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NEW FORMULAS FOR BERNOULLI POLYNOMIALS WITH APPLICATIONS OF MATRIX EQUATIONS AND LAPLACE TRANSFORM

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Abstract. We give a linear transformation on the polynomial ring of rational numbers. A matrix representation of this linear transformation based on standard bases is constructed. For some special cases of this matrix, matrix equations including inverse matrices related to the Bell polynomials and Diophantine equation are obtained. With the help of these equations, new formulas containing different polynomials with the Bernoulli polynomials are found. In order to compute these polynomials, a computational algorithm is given. Finally, by applying the Laplace transform to the generating function for the Bernoulli polynomials, we derive some novel formulas involving the Hurwitz zeta function and infinite series.

1. Introduction

It is well known that in recent years many different applications of linear transformations have been given not only in the algebraic field of mathematics but also in other applied sciences. To give an example of these different applications, geometric transformations implemented in computer graphics also occur; that is translation, rotation and scaling of 2D or 3D objects can be done using a transformation matrix. Therefore, linear transformations are also used as a tool to describe change. Many examples can be given for these, some of which are well known to be used in analysis as transformations corresponding to derivatives, transformations corresponding to integrals, or in relativity as a device to keep track of local transformations of reference frames. In addition, other implementation examples can be given, including compiler optimizations of nested loop code and parallelization of compiler techniques.

In recent years, it has been also seen that generating functions for the Bernoulli numbers and polynomials were given by different methods. For example, in complex

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analysis, generating functions of these numbers and polynomials can be constructed using the Cauchy derivative formula, Cauchy residue theorem and meromorphic functions. These numbers and polynomials are also used in algebraic topology and number theory, as criteria of regular prime numbers or their appearance in the Todd class which can be seen on complex vector bundles in topological space, the values of zeta functions on even integers, in Milnor's homotopy group related to the characteristic class. They can also appear in K-theory, the Euler-Maclaurin summation formula, Bessel functions, trigonometric functions, cylindrical functions, hypergeometric functions etc. In addition, it is also well-known that these numbers and polynomials are given by the Volkenborn integral on the set of p-adic integers. Perhaps their definitions are also given on other spaces or sets that we have not seen before. Therefore, the main motivation of this paper is to give different computational formulas of polynomials containing the Bernoulli polynomials using not only the linear transformation defined on the polynomials ring of rational numbers and its matrix equations, but also the Laplace transform and the Hurwitz zeta function.

We can use the following notations and definitions. Let $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote a set of positive integers, the ring of rational integers, a set of real numbers, and a set of complex numbers. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Bernoulli numbers and polynomials are respectively given by

$$F(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{1.1}$$

$$F(t, x) = e^{tx} F(t) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.2}$$

(cf. [**1**, **16**, **22**]).

The Stirling numbers of the first kind and the second kind, which are denoted by $S_1(n, k)$ and $S_2(m, n)$, respectively, are defined by

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n, k) \frac{t^n}{n!},$$

$$\frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} S_2(m, n) \frac{t^m}{m!} \tag{1.3}$$

(cf. [**5**, **16**, **22**]).

The array polynomials are defined by the following generating function:

$$\frac{(e^t - 1)^v e^{tx}}{v!} = \sum_{m=0}^{\infty} S_v^m(x) \frac{t^m}{m!} \tag{1.4}$$

(cf. [**3**, **4**, **17**]).

The results of this article, including all sections, are briefly stated as follows:

In Section 2, we defined $\mathbb{Q}$-linear transformation on the polynomial ring $\mathbb{Q}[x]$. By using this transformation, we give its matrix representation with respect to the basis $\{1, x, x^2, x^3, \dots\}$ of $\mathbb{Q}[x]$. By using matrix representation, we find some new classes of special polynomials involving Bernoulli polynomials and Bell polynomials. We also give a computational algorithm for the polynomials obtained with the help of this $\mathbb{Q}$-linear transformation.

In Section 3, we give linear transformation. By applying Cayley–Hamilton theorem, we also give inverse matrix formulas involving the Bell polynomials for this linear transformation. Using this inverse matrix formula, we derive matrix representation for the Bernoulli polynomials.

In Section 4, we give a derivative formula for linear transformation.

In Section 5, we define a new family of polynomials and its matrix. We give some applications of these polynomials. We also define a new family of polynomials by a matrix representation. By using this matrix with its inverse, we also derive the Bernoulli numbers.

In Section 6, applying the Laplace transform to generating function for the Bernoulli polynomials, we give not only infinite series representation for the Bernoulli polynomials, but also we derive some novel formulas including the Stirling numbers and the array polynomials.

2. A class of $\mathbb{Q}$-linear transformation on the polynomials ring $\mathbb{Q}[x]$

In this section, we define not only the following $\mathbb{Q}$-linear transformation on the polynomials ring $\mathbb{Q}[x]$, but also construct its matrix representation with respect to the basis $\{1, x, x^2, x^3, \dots\}$ of $\mathbb{Q}[x]$. By using this transformation and its matrix, we derive some new classes of special polynomials involving the Bernoulli polynomials, the Bell polynomials etc. We also give a computational algorithm for the polynomials obtained with the help of this $\mathbb{Q}$-linear transformation.

