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EXTENSION OF TURÁN-TYPE INEQUALITIES FOR POLAR DERIVATIVES OF POLYNOMIALS INTO INTEGRAL MEAN VERSION

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Abstract. Let $p(z)$ be a polynomial of degree n and let $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to a real or complex number α. If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k, k \geqslant 1$, then for a real or complex number α with $|\alpha| \geqslant k$, Aziz and Rather [J. Math. Ineq. Appl. **1** (1998), 231–238] proved

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant n\left(\frac{|\alpha| - k}{1 + k^n}\right) \max_{|z|=1} |p(z)|.$$

We first extend the above inequality into integral mean without applying subordination property. As an application of our result, we prove another integral mean inequality. Our results have interesting consequences to the earlier well-known inequalities.

1. Introduction and Statement of Results

Experimental observations and investigations in various fields of science and engineering are often converted into mathematical notations and mathematical models. Almost every branch of mathematics, from algebraic number theory and algebraic geometry to applied analysis, Fourier analysis, numerical analysis and computer sciences, has its own corpus of theory arising from the study of polynomials. Historically, the question relating to polynomials, for example, the solution of polynomial equations and the approximation by polynomials, give rise to some of the most important problems of the day. The well-known Russian mathematician Chebyshev (1821–1894) studied some properties of polynomials with the least deviation from a given continuous function and introduced the concept of best approximation in mathematical analysis. Various interesting inequalities in both directions relating the norm of the derivative and the polynomial itself play a key role in the literature for proving the inverse theorems in approximation theory and, of

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course, have their own intrinsic interests. The first result in this area was connected with some investigation of the well-known Russian chemist Mendeleev [**19**]. In fact, Mendeleev's problem was to determine $\max_{-1\leqslant x\leqslant 1}|p'(x)|$, where $p(x)$ is a quadratic polynomial of real variable x with real coefficients and satisfying $-1\leqslant p(x)\leqslant 1$ for $-1\leqslant x\leqslant 1$. He himself was able to prove that if $p(x)$ is a quadratic polynomial and $|p(x)|\leqslant 1$ on $[-1,1]$, then $|p'(x)|\leqslant 4$ on the same interval. Markov [**18**] generalized this result for a polynomial of degree n in the real axis. In fact, he proved that if $p(x)$ is an algebraic polynomial of degree at most n with real coefficients, then

$$\max_{-1\leqslant x\leqslant 1}|p'(x)|\leqslant n^2\max_{-1\leqslant x\leqslant 1}|p(x)|.$$

After about twenty years, Bernstein [**3**] needed the analogue of Markov's theorem for the unit disc in the complex plane instead of the interval $[-1,1]$ in order to prove the inverse theorem of approximation (see Borwein and Erdélyi [**5**, p. 241]). This leads to the famous well-known result known as Bernstein's inequality which states that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)|\leqslant n\max_{|z|=1}|p(z)|.$$

The above inequalities show how fast a polynomial of degree at most n or its derivative can change, and are of interest both in mathematic specially in approximation theory and in the application areas such a physical systems. Various analogues of these inequalities are known in which the underlying intervals, the sup-norms, and the family of functions are replaced by more general sets, norms, and families of functions, respectively. One such generalization is replacing sup-norm by factor involving integral mean.

Let $p(z)$ be a polynomial of degree n over the set of complex numbers and for real number $r>0$. We define

$$\|p\|_r=\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{1/r}.$$

If we take limit as $r\to\infty$ and make use of the well-known fact from analysis [**27**,**31**] that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{1/r}=\max_{|z|=1}|p(z)|,$$

we can suitably denote $\|p\|_\infty=\max_{|z|=1}|p(z)|$. On the other hand Turán's [**32**] classical inequality provides a lower bound estimate to the size of the derivative of a polynomial on the unit circle relative to the size of the polynomial itself when there is a restriction on its zeros. It states that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant 1$, then

$$\|p'\|_\infty\geqslant\frac{n}{2}\|p\|_\infty. \tag{1.1}$$

Inequality (1.1) is sharp and equality holds for $p(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$. Inequality (1.1) of Turán [**32**] has been of considerable interest and applications, and it would be of interest to seek its generalization for polynomials having all their zeros in $|z| \leqslant k$, $k > 0$. The case when $0 < k \leqslant 1$ was settled by Malik [**16**] and proved

$$\|p'\|_\infty \geqslant \frac{n}{1+k}\|p\|_\infty, \tag{1.2}$$

while the case when $k \geqslant 1$ by Govil [**9**] and proved

$$\|p'\|_\infty \geqslant \frac{n}{1+k^n}\|p\|_\infty. \tag{1.3}$$

Equality in (1.3) holds for $p(z) = z^n + k^n, k \geqslant 1$.

