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PSEUDO-LINEAR COMBINATION OF FUZZY METRICS

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Abstract. We explore a new fuzzy metric constructed from already defined fuzzy metrics over the same set using pseudo-linear combination. Operations used in pseudo-linear combination are triangular norm and conorm. The fuzzy space thus obtained is proved to be complete. Additional features related to this space are also presented. A fuzzy metric obtained in this way can be used to construct an image denoising procedure, from the fuzzy metrics used for the spatial distance and the color similarity measure between the pixels in the image. The goal is to enhance the sharpness and quality of the image, expressed and measured by the image quality index.

1. Introduction

The advent of Zadeh's theory of fuzzy sets in 1965 produced an extraordinary number of researches studies in the literature on fuzziness. It is safe to say that this concept is applied in all fields of science and technology. The concept of fuzzy metric space was first defined in 1975 by Kramosil and Mihalek [**8**]. Unlike classic metric space, the distance between two objects is not expressed as a definite real number in fuzzy metric spaces. Given that one fuzzy metric does not generate Hausdorff topology, George and Veeramani [**1**–**3**], introduced a new definition of fuzzy metrics by inserting more stringent conditions into the existing definition.

By modifying this approach, in the theory of fuzzy metric spaces, the results published by the group of authors Gregory, Morillas, Sapena, Romaguera [**4**–**6**] play a significant role. The work of Ralević, Karaklić, Pištinjat [**13**] also belongs to that class of works that introduce the notions T and S fuzzy metric spaces.

The second section contains a list of known notions and their properties, which are used in other sections. The term fuzzy metric space and the examples we use in applications are in the third section. It also looks at the main result regarding the pseudo-linear combination of fuzzy metrics and the structure of the corresponding space. A crucial part of image processing is removing noise from the image. Recently there has been a proliferation of filters that employ fuzzy metrics. One such

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is presented in the fourth section and is based on a pseudo-linear combination of the distances used in that area.

2. Preliminaries

Some of the properties of the norms (triangular norms and conorms) are listed (see e.g. Klir, Yuan [**7**]) to make it easier to follow the topic discussed in the paper. The concept of pseudo linear combination was also introduced, superdistributiveness and subdistributiveness (i.e., superdistributivity and subdistributivity) of norms and it is demonstrated that the class of such norms is not empty.

Definition 2.1. The binary operation $N\colon [0,1]^2 \to [0,1]$ is a *norm*, if N is nondecreasing (in both components), commutative and associative and has a neutral element $e \in \{0,1\}$.

If $e=1$, then N is the *triangular norm* (shorter t-norm), and instead of N we write T. If $e=0$, then N is the *triangular conorm* (shorter t-conorm), and instead of N we write S.

Example 2.1. The most used t-conorms are:

standard union $S_M(a,b) = \max(a,b) = a \vee b$;
probabilistic sum $S_P(a,b) = a+b-ab$;
bounded sum $S_L(a,b) = \min(1, a+b)$;
drastic union $S_W(a,b) = \left\{\begin{smallmatrix} \max(a,b), & \min(a,b)=0 \\ 1, & \text{else} \end{smallmatrix}\right.$,

and to them corresponding t-norms:

standard intersection $T_M(a,b) = \min(a,b) = a \wedge b$;
algebraic product $T_P(a,b) = ab$;
finite difference $T_L(a,b) = \max(0, a+b-1)$;
drastic intersection $T_W(a,b) = \left\{\begin{smallmatrix} \min(a,b), & \max(a,b)=1 \\ 0, & \text{else} \end{smallmatrix}\right.$.

For all $a, b \in [0,1]$

$$T(a,b) \leqslant \min\{a,b\}, \quad S(a,b) \geqslant \max\{a,b\},$$

holds. Clearly, $T(a,a) \leqslant a$, $S(a,a) \geqslant a$ for all $a \in [0,1]$.

A norm N is an *Archimedean* norm if $N(a,a) < a$ for all $a \in (0,1)$, for t-norm N and for all $a \in (0,1)$, $N(a,a) > a$ for t-conorm N. If $N(a,a) = a$ for all $a \in [0,1]$, then N is *idempotent* norm N.

If, in the definition of the norm, instead of the axiom of monotonicity, a strict monotonicity is valid, i.e., $a_1 < a_2 \wedge b_1 < b_2 \Rightarrow N(a_1,b_1) < N(a_2,b_2)$, for all $a_1, a_2, b_1, b_2 \in [0,1]$, then the norm is *strict.*

Definition 2.2. *The power of the norm* N is given by

$$N^1(a_1,a_2) = N(a_1,a_2), \quad N^n(a_1,\dots,a_n,a_{n+1}) = N(N^{n-1}(a_1,\dots,a_n),a_{n+1}) \ \ (n \geqslant 2).$$

Definition 2.3. *The power of element* (with respect norm N) $a \in [0,1]$ is defined by: $a^{(1)} = a$, $a^{(2)} = N(a,a)$, $a^{(n)} = N(a^{(n-1)},a)$ $(n \geqslant 3)$.