Let $\mathcal{L}_{a,b}^{c,d}\colon \mathbb{Q}[x] \to \mathbb{Q}[x]$, where $a, b, c, d \in \mathbb{Q}$. Note that since $\mathcal{L}_{a,b}^{c,d}$ is a linear transformation, $\mathcal{L}_{a,b}^{c,d}[0] = 0$. Let $P_n(x)$ be any polynomial defined on the polynomial ring $\mathbb{Q}[x]$. Then, we define the following generalized linear transformation on the polynomials ring $\mathbb{Q}[x]$:

$$\mathcal{L}_{a,b}^{c,d}[P_n(x)] = \int_{ax+b}^{cx+d} P_n(u)\, du, \tag{2.1}$$

Substituting $P_n(x) = x^n$ into (2.1), we obtain

$$\mathcal{L}_{a,b}^{c,d}[x^n] = \int_{ax+b}^{cx+d} u^n du = \frac{(cx+d)^{n+1} - (ax+b)^{n+1}}{n+1}. \tag{2.2}$$

Remark 2.1. Substituting $a = c = d = 1$ and $b = 0$ into (2.2), $\mathcal{L}_{1,0}^{1,1}[x^n]$ reduces to $\mathbb{Q}$-linear transformation, which is given by Arakawa et al. [**1**, p. 55].

Using (2.2) for any polynomial defined on the polynomial ring $\mathbb{Q}[x]$

$$P_n(x) = \sum_{j=0}^{n} \alpha_j x^j \tag{2.3}$$

we give the following formulas for the generalized linear transformation $\mathcal{L}_{a,b}^{c,d}$:

$$\begin{aligned}
\mathcal{L}_{a,b}^{c,d}[P_n(x)] &= \sum_{j=0}^{n}\binom{j+1}{0}\frac{\alpha_j}{j+1}(d^{j+1}-b^{j+1}) \\
&+\sum_{j=0}^{n}\binom{j+1}{1}\frac{\alpha_j}{j+1}(cd^j-ab^j)x \\
&+\sum_{j=1}^{n}\binom{j+1}{2}\frac{\alpha_j}{j+1}(c^2d^{j-1}-a^2b^{j-1})x^2 \\
&+\sum_{j=2}^{n}\binom{j+1}{3}\frac{\alpha_j}{j+1}(c^3d^{j-2}-a^3b^{j-2})x^3 \\
&+\sum_{j=3}^{n}\binom{j+1}{4}\frac{\alpha_j}{j+1}(c^4d^{j-3}-a^4b^{j-3})x^4+\dots \\
&+\sum_{j=k}^{n}\binom{j+1}{k-1}\frac{\alpha_j}{j+1}(c^{k+1}d^{j-k}-a^{k+1}b^{j-k})x^{k+1}+\cdots \\
&+\alpha_{n-1}\frac{\binom{n}{n}}{n}(c^nd-a^nb)x^n+\alpha_n\frac{\binom{n+1}{n+1}}{n+1}(c^{n+1}-a^{n+1})x^{n+1}.
\end{aligned} \tag{2.4}$$

With the aid of (2.4), $(n+1)\times(n+2)$ matrix representation of the generalized linear transformation $\mathcal{L}_{a,b}^{c,d}[P_n(x)]$ with respect to basis of $\{1, x, x^2, x^3, \dots\}$ is given by the following theorem.

THEOREM 2.1. *We have*

$$M\big\{\mathcal{L}_{a,b}^{c,d}[P_n(x)]\big\} = [M_{ij}], \tag{2.5}$$

where $i = 1, 2, 3, \dots, n+1$, $j = 0, 1, 2, \dots, n+1$, $M_{ij} = \binom{i}{j}\frac{c^jd^{i-j}-a^jb^{i-j}}{i}$ *for* $i \geqslant j$, *and* $M_{ij} = 0$ *for* $i < j$.

Substituting $a = c$ into (2.4), we get

$$\begin{aligned}
\mathcal{L}_{a,b}^{a,d}[P_n(x)] &= \sum_{j=0}^{n}\binom{j+1}{0}\frac{\alpha_j}{j+1}(d^{j+1}-b^{j+1}) + a\sum_{j=0}^{n}\binom{j+1}{1}\frac{\alpha_j}{j+1}(d^j-b^j)x \\
&+a^2\sum_{j=1}^{n}\binom{j+1}{2}\frac{\alpha_j}{j+1}(d^{j-1}-b^{j-1})x^2 \\
&+a^3\sum_{j=2}^{n}\binom{j+1}{3}\frac{\alpha_j}{j+1}(d^{j-2}-b^{j-2})x^3 \\
&+\cdots+a^n\alpha_{n-1}\frac{\binom{n+1}{n}}{n+1}(d-b)x^n.
\end{aligned}$$

Similarly to the above, the matrix representation with respect to the standard basis is given as follows

$$M\big\{\mathcal{L}_{a,b}^{a,d}[P_n(x)]\big\} = [M_{ij}], \tag{2.6}$$

where $i = 1, 2, 3, \dots, n+1$, $j = 0, 1, 2, \dots, n$, where $M_{ij} = \binom{i}{j} \frac{a^j(d^{i-j} - b^{i-j})}{i}$ for $i > j$, and $M_{ij} = 0$ for $i \leqslant j$.

It is noted that since all entries in the last column of matrix (2.5) are 0 except for one, this last column is omitted by taking $c = a$ and a new matrix is reduced to a square matrix.