For the first time, in 1984 Malik [**15**] extended inequality (1.1) proved by Turán [**32**] into integral mean and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for each $r > 0$ we have $\|1 + z\|_r\|p'\|_\infty \geqslant n\|p\|_r$. The result is sharp and equality holds for $p(z) = (z + 1)^n$.

In 1988 Aziz [**1**] obtained the integral mean extension of inequality (1.3) and proved

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for each $r \geqslant 1$ we have $\|1 + k^n z\|_r\|p'\|_\infty \geqslant n\|p\|_r$. The result is sharp and equality holds for $p(z) = \alpha z^n + \beta k^n, |\alpha| = |\beta|$.*

Further, in the same paper [**1**] he also established an integral mean extension of inequality (1.2) of Malik [**16**] and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for each $r > 0$,

$$\|1 + kz\|_r\|p'\|_\infty \geqslant n\|p\|_r. \tag{1.4}$$

Equality in (1.4) holds for the polynomial $p(z) = (\alpha z + \beta k)^n$ where $|\alpha| = |\beta|$.

Inequalities on ordinary derivative have been extended widely in the literature to polar derivative of polynomials. For a polynomial $p(z)$ of degree n and a real or complex number α, let $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to α.

Note that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$, and it generalizes the ordinary derivative in the sense that $\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z)$. Shah [**28**] extended Turán's inequality (1.1) to polar derivative of a polynomial $p(z)$ by proving

Theorem 1.2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for any complex number α with $|\alpha| \geqslant 1$,*

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant \frac{n(|\alpha| - 1)}{2} \max_{|z|=1} |p(z)|. \tag{1.5}$$

The result is sharp and extremal polynomial is $p(z) = (z - 1)^n$ with real $\alpha \geqslant 1$.

Aziz and Rather [**2**] first extended inequality (1.2) due to Malik [**16**] to the polar derivative and proved

Theorem 1.3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for every complex number α with $|\alpha| \geqslant k$, $k \leqslant 1$,*

$$\|D_\alpha p(z)\|_\infty \geqslant n\Big(\frac{|\alpha| - k}{1+k}\Big)\|p(z)\|_\infty. \tag{1.6}$$

In the same paper [**2**] they also extended (1.3) to polar derivative.

Theorem 1.4. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every real or complex number α with $|\alpha| \geqslant k$,*

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant n\frac{|\alpha|-k}{1+k^n} \max_{|z|=1} |p(z)|. \tag{1.7}$$

Inequality (1.7) is best possible and equality occurs for $p(z) = (z-k)^n$ with real $\alpha \geqslant k$.

Further, Govil and Mctume [**12**] improved Theorem 1.4 by involving $\min_{|z|=k} |p(z)|$.

Theorem 1.5. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ $k \geqslant 1$, then for every real or complex number α with $|\alpha| \geqslant 1+k+k^n$,*

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant n\frac{|\alpha|-k}{1+k^n} \max_{|z|=1} |p(z)| + n\left\{\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right\} \min_{|z|=k} |p(z)|. \tag{1.8}$$

There is enough literature which deals with the integral mean extensions of Turán-type inequalities concerning ordinary as well as polar derivative for polynomials having all its zeros in $|z| \leqslant k$, $k \leqslant 1$ (see [**7, 21, 33, 34**]). It would indeed be interesting to note that in such extensions, the techniques used are more or less similar. A typical example is that of Dewan et al. [**7**], where they proved the following integral mean extension of (1.6) due to Aziz and Rather [**2**].

Theorem 1.6. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for every complex number α with $|\alpha| \geqslant k$ and for each $r > 0$*

$$\|1+kz\|_r \|D_\alpha p\|_\infty \geqslant n(|\alpha|-k)\|p\|_r.$$

The result is sharp and equality holds for $p(z) = (z-k)^n$.