The following properties are easy to check (see [**12**]).

REMARK 2.1. If N is a norm, then N^n is nondecreasing, commutative and associative operation. If N is a strict norm, then N^n is an increasing function.

REMARK 2.2. If T is a t-norm, then:

$$T(a_1, a_2) = 1 \Leftrightarrow a_1 = a_2 = 1,$$
$$T^n(a_1, a_2, \ldots, a_{n+1}) = 1 \Leftrightarrow a_1 = \cdots = a_{n+1} = 1.$$

REMARK 2.3. If T is a strict t-norm, then:

$$T(a_1, a_2) = 0 \Leftrightarrow a_1 = 0 \vee a_2 = 0,$$
$$T^n(a_1, a_2, \ldots, a_{n+1}) = 0 \Leftrightarrow a_1 = 0 \vee \cdots \vee a_{n+1} = 0.$$

REMARK 2.4. If S is a t-conorm, then:

$$S(a_1, a_2) = 0 \Leftrightarrow a_1 = a_2 = 0,$$
$$S^n(a_1, a_2, \ldots, a_{n+1}) = 0 \Leftrightarrow a_1 = \cdots = a_{n+1} = 0.$$

REMARK 2.5. If S is a strict t-conorm, then:

$$S(a_1, a_2) = 1 \Leftrightarrow a_1 = 1 \vee a_2 = 1,$$
$$S^n(a_1, a_2, \ldots, a_{n+1}) = 1 \Leftrightarrow a_1 = 1 \vee \cdots \vee a_{n+1} = 1.$$

REMARK 2.6. From associativity and commutativity of N^{n-1}, we have

$$N^{n-1}(a_1, \ldots, a_n) = N^{k-1}\big(N^{n_1-1}(a_{p(1)}, \ldots, a_{p(n_1)}), \\ N^{n_2-1}(a_{p(n_1+1)}, \ldots, a_{p(n_1+n_2)}), \ldots, N^{n_k-1}(a_{p(n_1+\cdots+n_{k-1}+1)}, \ldots, a_{p(n)})\big),$$

where p is arbitrary permutation of set $I = \{1, \ldots, n\}$ and $n_1 + \cdots + n_k = n$.

So, $N^0(a) = a$, $a^{(2)} = N(a, a)$, $a^{(3)} = N(a^{(2)}, a) = N(N(a, a), a) = N^2(a, a, a)$, $a^{(4)} = N(a^{(3)}, a) = N(N^2(a, a, a), a) = N^3(a, a, a, a)$, $\ldots$,

$$\begin{aligned} a^{(n)} &= N^{n-1}(a, \ldots, a) \\ &= N^{k-1}(N^{n_1-1}(a, \ldots, a), N^{n_2-1}(a, \ldots, a), \ldots, N^{n_k-1}(a, \ldots, a)) \\ &= N^{k-1}(a^{(n_1)}, a^{(n_2)}, \ldots, a^{(n_k)}). \end{aligned}$$

DEFINITION 2.4. Let $N_i \colon [0,1]^2 \to [0,1]$, $i = 1, 2$, be two norms. We say that the *subdistributivity* of N_1 according to N_2 is valid, if for all $a, b, c \in [0,1]$:

$$N_2(N_1(a, b), N_1(a, c)) \geqslant N_1(a, N_2(b, c)).$$

DEFINITION 2.5. Let $N_i \colon [0,1]^2 \to [0,1]$, $i = 1, 2$, be two norms. We say that the *superdistributivity* of N_1 according to N_2 is valid, if for all $a, b, c \in [0,1]$:

$$N_2(N_1(a, b), N_1(a, c)) \leqslant N_1(a, N_2(b, c)).$$

LEMMA 2.1. *If S is a t-conorm, then the subdistributivity of operation* min *according to S hold, i.e.,* $S(\min\{a, b\}, \min\{a, c\}) \geqslant \min\{a, S(b, c)\}$.

PROOF. Because of commutativity of the operations min and S, it is sufficient to examine three cases: (1) $a \leqslant b \leqslant c$, (2) $b \leqslant a \leqslant c$, (3) $b \leqslant c \leqslant a$.

