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A NOTE ON CANCELLATION LAW FOR SOME CLASS OF UNBOUNDED SETS

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Abstract. The cancellation law plays a crucial role in possibility of embedings an algebraical structures into another richer structures such as semigroup into group, cone into linear space etc. We give some results concerning the cancellation law for convex sets. In particular in Theorem 2.1 we give some version of cancellation law for unbounded sets of topological vector space.

1. Introduction

The cancellation law plays a very important role in possibility of embedings an algebraical structures into another richer structures such as semigroup into a group, cone into linear space etc. (see [**5**, **6**]). Some generalizations of classical cancellation law for convex and compact sets are given in [**2**–**4**]. In this short note we give some results concerning the cancellation law for convex sets related to results given in [**1**, **2**]. In particular in Theorem 2.1 we give a version of cancellation law for unbounded sets of topological vector space.

Let X be a real topological vector space and let $A \subset X$. By *recession cone* of the set A we denote a set

$$\operatorname{recc}(A) = \{x \in X : tx + A \subset A \text{ for all } t > 0\}.$$

It is easy to observe that for any subset A of X the recession cone is a convex cone.

If the set A is convex, then $\operatorname{recc}(A) = \{x \in X : x + A \subset A\}$. It is well known that if the set A is bounded, then $\operatorname{recc}(A) = \{0\}$. Moreover if the set A is convex and the space is finite dimensional, then the converse is also true namely the condition $\operatorname{recc}(A) = \{0\}$ implies that A is bounded. But in infinite dimensional Banach spaces there always exists a closed convex and unbounded set such that $\operatorname{recc}(A) = \{0\}$ as it was shown in [**2**].

In this paper by the *asymptotic cone* of the set A we understand a set

$$A_\infty = \{x \in X : \text{ there exists } t_n > 0, t_n \to 0, x_n \in A \text{ such that } t_n x_n \to x\}.$$

It can be proved that if the set A is closed and convex then $A_\infty = \operatorname{recc}(A)$.

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We say that a subset B of X *absorbs* a subset A of X if for any $a \in A$ there exists $\lambda > 0$ such that $\lambda a \in B$. A subset A of X is called a *sequentially compact* if for any sequence $(x_n) \subset A$ there exists a subsequence $(x_{n_k}) \subset A$ convergent to some $x \in A$.

Urbański in [**7**] proved the following version of order cancellation law for convex subsets of topological vector space.

THEOREM 1.1. *Let X be a real topological vector space and let $A, B, C \subset X$. If B is bounded and C is closed and convex, then the inclusion $A + B \subset B + C$ implies that $A \subset C$.*

It is easy to observe that the above theorem is equivalent to the following.

THEOREM 1.2. *Let X be a real topological vector space and let $B, C \subset X$. If B is bounded and C is closed and convex then the inclusion $B \subset B + C$ implies that $0 \in C$.*

In this note we give some version of cancellation law in the case when the set B is unbounded which generalizes some results given in [**2**].

2. The Main Result

Now we prove the following theorem which is a version of cancellation law for unbounded sets.

THEOREM 2.1. *Let X be a topological vector space and $B, C \subset X$. If the set B is bounded and C is closed and convex or*

(a) *the set B is unbounded,*
(b) *the set C is closed and convex,*
(c) $(-B)_\infty \cap C_\infty = \{0\}$,
(d) *there exists a sequentially compact set $D \subset X, 0 \notin D$ such that the set D absorbs the set C,*

then from the inclusion $B \subset B + C$ follows that $0 \in C$.

PROOF. Take any $b_1 \in B$; then from the inclusion $B \subset B + C$ we get that there exists $b_2 \in B$ and $c_1 \in C$ such that $b_1 = b_2 + c_1$, and similarly for any $n \in \mathbb{N}$ from the inclusion $B \subset B + C$ we obtain the existence of elements $b_{n+1} \in B$ and $c_n \in C$ such that $b_n = b_{n+1} + c_n$. By summing these equalities we get the following

$$b_1 + b_2 + \cdots + b_n = b_2 + \cdots + b_{n+1} + c_1 + \cdots + c_n,$$

and hence $b_1 - b_{n+1} = c_1 + \cdots + c_n$. By dividing the above equality by n we obtain that $\frac{b_1}{n} - \frac{b_n}{n} = \frac{1}{n}(c_1 + \cdots + c_n)$. Now we consider two cases:

Case 1. The sequence (b_n) is bounded. In this case the sequence

$$z_n = \frac{1}{n}(c_1 + \cdots + c_n) = \frac{b_1}{n} - \frac{b_n}{n}$$

is convergent to 0 and since C is closed and convex therefore $0 \in C$.

Observe that if the set B is bounded, then only case 1 can hold.

Case 2. The sequence (b_n) is unbounded. Since there exists an sequentially compact set D, $0 \notin D$ that absorbs the set C, therefore there exists real numbers $\lambda_n > 0, \lambda_n \to 0$, such that $\lambda_n z_n \in D$. From the sequential compactness of the set D there exists subsequence (n_k) and $d \in D$, such that $\lambda_{n_k} z_{n_k} \to d \neq 0$ but then $d \in C_\infty$ and

$$d = \lim_{k\to\infty} \lambda_{n_k} z_{n_k} = \lim_{k\to\infty} \Big(\frac{\lambda_{n_k}}{n_k} b_1 - \frac{\lambda_{n_k}}{n_k} b_{n_k} \Big) = \lim_{k\to\infty} \frac{\lambda_{n_k}}{n_k} (-b_{n_k}) \in (-B)_\infty$$

but this contradicts to the assumption (c). □

REMARK 2.1. In the case when $(X, \|\cdot\|)$ is a finite dimensional Banach space, the set $D = \{x \in X : 1 \leqslant \|x\| \leqslant 2\}$ is sequentially compact set that absorbs the whole space except zero, the assumption (d) can be omitted. Therefore in finite dimensional topological vector space X the following property is true:

Let $A, B, C \subset X$. If the set B is bounded and C is closed and convex or

(a) *the set B is unbounded,*
(b) *the set C is closed and convex,*
(c) $(-B)_\infty \cap C_\infty = \{0\}$,

then the inclusion $A + B \subset B + C$ implies that $A \subset C$.

REMARK 2.2. If $(X, \|\cdot\|)$ is a reflexive Banach space then the closed unit ball is weakly sequentially compact (see [**1**]). So if we consider closed and convex subset C of $(X, \|\cdot\|)$ as a subset of topological vector space (X, τ), where τ is the weak topology on X, then the set is still closed and convex in this topological vector space. Moreover if the set $C_p = \overline{\text{conv}}\{\frac{x}{\|x\|} : x \in C\}$ (where $\overline{\text{conv}}$ denotes the closed convex envelope) can be separated by hyperplane from zero, then it is a sequentially compact set that absorbs the set C and zero does not belong to it. Therefore if X is a reflective space, then the following property is true:

Let $A, B, C \subset X$. If the set B is bounded and C is closed and convex or

(a) *the set B is unbounded,*
(b) *the set C is closed convex subset of X and the set $C_p = \overline{\text{conv}} \left\{ \frac{x}{||x||} : x \in C \right\}$ can by separated by hyperplane from zero,*
(c) $(-B)_\infty \cap C_\infty = \{0\}$,

then the inclusion $A + B \subset B + C$ implies that $A \subset C$.

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