In an attempt to obtain the integral versions of Turán-type inequalities of the above class of polynomials with $k \geqslant 1$ by following the usual techniques mentioned above, it was only in 2017 that Rather and Bhat [**25**] gave extension of Theorem 1.4 in integral mean setting. If we examine closely the paper due to Rather and Bhat [**25**], it may be noticed that even though they have proved three theorems, in which Theorems 1 and 2 are obtained in the same lines by applying two similar results of Rather et al. [**26**, Theorems 1 and 3]. Moreover, their Theorem 3 is a better form of the first theorem in the sense that in it, the factor $\max_{|z|=1} |D_\alpha p(z)|$ is replaced by integral mean of $|D_\alpha p(z)|$ for $|z| = 1$. Because of the similarities, their paper is mainly about Theorem 3. But in the current paper, we have proved the same result (Theorem 3, Corollary 3) in a simpler approach entirely based on some existing inequalities on polynomials. Some main differences in the proof of Theorem 3 of Rather and Bhat [**25**] with Theorem 2.1 of the current paper are:

1. They have obtained their inequality (26) by applying a result of De-Bruijin [**6**, Theorem 1, p. 1265] concerning an inequality for the derivatives between two polynomials on a convex region whereas the same inequality namely, (4.2) follows from Lemma 3.1.

2. They also applied the well-known Gauss–Lucas theorem in order to consider a rational function on which a well-known property of subordination [**14**, p. 422] is used to obtain an important integral mean inequality which plays a central role in deriving their desired result through further application of Holder's inequality, while all these concerns have been simply compensated with the use of a result due to Govil [**9**] (i.e., Lemma 3.2).

3. We also give an application of Theorem 2.1, which provides the integral analogue of a result proved by Govil and Mctume [**12**].

For a better insight into both Bernstein and Turán-type inequalities, one can refer the recently published monograph of Gardner et al. [**8**] (also see Marden [**17**], Milovanović et al. [**22**], Rahman and Schmeisser [**24**] and some recently published papers [**11**, **20**, **29**, **30**]).

2. Main Results

In mathematics, it is of interest to seek other methods of proof and in this regard, as mentioned above, in this paper, first we present an alternative proof of the result due to Rather and Bhat [**25**, Theoram 3] simply based on some existing inequalities on polynomials.

Theorem 2.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every real or complex number α with $|\alpha| \geqslant k$ and real number $r > 0$,*

$$\|D_\alpha p\|_r \geqslant \frac{|\alpha| - k}{2} \frac{n}{E_r} \|p\|_r, \quad \text{where} \quad E_r = \frac{\|1 + k^n z\|_r}{\|1 + z\|_r}. \tag{2.1}$$

Remark 2.1. Taking limit as $r \to \infty$ on both sides of (2.1) and noting that $E_r \to \frac{1+k^n}{2}$, then (2.1) reduces to (1.7).

Remark 2.2. Putting $k = 1$, Theorem 2.1 gives the following corollary which matches the integral mean version of Theorem 1.2.

Corollary 2.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for every real or complex number α with $|\alpha| \geqslant 1$ and real number $r > 0$,*

$$\|D_\alpha p\|_r \geqslant n \frac{|\alpha| - 1}{2} \|p\|_r. \tag{2.2}$$

Remark 2.3. Taking limit as $r \to \infty$, inequality (2.2) of Corollary 2.1 becomes inequality (1.5) due to Shah [**28**].

Further, as an application of Theorem 2.1, we prove the following result, which is the integral mean analogue of Theorem 1.5. More precisely, we obtain

Theorem 2.2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every real or complex number α, β with $|\alpha| \geqslant 1 + k + k^n$, $|\beta| < 1$ and for any $r > 0$,*

$$\|D_\alpha p(z) + n\beta m\|_r \geqslant \frac{|\alpha| - k}{E_r} \frac{n}{2} \|p(z) + \beta m\|_r, \tag{2.3}$$

where $m = \min_{|z|=k} |p(z)|$ and E_r is as defined in Theorem 2.1.