(1) $\Rightarrow$ $S(\min\{a,b\},\min\{a,c\}) = S(a,a) \geqslant a \geqslant \min\{a, S(b,c)\} \Leftrightarrow \top$;
(2) $\Rightarrow$ $S(\min\{a,b\},\min\{a,c\}) = S(b,a) \geqslant a \geqslant \min\{a, S(b,c)\} \Leftrightarrow \top$;
(3) $\Rightarrow$ $S(\min\{a,b\},\min\{a,c\}) = S(b,c) \geqslant \min\{a, S(b,c)\} \Leftrightarrow \top$. □

LEMMA 2.2. *If T is a t-norm, then the superdistributivity of operation* max *according to S hold, i.e.,* $T(\max\{a,b\},\max\{a,c\}) \leqslant \max\{a, T(b,c)\}$.

PROOF. Because of commutativity of the operations max and T, it is sufficient to examine three cases: (1) $a \leqslant b \leqslant c$, (2) $b \leqslant a \leqslant c$, (3) $b \leqslant c \leqslant a$.

(1) $\Rightarrow$ $T(\max\{a,b\},\max\{a,c\}) = T(b,c) \leqslant \max\{a, T(b,c)\} \Leftrightarrow \top$;
(2) $\Rightarrow$ $T(\max\{a,b\},\max\{a,c\}) = T(a,c) \leqslant a \leqslant \max\{a, T(b,c)\} \Leftrightarrow \top$;
(3) $\Rightarrow$ $T(\max\{a,b\},\max\{a,c\}) = T(a,a) \leqslant a \leqslant \max\{a, T(b,c)\} \Leftrightarrow \top$. □

Distributivity (superdistributivity and subdistributivity) of min according to max, and max according to min hold, i.e.,

$$\max\{\min\{a,b\},\min\{a,c\}\} = \min\{a,\max\{b,c\}\},$$
$$\min\{\max\{a,b\},\max\{a,c\}\} = \max\{a,\min\{b,c\}\}.$$

DEFINITION 2.6. Let $N_i\colon [0,1]^2 \to [0,1]$, $i = 1,2$, be two norms. The function $F\colon X \to [0,1]$, $X \neq \emptyset$, is (N_1, N_2) *pseudo-linear combination of functions* $f_j\colon X \to [0,1]$, $j = 1,\dots,n$, if there are constants $\alpha_j \in [0,1]$, $j = 1,\dots,n$, so for all $x \in [0,1]$ the following holds

$$F(x) = N_1^{n-1}\big(N_2(\alpha_1, f_1(x)), N_2(\alpha_2, f_2(x)), \dots, N_2(\alpha_n, f_n(x))\big).$$

3. Fuzzy metrics

This section will present a theorem for constructing new fuzzy metrics from existing fuzzy metrics as their pseudo-linear combination. The goal is for such metrics to be sufficiently good to use in image filtering to construct new ones for noise removing. Here we rely on research from [**13**] related to fuzzy S-metric and fuzzy T-metric. Relying on concepts from [**5**, **6**] we have proved, using appropriate assumptions, the completeness of the fuzzy metric space thus obtained.

DEFINITION 3.1. [**13**] Let $X \neq \emptyset$, S be a continuous t-conorm, T be a continuous t-norm, and $\boldsymbol{d}$ be a fuzzy set defined on $X \times X \times (0,+\infty)$, and satisfies the following conditions for all $x, y, z \in X$, $\alpha, \beta > 0$:

(1) (i) $\boldsymbol{d}(x,y,\alpha) \in [0,1)$, (ii) $\boldsymbol{d}(x,y,\alpha) \in (0,1]$;
(2) (i) $\boldsymbol{d}(x,y,\alpha) = 0 \Leftrightarrow x = y$, (ii) $\boldsymbol{d}(x,y,\alpha) = 1 \Leftrightarrow x = y$;
(3) (i), (ii), $\boldsymbol{d}(x,y,\alpha) = \boldsymbol{d}(y,x,\alpha)$;
(4) (i) $S(\boldsymbol{d}(x,y,\alpha), \boldsymbol{d}(y,z,\beta)) \geqslant \boldsymbol{d}(x,z,\alpha+\beta)$,
(ii) $T(\boldsymbol{d}(x,y,\alpha), \boldsymbol{d}(y,z,\beta)) \leqslant \boldsymbol{d}(x,z,\alpha+\beta)$;
(5) (i), (ii) $\boldsymbol{d}(x,y,\_)\colon (0,+\infty) \to [0,1]$ is a continuous function.