Next, by (2.6), we define a new family of polynomials by the following definition.

Definition 2.1. Let $\det\left(M\{\mathcal{L}_{a,b}^{a,d}[P_n(x)]\}\right) \neq 0$; we define $Q_n(x; a, b, a, d)$ polynomial sequences with respect to basis $\{1, x, x^2, x^3, \dots\}$ of $\mathbb{Q}[x]$ by

$$\begin{bmatrix} Q_0(x; a, b, a, d) \\ Q_1(x; a, b, a, d) \\ Q_2(x; a, b, a, d) \\ \vdots \\ Q_{n-1}(x; a, b, a, d) \\ Q_n(x; a, b, a, d) \end{bmatrix} = M^{-1}\{\mathcal{L}_{a,b}^{a,d}[P_n(x)]\} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \\ x^n \end{bmatrix}, \tag{2.7}$$

where M^{-1} denotes inverse of the matrix M.

Example 2.1. Putting $n = 1$ in (2.7), we get

$$M\{\mathcal{L}_{a,b}^{c,d}[P_1(x)]\} = \begin{bmatrix} d-b & c-a & 0 \\ \frac{d^2-b^2}{2} & cd-ab & c^2-a^2 \end{bmatrix},$$

$$\det\left(M\{\mathcal{L}_{a,b}^{a,d}[P_1(x)]\}\right) = a(b-d)^2 = ab^2 - 2abd + ad^2,$$

$$M^{-1}\{\mathcal{L}_{a,b}^{a,d}[P_1(x)]\} = \begin{bmatrix} \frac{-1}{b-d} & 0 \\ \frac{(b+d)}{2(ab-ad)} & \frac{-1}{ab-ad} \end{bmatrix}.$$

We have

$$\begin{bmatrix} Q_0(x; a, b, a, d) \\ Q_1(x; a, b, a, d) \end{bmatrix} = \begin{bmatrix} \frac{-1}{b-d} & 0 \\ \frac{(b+d)}{2a(b-d)} & \frac{-1}{a(b-d)} \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}.$$

From the above equation, we get the following polynomials

$$Q_0(x; a, b, a, d) = -\frac{1}{b-d}, \qquad Q_1(x; a, b, a, d) = \frac{(b+d)}{2a(b-d)} - \frac{1}{a(b-d)}x.$$

Example 2.2. Putting $n = 2$ in (2.7), we get

$$M\{\mathcal{L}_{a,b}^{c,d}[P_2(x)]\} = \begin{bmatrix} (d-b) & (c-a) & 0 & 0 \\ \frac{(d^2-b^2)}{2} & (cd-ab) & \frac{(c^2-a^2)}{2} & 0 \\ \frac{(d^3-b^3)}{3} & (cd^2-ab^2) & (c^2d-a^2b) & \frac{(c^3-a^3)}{3} \end{bmatrix}.$$

By using inverse matrix method, we get

$$M^{-1}\{\mathcal{L}_{a,b}^{a,d}[P_2(x)]\} = \begin{bmatrix} \frac{-1}{b-d} & 0 & 0 \\ \frac{(b+d)}{2a(b-d)} & \frac{-1}{a(b-d)} & 0 \\ \frac{-\left(\frac{b^2+d^2}{6} + \frac{2bd}{3}\right)}{a^2(b-d)} & \frac{(b+d)}{a^2(b-d)} & \frac{-1}{a^2(b-d)} \end{bmatrix}.$$

We have

$$\begin{bmatrix} Q_0(x;a,b,a,d) \\ Q_1(x;a,b,a,d) \\ Q_2(x;a,b,a,d) \end{bmatrix} = \begin{bmatrix} \frac{-1}{b-d} & 0 & 0 \\ \frac{(b+d)}{2a(b-d)} & \frac{-1}{a(b-d)} & 0 \\ \frac{-\left(\frac{b^2+d^2}{6}+\frac{2bd}{3}\right)}{a^2(b-d)} & \frac{(b+d)}{a^2(b-d)} & \frac{-1}{a^2(b-d)} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}.$$

Therefore from the above, we get the following polynomials:

$$Q_0(x;a,b,a,d) = \frac{-1}{b-d}$$

$$Q_1(x;a,b,a,d) = \frac{(b+d)}{2a(b-d)} - \frac{1}{a(b-d)}x$$

$$Q_2(x;a,b,a,d) = \frac{-\left(\frac{b^2+d^2}{6}+\frac{2bd}{3}\right)}{a^2(b-d)} + \frac{(b+d)}{a^2(b-d)}x + \frac{-1}{a^2(b-d)}x^2.$$

2.1. Algorithm. Here we give an algorithm (Algorithm 1) for obtaining the polynomials $Q_n(x;a,b,a,d)$ polynomial sequence defined in Definition 2.1.

Algorithm 1 The following procedure `FIND_Q_POLY_SEQ` will return an $(n+1)\times 1$ column matrix whose entries are the terms of the polynomial sequence $\{Q_0(x;a,b,a,d), Q_1(x;a,b,a,d), Q_2(x;a,b,a,d), \ldots, Q_{n-1}(x;a,b,a,d), Q_n(x;a,b,a,d)\}$.