Remark 2.4. We are interested to verify that Theorem 2.2 is the integral mean analogue of Theorem 1.5 due to Govil and Mctume [**12**] and we do as follows. Taking limit as $r \to \infty$ on both sides of inequality (2.3) of Theorem 2.2, we have

$$\max_{|z|=1} |D_\alpha p(z) + n\beta m| \geqslant n\frac{|\alpha| - k}{1 + k^n} \max_{|z|=1} |p(z) + \beta m|. \tag{2.4}$$

If z_0 be a point on $|z| = 1$ such that $|p(z_0)| = \max_{|z|=1} |p(z)|$, then (2.4) becomes

$$\max_{|z|=1} |D_\alpha p(z) + n\beta m| \geqslant n\frac{|\alpha| - k}{1 + k^n} |p(z_0) + \beta m|. \tag{2.5}$$

If we choose the argument of β such that $|p(z_0) + \beta m| = |p(z_0)| + |\beta| m$, then from (2.5), we have

$$\max_{|z|=1} |D_\alpha p(z) + n\beta m| \geqslant n\frac{|\alpha| - k}{1 + k^n} \{|p(z_0)| + |\beta| m\},$$

which implies

$$\max_{|z|=1} |D_\alpha p(z)| + n|\beta| m \geqslant n\frac{|\alpha| - k}{1 + k^n} \{|p(z_0)| + |\beta| m\}. \tag{2.6}$$

Inequality (2.6) is equivalent to

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant n\frac{|\alpha| - k}{1 + k^n} \max_{|z|=1} |p(z)| + n|\beta| \left\{ \frac{|\alpha| - (1 + k + k^n)}{1 + k^n} \right\} m, \tag{2.7}$$

and (2.7) further gives if we take $|\beta| \to 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant n\frac{|\alpha| - k}{1 + k^n} \max_{|z|=1} |p(z)| + n\left\{ \frac{|\alpha| - (1 + k + k^n)}{1 + k^n} \right\} m,$$

which is (1.8).

Remark 2.5. Dividing both sides of inequality (2.3) by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we have the following integral mean analogue of a best possible inequality proved by Govil [**10**].

Corollary 2.2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every real or complex number β with $|\beta| < 1$ and for any $r > 0$,*

$$\|p'\|_r \geqslant \frac{\|1 + z\|_r}{\|1 + k^n z\|_r} \|p(z) + \beta m\|_r,$$

where $m = \min_{|z|=k} |p(z)|$.

Remark 2.6. Following the similar arguments of Remark 2.4, it would be evident that Corollary 2.2 is the integral mean version of an inequality due to Govil [**10**].

Remark 2.7. Putting $k = 1$, Corollary 2.2 reduces to an improved integral mean analogue of inequality (1.1) of Turán [**32**].

EXAMPLE 2.1. Let $p(z) = z^4 + 4$, with all zeros $\{1 - i, 1 + i, -1 - i, -1 + i\}$ on the circle $|z| = \sqrt{2}$, so that Theorem 2.2 holds for $|z| \leqslant \sqrt{2}$. For this polynomial, we have $m = \min_{|z|=\sqrt{2}} |p(z)| = 0$. And we take any α such that $|\alpha| = 2$. For $r = \frac{1}{2}$, we have $\|p(z)\|_{1/2} = 4.036$ and $E_{1/2} = 3.465$. Then, it is easy to see that by inequality (2.3) for $m = 0$, we have $\|D_\alpha p\|_{\frac{1}{2}} \geqslant 1.365$. For $r = 2$, we have $\|p(z)\|_2 = 4.123$ and $E_2 = 2.915$. By inequality (2.3) for $m = 0$, we have $\|D_\alpha p\|_2 \geqslant 1.657$. For $r \to \infty$, we have $\|p(z)\|_\infty = 5$. By inequality (1.8) for $m = 0$, we have $\|D_\alpha p\|_\infty \geqslant 2.343$. Here, in this example, we illustrate the estimates of $\|D_\alpha p\|_r$ for different values of $r > 0$ rather than sup-norms.

3. Lemmas

We shall need the following lemmas in order to prove the above theorems. For a polynomial $p(z)$ of degree n we will use $\tilde{p}(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

LEMMA 3.1. (Malik [**16**]) *If $p(z)$ is a polynomial having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for $|z| = 1$ we have $|\tilde{p}'(z)| \leqslant k|p'(z)|$.*

LEMMA 3.2. (Govil [**9**]) *If $p(z)$ is a polynomial having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for $|z| = 1$ we have $|p'(z)| \geqslant \frac{n}{1+k^n}|p(z)|$.*

LEMMA 3.3. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for every $R \geqslant 1$ and $r > 0$ we have*

$$\|p(Rz)\|_r \leqslant E_r \|p\|_r, \quad \textit{where} \quad E_r = \frac{\|1 + R^n z\|_r}{\|1 + z\|_r}.$$

This lemma was proved by Boas and Rahman [**4**] for $r \geqslant 1$. Later, Rahman and Schmeisser [**23**] showed the validity for $0 < r < 1$ as well.