The fuzzy set $\boldsymbol{d}$ is called

(i) a *fuzzy S-metric* and a triple $(X, \boldsymbol{d}, S)$ is *fuzzy S-metric space* (where $\boldsymbol{d}$ satisfies axioms (1)–(5)(i));

(ii) a *fuzzy T-metric* and a triple $(X, \boldsymbol{d}, T)$ is *fuzzy T-metric space* (where $\boldsymbol{d}$ satisfies axioms (1)–(5)(ii)).

If instead of (1), it holds that $\boldsymbol{d}(x, y, \alpha) \in [0, 1]$, the fuzzy set $\boldsymbol{d}$ is a *fuzzy S-metric (fuzzy T-metric) in the broader sense*, and $(X, \boldsymbol{d}, S)$ ($(X, \boldsymbol{d}, T)$) is a *fuzzy S-metric (fuzzy T-metric) space in the broader sense.*

We will denote the fuzzy S-metric with $\boldsymbol{s}$ and the fuzzy T-metric with $\boldsymbol{t}$, and we will write the mark $\boldsymbol{d}$ if some statement is valid in both cases and use the term *fuzzy metric.*

Definition 3.2. [**13**] Fuzzy metric $\boldsymbol{d}$ is *stationary* on X if $\boldsymbol{d}$ does not depend on α, i.e. if for all fixed $x, y \in X$, the function $\boldsymbol{d}_{x,y}(\alpha) = \boldsymbol{d}(x, y, \alpha)$ is a constant.

Remark 3.1. Fuzzy S-metric $\boldsymbol{s}(x, y, \_)\colon (0, +\infty) \to [0, 1]$ is nondecreasing function, and fuzzy T-metric $\boldsymbol{S}(x, y, \_)\colon (0, +\infty) \to [0, 1]$ is nonincreasing function.

Example 3.1. [**5, 13**] The mapping $\mathbf{t}_K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $\mathbf{t}_K(x, y) = \frac{\min\{x,y\}+K}{\max\{x,y\}+K}$, where $K > 0$, is a fuzzy T-metric with respect to multiplication, and $\mathbf{s}_K(x, y) = \frac{|x-y|}{\max(x,y)+K}$ is a fuzzy S-metric with respect to the algebraic sum, $S(x, y) = 1 - (1 - x)(1 - y) = x + y - xy$, dual to T with respect to the standard fuzzy complement.

Example 3.2. [**13**] The mapping $\mathbf{t}_p\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $p > 0$, defined by

$$\mathbf{t}_p(x, y) = \frac{(\frac{1}{2}(x^p + y^p))^{1/p} + K}{\max\{x, y\} + K},$$

where $K > 0$, is a fuzzy T-metric with respect to multiplication. Specially, for $p = 1$ that is $\mathbf{t}_1(x, y) = \frac{\frac{1}{2}(x+y)+K}{\max\{x,y\}+K}$, and $\mathbf{s}_1(x, y) = \frac{|x-y|}{2(\max(x,y)+K)}$ is the fuzzy S-metric with respect to the algebraic sum, dual to T with respect to standard fuzzy complement.

Example 3.3. [**5**] If (X, d) is a metric space, then the mapping $\mathbf{t}\colon X \times X \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\mathbf{t}(x, y, t) = \frac{t}{t + d(x, y)},$$

and its dual (with respect to the standard fuzzy complement) $\mathbf{s}(x, y, t) = 1 - \mathbf{t}(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$ is a fuzzy S-metric with respect to the algebraic sum.

Definition 3.3. Let $(X, \boldsymbol{d}_i)$, $i = 1, 2$ be (i) fuzzy T-metric spaces, (ii) fuzzy S-metric spaces. The function $f\colon X \to Y$, $D \subset X$ is *continuous* in $x_0 \in D$ if

(i) $(\forall \varepsilon \in (0, 1))(\exists \delta \in (0, 1))(\forall x \in D)\, \boldsymbol{d}_1(x, x_0, \alpha) > 1 - \delta \Rightarrow \boldsymbol{d}_2(f(x), f(x_0), \alpha) > 1 - \varepsilon.$

(ii) $(\forall \varepsilon \in (0, 1))(\exists \delta \in (0, 1))(\forall x \in D)\, \boldsymbol{d}_1(x, x_0, \alpha) < \delta \Rightarrow \boldsymbol{d}_2(f(x), f(x_0), \alpha) < \varepsilon.$

for each $\alpha > 0$.