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procedure `FIND_Q_POLY_SEQ`(a, b, c, d : rational numbers, $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$: a list of n rational numbers)

  Local variables: n, `PolyP`

  Step 1: Assign initial values to local variables

  $n \leftarrow$ `size`$\{\alpha_0, \alpha_1, \ldots, \alpha_n\} - 1$

  `PolyP`$\leftarrow \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$ ▷ See (2.3)

  Step 2: Apply the generalized linear transformation $\mathcal{L}_{a,b}^{c,d}$ to the polynomial `PolyP`. ▷ See (2.1)

  Step 3: Write the result of Step 2 in the form provided by (2.4).

  Step 4: Create an $(n+1)\times(n+2)$ matrix representation with respect to basis of $\{1, x, x^2, x^3, \ldots\}$ from the form obtained in Step 3. ▷ See (2.5)

  Step 5: Substitute $a = c$ into the matrix obtained in Step 4, and by deleting its last column extract its $(n+1)\times(n+1)$ submatrix M. ▷ See (2.6)

  if $\det(M) \neq 0$ **then**

    Step 6: Find the inverse M^{-1} of the submatrix M.

    Step 7: Multiply the inverse matrix M^{-1} by the $(n+1)\times 1$ column matrix of the standard basis $\{1, x, x^2, \ldots, x^n\}$ ▷ See (2.7)

    Return the product matrix obtained in Step 7 as an $(n+1)\times 1$ column matrix of the polynomial sequence $\{Q_0(x;a,b,a,d), Q_1(x;a,b,a,d), Q_2(x;a,b,a,d), \ldots, Q_{n-1}(x;a,b,a,d),$ $Q_n(x;a,b,a,d)\}$. ▷ Definition 2.1

  else

    Return `exit`

  end if

end procedure

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3. Applications of the generalized linear transformation $\mathcal{L}_{a,b}^{c,d}[P_n(x)]$

In this section, we give some special values of the generalized linear transformation $\mathcal{L}_{a,b}^{c,d}[P_n(x)]$. By applying the Cayley–Hamilton theorem, we also give inverse matrix formulas involving the Bell polynomials. By aid of the inverse matrix formula, we give matrix representation for the Bernoulli polynomials.
Substituting $a = c$ and $d = b + 1$ into (2.4), we also get

$$\begin{aligned}(3.1)\qquad \mathcal{L}_{a,b}^{a,b+1}[P_n(x)] &= \sum_{j=0}^{n} \binom{j+1}{0} \frac{\alpha_j}{j+1} (((b+1)^{j+1} - b^{j+1}) \\ &+ a(j+1)((b+1)^j - b^j)x) \\ &+ a^2 \sum_{j=1}^{n} \binom{j+1}{2} \frac{\alpha_j}{j+1} ((b+1)^{j-1} - b^{j-1})x^2 + \dots \\ &+ a^n \sum_{j=n-1}^{n} \binom{j+1}{n} \frac{\alpha_j}{j+1} ((b+1)^{j-n+1} - b^{j-n+1})x^n.\end{aligned}$$

We assume that $j-n+1 \geqslant 0$, otherwise when $j-n+1 < 0$, these power's numbers omitted in the related sums.
With the aid of (3.1), $(n+1)\times(n+1)$ matrix representation of the linear transformation $\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]$ with respect to the basis of $\{1, x, x^2, x^3, \dots\}$ is given by the following corollary.

Corollary 3.1. *We have*

$$(3.2)\qquad M\big\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\big\} = [M_{ij}],$$

where $i = 1, 2, 3, \dots, n+1$, $j = 0, 1, 2, \dots, n$, $M_{ij} = \binom{i}{j} \frac{a^j((b+1)^{i-j} - b^{i-j})}{i}$ *for* $i > j$, *and* $M_{ij} = 0$ *for* $i \leqslant j$.

By applying the Cayley–Hamilton theorem to (3.2) and using inverse matrix formula, given in formula (cf. [**12**, (B.10) and (B.11)]), we show that inverse matrix of the matrix $M\big\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\big\}$ is given by the following theorem.

Lemma 3.1. *If* $c = a$, $M\big\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\big\}$ *is a diagonal* $n+1$ *dimension of matrix, then we have*

$$\begin{aligned}(3.3)\ M^{-1}\big\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\big\} &= \frac{1}{\det\big(M\big\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\big\}\big)} \sum_{j=0}^{n} M^j \big\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\big\} \\ &\times \sum_{a_1,a_2,\dots,a_n} \prod_{v=1}^{n} \frac{(-1)^{a_v+1}}{v^{a_v} a_v!} \big(\operatorname{tr}\big(M^v\big\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\big\}\big)\big)^{a_v}\end{aligned}$$

where $\det(M\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\})$ *and* $\operatorname{tr}(M^v\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\})$ *are the determinant and the trace of* $M^v\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\}$ *respectively, and* $\sum_{a_1,a_2,a_3,\dots,a_n}$ *is taken over* j *and the sets of all* $a_v \geqslant 0$ *satisfying* $j + \sum_{m=0}^{n} m.a_m = n$.