LEMMA 3.4. *If $p(z)$ is a polynomial of degree n, then for every $R \geqslant 1$ and $r > 0$,*

$$\|p(Rz)\|_r \leqslant R^n \|p\|_r. \tag{3.1}$$

As far as Lemma 3.4 is concerned, it is difficult to trace its origin. It was deduced from the well-known result of Hardy [**13**], according to which for every function $f(z)$ analytic in $|z| < t_0$, and for every $r > 0$ we have

$$\left\{ \int_0^{2\pi} |f(e^{i\theta})|^r d\theta \right\}^{1/r}$$

is a nondecreasing function of t for $0 < t < t_0$. If $p(z)$ is a polynomial of degree n, then $f(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ is again a polynomial, that is, an entire function and by Hardy's result for $r > 0$,

$$\left\{ \int_0^{2\pi} |f(te^{i\theta})|^r d\theta \right\}^{1/r} \leqslant \left\{ \int_0^{2\pi} |f(e^{i\theta})|^r d\theta \right\}^{1/r},$$

for $t = \frac{1}{R} \leqslant 1$. This is equivalent to (3.1).

4. Proofs of Theorems

Proof of Theorem 2.1. By hypothesis $p(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then the polynomial $R(z) = p(kz)$ has all its zeros in $|z| \leqslant 1$. It is easy to see that for $|z| = 1$

$$|\tilde{R}'(z)| = |nR(z) - zR'(z)|, \tag{4.1}$$

where $\tilde{R}(z) = z^n \overline{R\big(\frac{1}{\bar{z}}\big)}$.

Further applying Lemma 3.1 for $k = 1$ to $R(z)$, we have for $|z| = 1$

$$|\tilde{R}'(z)| \leqslant |R'(z)|. \tag{4.2}$$

Now for $\left|\frac{\alpha}{k}\right| \geqslant 1$ and $|z| = 1$, we have

$$\begin{aligned} |D_{\alpha/k}R(z)| &= \Big|nR(z) + \Big(\frac{\alpha}{k} - z\Big)R'(z)\Big| \\ &\geqslant \Big|\frac{\alpha}{k}\Big||R'(z)| - |nR(z) - zR'(z)| \\ &= \Big|\frac{\alpha}{k}\Big||R'(z)| - |\tilde{R}'(z)| \quad \text{[by (4.1)]} \\ &\geqslant \Big(\Big|\frac{\alpha}{k}\Big| - 1\Big)|R'(z)| \qquad \text{[by (4.2)]}. \end{aligned} \tag{4.3}$$

Further, applying Lemma 3.2 with $k = 1$ to $R(z)$, we have for $|z| = 1$

$$|R'(z)| \geqslant \frac{n}{2}|R(z)|. \tag{4.4}$$

Using (4.4) in inequality (4.3), we have

$$|D_{\alpha/k}R(z)| \geqslant \frac{|\alpha| - k}{k}\frac{n}{2}|R(z)|.$$

Replacing $R(z)$ by $p(kz)$ in the above inequality, we get

$$\Big|np(kz) + \Big(\frac{\alpha}{k} - z\Big)kp'(kz)\Big| \geqslant \frac{|\alpha| - k}{k}\frac{n}{2}|p(kz)|,$$

which is equivalent to

$$|np(kz) + (\alpha - kz)p'(kz)| \geqslant \frac{|\alpha| - k}{k}\frac{n}{2}|p(kz)|,$$

therefore, for any $\theta \in [0, 2\pi)$ and $r > 0$, we have

$$|D_\alpha p(ke^{i\theta})|^r \geqslant \Big(\frac{|\alpha| - k}{k}\frac{n}{2}\Big)^r |p(ke^{i\theta})|^r, 0 \leqslant \theta < 2\pi,$$

and hence

$$\bigg\{\int_0^{2\pi} |D_\alpha p(ke^{i\theta})|^r d\theta\bigg\}^{1/r} \geqslant \frac{|\alpha| - k}{k}\frac{n}{2}\bigg\{\int_0^{2\pi} |p(ke^{i\theta})|^r d\theta\bigg\}^{1/r}. \tag{4.5}$$