THEOREM 3.1. *Let T and S are continuous strict triangular norm and conorm, respectively, where the superdistributivity of S according to T hold. If $\boldsymbol{d}_i\colon X_i\times X_i \to (0,1]$, $i \in I = \{1,\dots,n\}$, $n \in \mathbb{N}$, are fuzzy T-metrics with respect to the norm T and $\kappa_i \in [0,1)$, $i \in I$, then the function $\boldsymbol{d}$ defined by*

$$\begin{aligned}\boldsymbol{d}(x,y,\alpha) &= T^{n-1}(S(\kappa_1,\boldsymbol{d}_1(x_1,y_1,\alpha)),\dots,S(\kappa_n,\boldsymbol{d}_n(x_n,y_n,\alpha))),\\ x &= (x_1,\dots,x_n),\ y=(y_1,\dots,y_n) \in X = X_1\times\cdots\times X_n,\end{aligned} \tag{3.1}$$

is the fuzzy T-metric with respect to the norm T. If T is not a strict norm, then $\boldsymbol{d}$ is a fuzzy metric in a broader sense.

PROOF. $\kappa_i \in [0,1)$, $\boldsymbol{d}_i(x_i,y_i,\alpha) \in (0,1]$, $i \in I \ \Rightarrow\ S(\kappa_i,\boldsymbol{d}_i(x_i,y_i,\alpha)) \in [0,1]$, $i \in I \ \Rightarrow\ \boldsymbol{d}(x,y,\alpha) \in [0,1]$.

If it is $\boldsymbol{d}(x,y,\alpha) = 0$, based on Remark 2.3 ($T$ strictly) must for some i be $S(\kappa_i,\boldsymbol{d}_i(x_i,y_i,\alpha)) = 0$. But because of Remark 2.4: $\kappa_i = 0$ and $\boldsymbol{d}_i(x_i,y_i,\alpha) = 0 \Leftrightarrow \perp$. So, $\boldsymbol{d}(x,y,\alpha) \in (0,1]$.

From Remark 2.2 it follows $\boldsymbol{d}(x,y,\alpha) = 1 \Leftrightarrow (\forall i \in I)\ S(\kappa_i,\boldsymbol{d}_i(x_i,y_i,\alpha)) = 1$, and from Remark 2.5 we have $(\forall i \in I)\ (\kappa_i = 1 \vee \boldsymbol{d}_i(x_i,y_i,\alpha) = 1)$. But, $(\forall i \in I)$ $\kappa_i \neq 1$, imply $(\forall i \in I)\ \boldsymbol{d}_i(x_i,y_i,\alpha) = 1$, and then $x_i = y_i$, for each $i \in I$, i.e., $x = y$.

$$\begin{aligned}\boldsymbol{d}(x,x,\alpha) &= T^{n-1}(S(\kappa_1,\boldsymbol{d}_1(x_1,x_1,\alpha)),\dots,S(\kappa_n,\boldsymbol{d}_n(x_n,x_n,\alpha)))\\ &= T^{n-1}(S(\kappa_1,1),\dots,S(\kappa_n,1)) = T^{n-1}(1,\dots,1) = 1,\\ \boldsymbol{d}(x,y,\alpha) &= T^{n-1}(S(\kappa_1,\boldsymbol{d}_1(x_1,y_1,\alpha)),\dots,S(\kappa_n,\boldsymbol{d}_n(x_n,y_n,\alpha)))\\ &= T^{n-1}(S(\kappa_1,\boldsymbol{d}_1(y_1,x_1,\alpha)),\dots,S(\kappa_n,\boldsymbol{d}_n(y_n,x_n,\alpha))) = \boldsymbol{d}(y,x,\alpha).\end{aligned}$$

Using commutativity and associativity T^{n-1}, from Remark 2.6:

$$\begin{aligned}T(\boldsymbol{d}(x,z,\alpha),\boldsymbol{d}(z,y,\beta)) &= T\big(T^{n-1}(S(\kappa_1,\boldsymbol{d}_1(x_1,z_1,\alpha)),\dots,S(\kappa_n,\boldsymbol{d}_n(x_n,z_n,\alpha))),\\ &\qquad T^{n-1}(S(\kappa_1,\boldsymbol{d}_1(z_1,y_1,\beta)),\dots,S(\kappa_n,\boldsymbol{d}_n(z_n,y_n,\beta)))\big)\\ &= T^{2n-1}\big(S(\kappa_1,\boldsymbol{d}_1(x_1,z_1,\alpha)),\dots,S(\kappa_n,\boldsymbol{d}_n(x_n,z_n\alpha)),\\ &\qquad S(\kappa_1,\boldsymbol{d}_1(z_1,y_1,\beta)),\dots,S(\kappa_n,\boldsymbol{d}_n(z_n,y_n,\beta))\big)\\ &= T^{2n-1}\big(S(\kappa_1,\boldsymbol{d}_1(x_1,z_1,\alpha)),S(\kappa_1,\boldsymbol{d}_1(z_1,y_1,\beta)),\\ &\qquad \dots,S(\kappa_n,\boldsymbol{d}_n(x_n,z_n\alpha)),S(\kappa_n,\boldsymbol{d}_n(z_n,y_n,\beta))\big)\\ &= T^{n-1}\big(T(S(\kappa_1,\boldsymbol{d}_1(x_1,z_1,\alpha)),S(\kappa_1,\boldsymbol{d}_1(z_1,y_1,\beta))),\\ &\qquad \dots,T(S(\kappa_n,\boldsymbol{d}_n(x_n,z_n,\alpha)),S(\kappa_n,\boldsymbol{d}_n(z_n,y_n,\beta)))\big).\end{aligned}$$