LEMMA 3.2. *If* $c = a$, $M\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\}$ *is a diagonal matrix, then*

$$M^{-1}\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\} = \frac{1}{\det\left(M\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\}\right)} \times \sum_{j=1}^{n} M^{j-1}\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\} \frac{(-1)^{n-1}}{(n-j)!} B_{n-j}(w_1, w_2, \ldots, w_{n-j}), \tag{3.4}$$

where $w_k = -(k-1)!\,\mathrm{tr}\left\{M^k\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\}\right\}$ *and* $B_n(w_1, w_2, \ldots, w_n)$ *denotes the* n*th complete exponential Bell polynomials, defined by* (*see* [**2**, **5**])

$$B_n(w_1, w_2, \ldots, w_n) = n! \sum_{n=1k_1+2k_2+\cdots+nk_n} \prod_{j=1}^{n} \frac{w_j^{k_j}}{(j!)^{k_j} k_j!}.$$

By using equations (3.3) and (3.4), we give a new family of polynomials by the following theorem.

THEOREM 3.1. *We have*

$$\begin{bmatrix} Q_0(x; a, b, a, b+1) \\ Q_1(x; a, b, a, b+1) \\ Q_2(x; a, b, a, b+1) \\ \vdots \\ Q_{n-1}(x; a, b, a, b+1) \\ Q_n(x; a, b, a, b+1) \end{bmatrix} = M^{-1}\{\mathcal{L}_{a,b}^{a,b+1}[P_n(x)]\} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \\ x^n \end{bmatrix}.$$

Under the conditions $a = 1, b = 0$ in equation (3.2), the Bernoulli polynomials are also given by the following corollary.

COROLLARY 3.2. *We have*

$$\begin{bmatrix} B_0(x) \\ B_1(x) \\ B_2(x) \\ \vdots \\ B_{n-1}(x) \\ B_n(x) \end{bmatrix} = M^{-1}\{\mathcal{L}_{1,0}^{1,1}[x^n]\} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \\ x^n \end{bmatrix}, \tag{3.5}$$

where

$$M^{-1}\{\mathcal{L}_{1,0}^{1,1}[x^n]\} = \sum_{j=0}^{n} \frac{M^j\{\mathcal{L}_{1,0}^{1,1}[x^n]\}}{\det(M\{\mathcal{L}_{1,0}^{1,1}[x^n]\})} \sum_{a_1, a_2, \ldots, a_n} \prod_{v=1}^{n} \frac{(-1)^{a_v+1}}{v^{a_v} a_v!} \left(\mathrm{tr}\left(M^v\{\mathcal{L}_{1,0}^{1,1}[x^n]\}\right)\right)^{a_v}.$$

REMARK 3.1. When $a = 1$ and $b = 0$, with the aid of (3.5), we see that

$$B_n(x) = \left(\mathcal{L}_{1,0}^{1,1}[x^n]\right)^{-1}$$

(*cf.* [**1**, p. 55]), where $B_n(x)$ is the inverse image under the $\mathcal{L}$ transformation of x^n.

4. Derivative formula for generalized linear transformation $\mathcal{L}_{a,b}^{c,d}[P_n(x)]$

Now, we give derivative formula of the linear transformation $\mathcal{L}_{a,b}^{c,d}[P_n(x)]$. Taking derivative of $\mathcal{L}_{a,b}^{c,d}[P_n(x)]$ with respect to x, we obtain

$$\begin{aligned}\frac{d}{dx}\left\{\mathcal{L}_{a,b}^{c,d}[P_n(x)]\right\} &= \sum_{j=0}^{n}\binom{j+1}{1}\frac{\alpha_j(cd^j-b^ja)}{j+1}\\&+2\sum_{j=1}^{n}\binom{j+1}{2}\frac{(c^2d^{j-1}-a^2b^{j-1})\alpha_j}{j+1}x\\&+3\sum_{j=2}^{n}\binom{j+1}{3}\frac{(c^3d^{j-2}-a^3b^{j-2})\alpha_j}{j+1}x^2\\&+4\sum_{j=3}^{n}\binom{j+1}{4}\frac{(c^4d^{j-3}-a^4b^{j-3})\alpha_j}{j+1}x^3\\&+\cdots+\sum_{j=k}^{n}\binom{j+1}{k-1}\frac{(k+1)(c^{k+1}d^{j-k}-a^{k+1}b^{j-k})\alpha_j}{j+1}x^k\\&+\cdots+(c^nd-a^nb)\alpha_{n-1}x^{n-1}+(c^{n+1}-a^{n+1})\alpha_nx^n.\end{aligned}$$

Putting $a=c=d=1$ and $b=0$ in the above equation, we have

$$\begin{aligned}\frac{d}{dx}\left\{\mathcal{L}_{1,0}^{1,1}[P_n(x)]\right\} &= \sum_{j=0}^{n}\binom{j+1}{1}\frac{\alpha_j}{j+1}+2\sum_{j=1}^{n}\binom{j+1}{2}\frac{\alpha_j}{j+1}x\\&+3\sum_{j=2}^{n}\binom{j+1}{3}\frac{\alpha_j}{j+1}x^2+4\sum_{j=3}^{n}\binom{j+1}{4}\frac{\alpha_j}{j+1}x^3\\&+\cdots+\sum_{j=k}^{n}\binom{j+1}{k-1}\frac{(k+1)\alpha_j}{j+1}x^k+\cdots+\alpha_{n-1}x^{n-1}.\end{aligned}$$

Combining (3.5) with the above equation and its matrix representation, after some elementary calculations, we have the following well-known derivative formula for the Bernoulli polynomials: $\frac{d}{dx}\{B_n(x)\}=nB_{n-1}(x)$ (cf. [**22**]).