Since $R(z)$ has all its zeros in $|z| \leqslant 1$, then $\tilde{R}(z)$ is a polynomial of degree at most n having no zeros in $|z| < 1$ and applying Lemma 3.3 with $R = k \geqslant 1$ to $\tilde{R}(z)$, we get

$$\left\{\int_0^{2\pi}|\tilde{R}(ke^{i\theta})|^r d\theta\right\}^{1/r}\leqslant E_r\left\{\int_0^{2\pi}|\tilde{R}(e^{i\theta})|^r d\theta\right\}^{1/r}, \tag{4.6}$$

where

$$E_r=\left\{\int_0^{2\pi}|1+k^n e^{i\theta}|^r d\theta\right\}^{1/r}\left\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\right\}^{-1/r}.$$

Now, it can be easily obtained that $|\tilde{R}(ke^{i\theta})|=k^n|p(e^{i\theta})|$ and $|\tilde{R}(e^{i\theta})|=|p(ke^{i\theta})|$. With the above two relations, (4.6) takes the form

$$k^n\left\{\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{1/r}\leqslant E_r\left\{\int_0^{2\pi}|p(ke^{i\theta})|^r d\theta\right\}^{1/r}. \tag{4.7}$$

Since $D_\alpha p(z)$ is a polynomial of degree at most $(n-1)$, on applying Lemma 3.4 to $D_\alpha p(z)$ with $R=k\geqslant 1$, we have

$$\frac{1}{k^{n-1}}\left\{\int_0^{2\pi}|D_\alpha p(ke^{i\theta})|^r d\theta\right\}^{1/r}\leqslant\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^r d\theta\right\}^{1/r}. \tag{4.8}$$

Using (4.8) in (4.5), we get

$$k^{n-1}\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^r d\theta\right\}^{1/r}\geqslant\frac{|\alpha|-k}{k}\frac{n}{2}\left\{\int_0^{2\pi}|p(ke^{i\theta})|^r d\theta\right\}^{1/r}. \tag{4.9}$$

Combining (4.7) and (4.9), we have

$$\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^r d\theta\right\}^{1/r}\geqslant\frac{|\alpha|-k}{E_r}\frac{n}{2}\left\{\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{1/r},$$

which completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Without loss of generality, we can assume that $p(z)$ has all its zeros in $|z|<k$, $k\geqslant 1$, for if $p(z)$ has a zero on $|z|=k$, then $m=0$ and, in view of Theorem 2.1, the theorem follows trivially. Since $p(z)$ has all its zeros in $|z|<k$, $k\geqslant 1$, by Rouche's theorem, for every real or complex number β with $|\beta|<1$, the polynomial $F(z)=p(z)+\beta m$ also has all its zeros in $|z|<k, k\geqslant 1$. We apply Theorem 2.1 to the polynomial $F(z)$, thus for $|\beta|<1$ and any $r>0$,

$$\begin{aligned}\left\{\int_0^{2\pi}|D_\alpha\{p(e^{i\theta})+\beta m\}|^r d\theta\right\}^{1/r}&\\ \geqslant\frac{|\alpha|-k}{E_r}\frac{n}{2}&\times\left\{\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\right\}^{1/r},\end{aligned} \tag{4.10}$$

where E_r is as defined in Theorem 2.1. Also (4.10) is equivalent to

$$\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})+n\beta m|^r d\theta\right\}^{1/r}\geqslant\frac{|\alpha|-k}{E_r}\frac{n}{2}\left\{\int_0^{2\pi}|p(e^{i\theta})+\beta m|^r d\theta\right\}^{1/r},$$

and hence the proof of Theorem 2.2 is complete. □

Conclusion

Studying the extremal problems of functions of a complex variable and generalizing the classical polynomial inequalities is typical in geometric function theory. In the past few years, a series of papers related both to Bernstein and Turán-type inequalities have been published and significant advances in terms of extension, improvement as well as generalization have been achieved in different directions. One such generalization is replacing the sup-norm by a factor involving integral means. These types of inequalities are of interest both in mathematics and in the application areas such as physical systems. More precisely, the author contributes a vital work in establishing integral mean extension of some Turán-type inequalities for the polar derivatives of a class of polynomials by following some new approach.

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