(x;u)+\mathcal{F}_{n-k}(x;u)\\&=\sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}\left((-u)\mathcal{F}_{l-k}(u)\ \mathcal{F}_{n-l}(u)+2u\mathcal{F}_{n-k}(u)\right)\mathcal{F}_k(x;u)\mathcal{F}_n(x;u),\end{aligned}$$

$$\mathcal{F}_n(x;u)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{F}_{n-k}(u)\mathcal{F}_k(x;u)\quad(n\in\mathbb{Z}_+).$$

Performing the operation ($\mathcal{O}$) on both sides of the above equations yields the results for the extended Hermite–Frobenius–Euler polynomials ${}_{\nu H}\mathcal{F}_n(x,y,z;u;\alpha)$

$$\begin{aligned}&u_{\nu H}\mathcal{F}_n(x,y,z;u^{-1};\alpha)+{}_{\nu H}\mathcal{F}_n(x,y,z;u;\alpha)\\&\qquad=(1+u)\sum_{k=0}^{n}\binom{n}{k}\mathcal{F}_{n-k}(u^{-1})_{\nu H}\mathcal{F}_k(x,y,z;u;\alpha),\end{aligned}$$

$$\begin{aligned}&\frac{1}{n+1}{}_{\nu H}\mathcal{F}_k(x,y,z;u;\alpha)+{}_{\nu H}\mathcal{F}_{n-k}(x,y,z;u;\alpha)\\&\quad=\sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}\left((-u)\mathcal{F}_{l-k}(u)\ \mathcal{F}_{n-l}(u)+2u\mathcal{F}_{n-k}(u)\right)\\&\qquad\qquad{}_{\nu H}\mathcal{F}_k(x,y,z;u;\alpha)_{\nu H}\mathcal{F}_n(x,y,z;u;\alpha)\end{aligned}$$

$${}_{\nu H}\mathcal{F}_n(x,y,z;u;\alpha)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{F}_{n-k}(u)\ {}_{\nu H}\mathcal{F}_k(x,y,z;u;\alpha).$$

In the next section, the quasi-monomial properties for the Hermite Apostol type-Frobenius–Euler polynomials and for their extended forms are derived.

3. Quasi-monomial properties

The combination of monomiality principle along with operational techniques in the case of multi-variable special polynomials yields new mechanism of analysis for the solutions of a large class of partial differential equations usually experienced in physical problems. The operational methods open new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of special polynomials. The concept of monomiality principle arises from the idea of poweroid suggested by Steffensen [**16**]. This idea is reformulated and systematically used by Dattoli [**7**]. Ben Cheikh [**6**] has shown that every polynomial set is quasi-monomial and the properties of a given polynomial set may be deduced from the quasi-monomiality.

According to the monomiality principle, there exist two operators $\hat{M}$ and $\hat{P}$ playing, respectively, the role of multiplicative and derivative operators for a polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$, that is, $\hat{M}$ and $\hat{P}$ satisfy the following identities, for all $n \in \mathbb{N}$

$$\hat{M}\{p_n(x)\} = p_{n+1}(x), \tag{3.1}$$

$$\hat{P}\{p_n(x)\} = n\ p_{n-1}(x). \tag{3.2}$$

The polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is called a quasi-monomial. These multiplicative and derivative operators satisfy the commutation relation $[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}$ and therefore exhibits a Weyl group structure.

If the polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial, its properties can be established from those of the $\hat{M}$ and $\hat{P}$ operators. In fact the following holds

(i) If $\hat{M}$ and $\hat{P}$ have differential realizations, then the polynomials $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}\{p_n(x)\} = n\, p_n(x). \tag{3.3}$$

(ii) Assuming that $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as

$$p_n(x) = \hat{M}^n\{1\}. \tag{3.4}$$

(iii) In view of identity (3.4), the exponential generating function of $p_n(x)$ can be cast in the form

$$e^{t\hat{M}}\{1\} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \quad |t| < \infty.$$

In order to frame the polynomials ${}_H\mathcal{F}_n(x, y, z; \lambda; u)$ within the context of monomiality principle, the following result is proved.

Theorem 3.1. *The Hermite–Apostol type Frobenius–Euler polynomials ${}_H\mathcal{F}_n(x, y, z; \lambda; u)$ are quasi-monomial with respect to the following multiplicative and derivative operators*

$$\hat{M}_{H\mathcal{F}} = x + 2y\partial_x + 3z\partial_x^2 - \frac{\lambda e^{\partial_x}}{\lambda e^{\partial_x} - u} \tag{3.5}$$

$$\hat{P}_{H\mathcal{F}} = \partial_x; \quad \partial_x := \frac{\partial}{\partial x}, \tag{3.6}$$

respectively.

Proof. Differentiating (1.2) partially with respect to t, we get

$$\left(x + 2yt + 3zt^2 - \frac{\lambda e^t}{\lambda e^t - u}\right)\left(\frac{1-u}{\lambda e^t - u}\right) = \sum_{n=0}^{\infty} {}_H\mathcal{F}_{n+1}(x, y, z; \lambda; u)\frac{t^n}{n!}. \tag{3.7}$$