From the superdistributivity of S according to T:

$$T(S(\kappa_i,\boldsymbol{d}_i(x_i,z_i,\alpha)),S(\kappa_i,\boldsymbol{d}_i(z_i,y_i,\beta))) \leqslant S(\kappa_i,T(\boldsymbol{d}_i(x_i,z_i,\alpha),\boldsymbol{d}_i(z_i,y_i,\beta))),\quad i\in I,$$

and from triangle inequality for $\boldsymbol{d}_i$, $i \in I$, and the monotonicity of S:

$$S(\kappa_i,T(\boldsymbol{d}_i(x_i,z_i,\alpha),\boldsymbol{d}_i(z_i,y_i,\beta))) \leqslant S(\kappa_i,\boldsymbol{d}_i(x_i,y_i,\alpha+\beta)).$$

Now, because of the monotonicity of T^{n-1}, it follows

$$T(\boldsymbol{d}(x,z,\alpha),\boldsymbol{d}(z,y,\beta)) \leqslant T^{n-1}(S(\kappa_1,\boldsymbol{d}_1(x_1,y_1,\alpha+\beta)),\dots,$$

$$S(\kappa_n, \boldsymbol{d}_n(x_n, y_n, \alpha+\beta))) = \boldsymbol{d}(x, y, \alpha+\beta).$$

Triangular norm T, triangular conorm S and the metrics $\boldsymbol{d}_i(x, y, \_)\colon (0,+\infty) \to [0,1]$, $i \in I$, are continuous, then $\boldsymbol{d}(x, y, \_)\colon (0,+\infty) \to [0,1]$ is a continuous function. □

Analogously, [**1**]:

Definition 3.4. Let $(X, \boldsymbol{d})$ be (i) fuzzy T-metric space, (ii) fuzzy S-metric space. Then a sequence $\{x_n\}$ in X, *converges* to $x \in X$ if for each $\varepsilon \in (0,1)$ and each $\alpha > 0$, there is $n_0 \in \mathbb{N}$, for all $n \geqslant n_0$, such that

$$\text{(i) } \boldsymbol{d}(x_n, x, \alpha) > 1-\varepsilon, \quad \text{(ii) } \boldsymbol{d}(x_n, x, \alpha) < \varepsilon.$$

Definition 3.5. Let $(X, \boldsymbol{d})$ be fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be *Cauchy* if for each $\varepsilon \in (0,1)$ and each $\alpha > 0$, there is $n_0 \in \mathbb{N}$, for all $n, m \geqslant n_0$, such that

$$\text{(i) } \boldsymbol{d}(x_n, x_m, \alpha) > 1-\varepsilon, \quad \text{(ii) } \boldsymbol{d}(x_n, x_m, \alpha) < \varepsilon.$$

Definition 3.6. The fuzzy metric space $(X, \boldsymbol{d})$ be is *complete*, if every Cauchy series is convergent.

We give a sufficient condition for the space completeness the fuzzy metric of which is defined in the previous theorem.

Theorem 3.2. *Let T and S are continuous strict triangular norm and conorm, respectively, where the superdistributivity of S according to T hold. If $(X_i, \boldsymbol{d}_i)$, $i \in I = \{1, \dots, n\}$, $n \in \mathbb{N}$, are complete fuzzy T-metrics spaces with respect to the norm T and $\kappa_i \in [0,1)$, $i \in I$, then $(X, \boldsymbol{d})$ is complete fuzzy T-metric space, where the function $\boldsymbol{d}\colon X^2 \to [0,1]$, $X = X_1 \times \dots \times X_n$, defined by* (3.1).