5. Another family of polynomials and their matrix representation

In this section, in line with the method of the previous section, we give another new family of polynomials. By using these polynomials, we derive a matrix representation of the coefficients of these polynomials. We show that special values of this matrix with its inverse produce the Bernoulli numbers.

By using (2.2), we set the following polynomials:

$$\begin{aligned}Y_n(x;a,b,c,d) &:= (cx+d)^{n+1}-(ax+b)^{n+1}\\&=\sum_{j=0}^{n+1}\binom{n+1}{j}(c^jd^{n+1-j}-a^jb^{n+1-j})x^j.\end{aligned}\tag{5.1}$$

Due to the generalized linear transformation given in (2.2), the above relation can be easily represented by the following integral formula:

$$Y_n(x;a,b,c,d) = (n+1)\mathcal{L}_{a,b}^{c,d}[x^n] = (n+1)\int_{ax+b}^{cx+d} u^n du.$$

By (5.1), some values of $Y_n(x;a,b,c,d)$ are computed as follows:

$$\begin{aligned}
Y_0(x;a,b,c,d) &= (c-a)x+d-b\\
Y_1(x;a,b,c,d) &= (c^2-a^2)x^2+2(cd-ab)x+(d^2-b^2)\\
Y_2(x;a,b,c,d) &= (c^3-a^3)x^3+3(c^2d-a^2b)x^2+3(cd^2-ab^2)x+(d^3-b^3)\\
&\vdots\\
Y_n(x;a,b,c,d) &= (c^{n+1}-a^{n+1})x^{n+1}+\binom{n+1}{n}(c^nd-a^nb)x^n+\cdots\\
&\quad+(d^{n+1}-b^{n+1}).
\end{aligned}$$

We now define matrix representation by aid of coefficients of the above polynomials as follows:

$$M\{Y_n(x;a,b,c,d)\} = [M_{ij}], \tag{5.2}$$

where $i = 1,2,3,\ldots,n+1$, $j = 0,1,2,\ldots,n+1$, $M_{ij} = \binom{i}{j}(c^jd^{i-j} - a^jb^{i-j})$ for $i \geqslant j$, and $M_{ij} = 0$ for $i < j$. Thus we get

$$\begin{bmatrix} Y_0(x;a,b,a,d)\\ Y_1(x;a,b,a,d)\\ Y_2(x;a,b,a,d)\\ \vdots\\ Y_{n-1}(x;a,b,a,d)\\ Y_n(x;a,b,a,d)\end{bmatrix} = M^{-1}\{Y_n(x;a,b,a,d)\}\begin{bmatrix}1\\ x\\ x^2\\ \vdots\\ x^{n-1}\\ x^n\end{bmatrix}.$$

The first row of the inverse of the matrix $M\{Y_n(x;a,b,a,d)\}$ given by equation (5.2) can also be considered as the generating function for a special family of numbers, including Bernoulli numbers maybe other certain family of special numbers. That is, substituting $a = c = d = 1$ and $b = 0$ into inverse matrix $M^{-1}\{Y_n(x;1,0,1,1)\}$, given by equation (5.2), all entries of the matrix $M^{-1}\{Y_n(x;1,0,1,1)\}$ are reduced to the Bernoulli numbers, respectively.

Some well-known examples are given as follows:

$$M\{Y_6(x;1,0,1,1)\} = \begin{bmatrix}1&0&0&0&0&0&0\\1&2&0&0&0&0&0\\1&3&3&0&0&0&0\\1&4&6&4&0&0&0\\1&5&10&10&5&0&0\\1&6&15&20&15&6&0\\1&7&21&35&35&21&7\end{bmatrix}.$$

By using (3.3) inverse matrix method we get

$$M^{-1}\left\{Y_6(x;1,0,1,1)\right\}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0\\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0\\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} & 0 & 0\\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & -\frac{1}{2} & \frac{1}{6} & 0\\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{7} \end{bmatrix},$$

where we obtained the Bernoulli numbers up to $n=6$ in the first column of the $M\{Y_6(x;a,b,a,d)\}$ matrix. For details, see A027642; and also see [**9**–**21**]. On the other hand, other columns or rows of the above matrix also represent different number families. These number sequences may be also in the class of other known families of special number. For this, it is recommended to examine Sloane's On-Line Encyclopedia of Integer Sequences (OEIS). See, for details, [**20**].

6. Applications of the Laplace transform to generating function for the Bernoulli polynomials

In this section, by applying the Laplace transform and the Mellin transformation to (1.2), we give a relation between the Bernoulli polynomials and the Hurwitz zeta function. We also give infinite series representation involving the Bernoulli polynomials.