Now, using identity

$$\partial_x\{{}_H\mathcal{F}_n(x, y, z; \lambda; u)\} = t\{{}_H\mathcal{F}_n(x, y, z; \lambda; u)\} \tag{3.8}$$

and generating equation (1.2) in the l.h.s of (3.7), it follows that

$$\Big(x+2y\partial_x+3z\partial_x^2-\frac{\lambda e^{\partial_x}}{\lambda e^{\partial_x}-u}\Big)\sum_{n=0}^{\infty}{}_H\mathfrak{F}_n(x,y,z;\lambda;u)=\sum_{n=0}^{\infty}{}_H\mathfrak{F}_{n+1}(x,y,z;\lambda;u),$$

which on equating the coefficients of the same powers of t in both sides, yields (3.5). Again, in view of generating function (1.2) and identity (3.8), it follows that

$$\partial_x\Big\{\sum_{n=0}^{\infty}{}_H\mathfrak{F}_n(x,y,z;\lambda;u)\frac{t^n}{n!}\Big\}=\sum_{n=1}^{\infty}{}_H\mathfrak{F}_{n-1}(x,y,z;\lambda;u)\frac{t^n}{(n-1)!}.$$

Rearranging the terms in the above equation and then equating the coefficients of same powers of t in both sides of the resultant equation, assertion (3.6) follows. □

Remark 3.1. Using (3.5) and (3.6) in (3.3), the following differential equation for the Hermite–Apostol type Frobenius–Euler polynomials ${}_H\mathfrak{F}_n(x,y,z;\lambda;u)$ is obtained

$$\Big(x\partial_x+2y\partial_x^2+3z\partial_x^3-\frac{\lambda e^{\partial_x}}{\lambda e^{\partial_x}-u}\partial_x-n\Big){}_H\mathfrak{F}_n(x,y,z;\lambda;u)=0.$$

Next, by using integral transforms and quasi-monomiality of Hermite–Apostol type Frobenius–Euler polynomials, we show that the extended Hermite–Apostol type Frobenius–Euler polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)$ are quasi-monomials.

Consider the operation:

(Θ) Replacement of y by yt and z by zt, multiplication by $\frac{1}{\Gamma(\nu)}e^{-at}t^{\nu-1}$ and then integration with respect to t from $t=0$ to $t=\infty$.

Now, operating (Θ) on (3.5) and (3.6) and then using (2.5) and further in view of recurrence relations (3.1) and (3.2), we find that the polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;u;\lambda;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators

$$\hat{M}_{\nu H\mathfrak{F}}=x+2y\partial_x\partial_\alpha+3z\partial_x^2\partial_\alpha-\frac{\lambda e^{\partial_x}}{\lambda e^{\partial_x}-u}, \tag{3.9}$$

$$\hat{P}_{\nu H\mathfrak{F}}=\partial_x, \tag{3.10}$$

respectively. Further, use of (3.9) and (3.10) in (3.3) yields the following differential equation for the extended Hermite–Apostol type Frobenius–Euler polynomials ${}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)$

$$\Big(x\partial_x+2y\partial_x^2\partial_\alpha+3z\partial_x^3\partial_\alpha-\frac{\lambda e^{\partial_x}}{\lambda e^{\partial_x}-u}\partial_x-n\Big){}_{\nu H}\mathfrak{F}_n(x,y,z;\lambda;u;\alpha)=0.$$

We present certain special cases of the 3VHATFEP in Table 1, in which $\sum$ means $\sum_{n=0}^{\infty}$.

The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional order operators of operational calculus. To bolster the contention of using this approach, the extended form of hybrid type polynomials are introduced. The generating function and recurrence relations for the extended Laguerre–Appell polynomials are derived here. These results may be useful in the investigation of other useful properties of these polynomials and may

TABLE 1. Special cases of ${}_H\mathfrak{F}_n(x,y,z;\lambda;u)$

S. No.	Cases	Name of polynomial	Generating function
I.	$\lambda=1$,	Hermite Frobenius-Euler polynomials	$\left(\frac{1-u}{e^t-u}\right)e^{xt+yt^2+zt^3}=\sum {}_H\mathfrak{F}_n(x,y,z;u)\frac{t^n}{n!}$
	$u=-1$, $\lambda=1$	Hermite-Euler polynomials	$\left(\frac{2}{e^t+1}\right)e^{xt+yt^2+zt^3}=\sum {}_HE_n(x,y,z)\frac{t^n}{n!}$
II.	$z=0$	2-variable Hermite-Apostol type Frobenius Frobenius-Euler polynomials	$\left(\frac{1-u}{\lambda e^t-u}\right)e^{xt+yt^2}=\sum {}_H\mathfrak{F}_n(x,y;u;\lambda)\frac{t^n}{n!}$
	$z=0$, $\lambda=1$	2-variable Hermite-Frobenius-Euler polynomials	$\left(\frac{1-u}{e^t-u}\right)e^{xt+yt^2}=\sum {}_H\mathfrak{F}_n(x,y;u)\frac{t^n}{n!}$
III.	$x=2x$, $y=-1;\ z=0$	Hermite-Apostol type Frobenius-Euler polynomials	$\left(\frac{1-u}{\lambda e^t-u}\right)e^{2xt-t^2}=\sum {}_H\mathfrak{F}_n(x;\lambda;u)\frac{t^n}{n!}$
	$x=2x,\ y=-1$, $z=0;\ \lambda=1$	Hermite Frobenius-Euler polynomials	$\left(\frac{1-u}{e^t-u}\right)e^{2xt-t^2}=\sum {}_H\mathfrak{F}_n(x;u)\frac{t^n}{n!}$

have applications in physics. Operational methods can be exploited to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of special functions. The use of operational techniques in the study of special functions provide explicit solutions for the families of partial differential equations including heat and D′Alembert type equations. The method proposed in this article can be used in combination with the monomiality principle as a useful tool in analysing the solutions of a wide class of partial differential equations often encountered in physical problems.

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(Received 22 12 2020)
(Revised 27 04 2024)

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