Proof. Let $\{x^{(k)}\}$, $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$, be a Cauchy sequence in $(X, \boldsymbol{d})$, i.e.,

(3.2) $(\forall \eta \in (0,1))(\exists k_0 \in \mathbb{N})(\forall k, \ell \in \mathbb{N})\; k, \ell \geqslant k_0 \Rightarrow \boldsymbol{d}(x^{(k)}, x^{(\ell)}, \alpha) > 1-\eta, \quad \alpha > 0.$

Suppose (see Remark 3.2) $\varepsilon < 1 - \max\{\kappa_i \mid i \in I\}$, i.e., $\varepsilon < 1 - \kappa_i$, $i \in I$. Now, from

$$T^{n-1}(S(\kappa_1, \boldsymbol{d}_1(x_1^{(k)}, x_1^{(l)}, \alpha)), \dots, S(\kappa_n, \boldsymbol{d}_n(x_n^{(k)}, x_n^{(l)}, \alpha))) \leqslant$$
$$\min\{S(\kappa_1, \boldsymbol{d}_1(x_1^{(k)}, x_1^{(\ell)}, \alpha)), \dots, S(\kappa_n, \boldsymbol{d}_n(x_n^{(k)}, x_n^{(\ell)}, \alpha))\},$$

it follows $S(\kappa_i, \boldsymbol{d}_i(x_i^{(k)}, x_i^{(\ell)}, \alpha)) > 1-\eta$, for all $i \in I$, and for all $k, \ell \geqslant k_0$.

From the continuity of the strict t-conorm S follows the continuity its projections, i.e., functions $\sigma\colon [0,1] \to [0,1]$ defined by $y = \sigma(x) = S(\kappa_i, x)$, $\kappa_i \in (0,1]$ as well as that σ is an increasing function. It is also valid that it is $\sigma(0) = S(\kappa_i, 0) = \kappa_i$ and $\sigma(1) = S(\kappa_i, 1) = 1$, i.e., $\sigma([0,1]) = [\kappa_i, 1]$. But then σ has an inverse the function $\sigma^{-1}\colon [\kappa_i, 1] \to [0,1]$ which is also continuous and increasing. From continuity at $x_0 = 1$ $(y_0 = 1)$, is $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in [\kappa_i, 1])|y - y_0| = |S(\kappa_i, x) - 1| = 1 - S(\kappa_i, x) < \delta \Rightarrow |x - 1| = 1 - x < \varepsilon$, i.e.,

$$(\forall \varepsilon > 0)(\exists \delta > 0) S(\kappa_i, x) > 1-\delta \Rightarrow x > 1-\varepsilon.$$

Hence, for arbitrary $\varepsilon \in (0, \max \kappa_i)$, there exists $\delta_i > 0$, assuming in (3.2) that it is $\eta = \delta_i$, we get there is $k_0(i) \in \mathbb{N}$ so that it is true:

$$S(\kappa_i, \boldsymbol{d}_i(x_i^{(k)}, x_i^{(\ell)}, \alpha)) > 1 - \delta_i \Rightarrow \boldsymbol{d}_i(x_i^{(k)}, x_i^{(\ell)}, \alpha) > 1 - \varepsilon,$$

for all $k, \ell \geqslant k_0(i)$ and $i \in I$. Therefore, the sequences $\{x_i^{(k)}\}$, $i \in I$ are Cauchy ones.

The completeness of spaces $(X_i, \boldsymbol{d}_i)$, $i \in I = \{1, \dots, n\}$, implies the convergence of sequences $x_1^{(k)}, \dots, x_n^{(k)}$ to $x_1, \dots, x_n$, respectively, i.e., for $i \in I$:

$$(\forall \varepsilon_i \in (0,1))(\exists k_i \in \mathbb{N})(\forall k \in \mathbb{N})\ k \geqslant k_i \Rightarrow \boldsymbol{d}_i(x_i^{(k)}, x_i, \alpha) > 1 - \varepsilon_i, \quad \alpha > 0. \tag{3.3}$$

Let's show that $x^{(k)} \to x = (x_1, \dots, x_n)$, i.e.,

$$(\forall \varepsilon \in (0,1))(\exists k_0 \in \mathbb{N})(\forall k \in \mathbb{N})\ k \geqslant k_0 \Rightarrow \boldsymbol{d}(x^{(k)}, x, \alpha) > 1 - \varepsilon, \quad \alpha > 0. \tag{3.4}$$