Let $x\in\mathbb{C}$ with $x=a+ib$ and $\bar{x}=a-ib$ with $a>0$ and $b>0$. We set

$$F(-u,\bar{x})=\sum_{n=0}^{\infty}(-1)^n B_n(a-ib)\frac{u^n}{n!}.$$

Therefore

$$\frac{ue^{-ua}}{1-e^{-u}}=e^{-uib}\sum_{n=0}^{\infty}(-1)^n B_n(a-ib)\frac{u^n}{n!}, \tag{6.1}$$

which can be also written as

$$e^{-ua}\sum_{n=1}^{\infty}\frac{(1-e^{-u})^{n-1}}{n}=e^{-uib}\sum_{n=0}^{\infty}(-1)^n B_n(a-ib)\frac{u^n}{n!}. \tag{6.2}$$

Combining the above equation with (1.3), we get

$$\sum_{m=0}^{\infty}\sum_{v=0}^{m}\binom{m}{v}\sum_{n=1}^{v}\frac{(-1)^{m+n-1}(n-1)!a^{m-v}S_2(v,n-1)}{n}\frac{u^m}{m!}$$
$$=\sum_{m=0}^{\infty}(-1)^m\sum_{v=0}^{m}\binom{m}{v}(ib)^{m-v}B_v(a-ib)\frac{u^m}{m!}.$$

Substituting the following well-known formula into the right-hand side of the above equation

$$\sum_{v=0}^{m}\binom{m}{v}x^{m-v}B_v(y)=B_m(x+y),$$

we obtain

$$\sum_{m=0}^{\infty}\sum_{v=0}^{m}\binom{m}{v}\sum_{n=1}^{v}\frac{(-1)^{m+n-1}(n-1)!a^{m-v}S_2(v,n-1)}{n}\frac{u^m}{m!}=\sum_{m=0}^{\infty}(-1)^m B_m(a)\frac{u^m}{m!}.$$

By comparing the coefficients $\frac{u^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

THEOREM 6.1. *Let $m \in \mathbb{N}$. Then we have*

$$B_m(a)=\sum_{v=1}^{m}\binom{m}{v}\sum_{n=1}^{v}\frac{(-1)^{n-1}(n-1)!a^{m-v}S_2(v,n-1)}{n}. \tag{6.3}$$

By combining (1.4) with (6.2), by the same method of Cakić and Milovanović [**3**], Chang and Ha [**4**], Simsek [**17**], we have

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n-1)!}{n}S_{n-1}^{m}(a)\frac{(-u)^m}{m!}=\sum_{m=0}^{\infty}\sum_{v=0}^{m}\binom{m}{v}(ib)^{m-v}B_v(a-ib)\frac{(-u)^m}{m!}.$$

After some calculation, we have the following result, which was also proved in [**4**]:

$$\sum_{n=1}^{m}\frac{(-1)^{n-1}(n-1)!}{n}S_{n-1}^{m}(x)=\sum_{v=0}^{m}\binom{m}{v}(ib)^{m-v}B_v(a-ib). \tag{6.4}$$

Combining the above equation with (6.3) yields

$$\sum_{v=1}^{m}\sum_{n=1}^{v}\frac{(-1)^{n-1}\binom{m}{v}(n-1)!a^{m-v}S_2(v,n-1)}{n}=\sum_{n=0}^{m}\frac{(-1)^n n!}{n+1}S_n^m(x). \tag{6.5}$$

Combining (6.4) and (6.5) with

$$\frac{(-1)^n n!}{n+1}=\sum_{v=0}^{n}S_1(n,v)B_v,$$

(cf. [**5**, **11**, **16**, **19**]), we arrive at the following theorem:

THEOREM 6.2. *Let $m \in \mathbb{N}_0$. Then we have*

$$B_m(a)=\sum_{n=0}^{m}\sum_{v=0}^{n}S_1(n,v)B_vS_n^m(a),$$

$$\sum_{v=0}^{m}\binom{m}{v}(ib)^{m-v}B_v(a-ib)=\sum_{n=0}^{m}\sum_{v=0}^{n}S_1(n,v)B_vS_n^m(a).$$

We note that a different proof of (6.5) was also given by Chang and Ha [**4**].

By applying the Laplace transform to generating functions involving Bernoulli polynomials, recently many interesting studies have been published (cf. [**7**, **8**, **10**, **22**]). Therefore, by using (6.1), we have

$$\sum_{n=0}^{\infty}ue^{-u(a+n)}=e^{-uib}\sum_{n=0}^{\infty}(-1)^nB_n(a-ib)\frac{u^n}{n!}.$$

By integrating the above equation with respect to u from 1 to ∞, we get

$$\sum_{n=0}^{\infty}\int_0^{\infty} ue^{-u(a+n)}du=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}B_n(a-ib)\int_0^{\infty}u^n e^{-ibu}du \tag{6.6}$$

with $a>0$. Then, applying the Laplace transform of the function $g(u)=u^n$:

$$\mathcal{L}\{g(u)\}=\frac{n!}{y^{n+1}}$$

(where $y>0$) to both sides of (6.6), we obtain the following theorem:

Theorem 6.3. *Let $b>0$ and $a>0$. Then we have*

$$\zeta(2,a)=\sum_{n=0}^{\infty}\frac{i^{n-1}B_n(a-ib)}{b^{n+1}}. \tag{6.7}$$

By using (6.1), we have

$$ue^{-au}=(e^{-ibu}-e^{-(1+ib)u})\sum_{n=0}^{\infty}(-1)^n B_n(a-ib)\frac{u^n}{n!}.$$

Applying the Laplace transform to the above equation, for $a>0$ and $b>0$, we get

$$\begin{aligned}\int_0^{\infty} ue^{-au}du=&\sum_{n=0}^{\infty}(-1)^n B_n(a-ib)\frac{1}{n!}\int_0^{\infty}u^n e^{-ibu}du\\&-\sum_{n=0}^{\infty}(-1)^n B_n(a-ib)\frac{1}{n!}\int_0^{\infty}u^n e^{-(1+ib)u}du.\end{aligned}$$

After some calculations, for $b>1$, we arrive at the following theorem:

Theorem 6.4. *Let $a>0$ and $b>1$. Then we have*

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^n\binom{n}{j}\frac{((1+ib)^{n+1}-(ib)^{n+1})(a-ib)^{n-j}B_j}{(ib-b^2)^{n+1}}=\frac{1}{a^2}. \tag{6.8}$$

Remark 6.1. Putting $n=1$ into equation (17) in [**7**], a relationship can be established with equation (6.8).

Substituting $a=1$ into (6.7), and combining with the following known formula

$$\zeta(2):=\zeta(2,1)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},$$

we get the following corollary:

Corollary 6.1. *Let $b>1$ and $a>0$. Then we have*

$$\sum_{n=0}^{\infty}\frac{i^{n-1}B_n(a-ib)}{b^{n+1}}=\frac{\pi^2}{6}.$$

Replacing t, x by $-t$, $y-x$ in (1.2) yields

$$F(-t, y-x) = \frac{-te^{-t(y-x)}}{e^{-t}-1} = \sum_{n=0}^{\infty}(-1)^n B_n(y-x)\frac{t^n}{n!}.$$

Taking partial derivative k times with respect to y in the above equation, we get

$$\frac{\partial^k}{\partial y^k}\{F(-t, y-x)\} = \frac{(-1)^{k+1}t^{k+1}e^{-t(y-x)}}{e^{-t}-1}.$$

After some calculations, we obtain

$$\frac{(-1)^{k+1}t^{k+1}e^{-ty}}{e^{-t}-1} = e^{-tx}\sum_{n=0}^{\infty}\frac{(-1)^n \frac{\partial^k}{\partial y^k}\{B_n(y-x)\}}{n!}t^n.$$

Integrating the above equation with respect to u from 1 to ∞, we get

$$(-1)^{k+1}\int_0^{\infty}\frac{t^{k+1}e^{-ty}}{e^{-t}-1}dt = \sum_{n=0}^{\infty}\frac{(-1)^n \frac{\partial^k}{\partial y^k}\{B_n(y-x)\}}{n!}\int_0^{\infty}t^n e^{-tx}dt,$$

which yields

$$(-1)^{k+1}(k+1)!\sum_{n=0}^{\infty}\frac{1}{(y+n)^{k+2}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{x^{n+1}}\frac{\partial^k}{\partial y^k}\{B_n(y-x)\}.$$

After some calculations, we arrive at the following theorem:

THEOREM 6.5. *Let $k \in \mathbb{N}_0$. Let $y > 0$. Then we have*

$$\zeta(k+2, y) = \frac{(-1)^{k+1}}{(k+1)!}\sum_{n=0}^{\infty}\frac{(-1)^n}{x^{n+1}}\frac{\partial^k}{\partial y^k}\{B_n(y-x)\}. \tag{6.9}$$

Substituting $k = 0$ into (6.9), we arrive at (6.7).
Let the ratio of two convergent series be given such that when the first term of the series in the denominator is different from zero, using (1.1), we have the following result:

$$\frac{1 + 0t + 0t^2 + \cdots}{1 + \frac{1}{2!}t + \frac{1}{3!}t^2 + \frac{1}{4!}t^3 + \cdots} = \sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

From the above equation, we get

$$1 = \Big(1 + \frac{1}{2!}t + \frac{1}{3!}t^2 + \frac{1}{4!}t^3 + \cdots\Big)\Big(B_0 + \frac{B_1}{1!}t + \frac{B_2}{2!}t^2 + \frac{B_3}{3!}t^3 + \cdots\Big).$$

Using the Cauchy product rule on the right-hand side of the above equation yields an infinite system of linear equations with unknown B_k. So, this system is of a special form since every $k \in \mathbb{N}_0$ the first $k+1$ equations contain only the first $k+1$ unknowns B_k. Therefore, the solution of the system of linear equations is given by the following determinant, by which the Bernoulli numbers are calculated outside the influence of their generating function:

$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & \frac{1}{2!} \end{vmatrix}$$

(cf. [**14**, p. 149], [**6**]).

7. Conclusion

We defined $\mathbb{Q}$-linear transformation on the polynomials ring $\mathbb{Q}[x]$. By using this transformation, we gave its matrix representation with respect to the basis $\{1, x, x^2, x^3, \dots\}$ of $\mathbb{Q}[x]$. Using matrix representation, we obtained some new classes of special polynomials involving Bernoulli polynomials and Bell polynomials. We also defined other linear transformations. By applying the Cayley–Hamilton theorem, we gave their inverse matrix formulas involving the Bell polynomials. Using this inverse matrix formula, we gave matrix representation for the Bernoulli polynomials. We gave a computational algorithm for the polynomials obtained with help of this $\mathbb{Q}$-linear transformation. Moreover, applying the Laplace transform to generating function for the Bernoulli polynomials, we obtained both infinite series representation for the Bernoulli polynomials and many new formulas associated with the Stirling numbers and the array polynomials.
The results of this article may potentially be used both in applied sciences and in many branches of mathematics.

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