As the function $F\colon [0,1]^n \to [0,1]$, is defined by

$$F(a_1, \dots, a_n) = T^{n-1}(S(\kappa_1, a_1), \dots, S(\kappa_n, a_n))$$

continuous, as a composition of continuous functions, due to $F(1, \dots, 1) = 0$ holds $(\forall \varepsilon > 0)(\exists \delta_i > 0, i \in I)(\forall (a_1, \dots, a_n) \in [0,1]^n)(a_1, \dots, a_n) \in (1 - \delta_1, 1 + \delta_1) \times \cdots \times (1 - \delta_n, 1 + \delta_n) \Rightarrow F(a_1, \dots, a_n) \in (1 - \varepsilon, 1 + \varepsilon)$, i.e.,

$$\begin{aligned}&(\forall \varepsilon > 0)(\exists \delta_i > 0, i \in I)(\forall (a_1, \dots, a_n) \in [0,1]^n)\\ &\qquad\qquad a_i \in (1 - \delta_i, 1], i \in I \Rightarrow F(a_1, \dots, a_n) \in (1 - \varepsilon, 1].\end{aligned}$$

Taking that $\varepsilon_i = \delta_i < 1 - \kappa_i$, $i \in I$ in (3.3) follows there exists $k_i \in \mathbb{N}$, such that $\boldsymbol{d}_i(x_i^{(k)}, x_i, \alpha) > 1 - \varepsilon_i$, for all $k \geqslant k_i$, $\alpha > 0$, $i \in I$, i.e., for arbitrary $\varepsilon \in (0,1)$, exists $k_0 = \max\{k_1, \dots, k_n\} \in \mathbb{N}$ so that for all $k \geqslant k_0$, from $a_i = \boldsymbol{d}_i(x_i^{(k)}, x_i, \alpha) > 1 - \varepsilon_i$, follows $F(a_1, \dots, a_n) = \boldsymbol{d}(x^{(k)}, x, \alpha) > 1 - \varepsilon$, i.e., we get (3.4).

Therefore, every Cauchy sequence of the space $(X, \boldsymbol{d})$ is convergent, i.e., space is complete. □

REMARK 3.2. The function $f\colon X \to Y$, $D \subset X$ (where $(X, \boldsymbol{d}_1)$ and $(X, \boldsymbol{d}_2)$ are fuzzy T-metric spaces) is continuous in $x_0 \in D$ if there exists $\varepsilon_0 \in (0,1)$ such that for each $\alpha > 0$, valid

$$(\forall \varepsilon \in (0, \varepsilon_0])(\exists \delta \in (0,1))(\forall x \in D)\ \boldsymbol{d}_1(x, x_0, \alpha) > 1 - \delta \Rightarrow \boldsymbol{d}_2(f(x), f(x_0), \alpha) > 1 - \varepsilon.$$

THEOREM 3.3. *Let T and S are continuous strict triangular norm and conorm, respectively, where the subdistributivity of T according to S hold. If $\boldsymbol{d}_i\colon X_i \times X_i \to [0,1)$, $i \in I = \{1, \dots, n\}$, $n \in \mathbb{N}$, are fuzzy S-metrics with respect to the conorm S, and $\kappa_i \in (0,1]$, $i \in I$, then $\boldsymbol{d}$ defined by*

$$\begin{aligned}\boldsymbol{d}(x, y, \alpha) &= S^{n-1}(T(\kappa_1, \boldsymbol{d}_1(x_1, y_1, \alpha)), \dots, T(\kappa_n, \boldsymbol{d}_n(x_n, y_n, \alpha))),\\ &\quad x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in X = X_1 \times \cdots \times X_n,\end{aligned} \tag{3.5}$$

is the fuzzy S-metric with respect to the conorm S. If S is not a strict norm, then $\boldsymbol{d}$ is a fuzzy metric in a broader sense.

THEOREM 3.4. *Let T and S zzbe continuous strict triangular norm and conorm, respectively, where the subdistributivity of T according to S hold. If $(X_i, \boldsymbol{d}_i)$, $i \in I = \{1, \dots, n\}$, $n \in \mathbb{N}$, are complete fuzzy S-metrics spaces with respect to the conorm S and $\kappa_i \in (0, 1]$, $i \in I$, then $(X, \boldsymbol{d})$ is complete fuzzy S-metric space, where the function $\boldsymbol{d}\colon X^2 \to [0, 1]$, $X = X_1 \times \cdots \times X_n$, defined by* (3.5).

4. Filtering images using fuzzy metrics

The distance between objects is a key feature in image processing, especially image filtering. The distance defined above over the non-empty set X is the mapping $d\colon X \times X \to \mathbb{R}_0^+$ which is symmetric. Metrics are most often used, but also the so-called similarities (that is, differences) where it is in the definition metrics triangle inequality replaced by $d(x, z) \geqslant T(d(x, y), d(y, z))$ or $d(x, z) \leqslant S(d(x, y), d(y, z))$, (for some triangulated norm T, i.e., conorm S). Due to the nature of the imaging, there has been much research over in the last two decades where a fuzzy metric replaces the metric.

When filtering digital color images (with color components red, green and blue (RGB)), the pixels will be denoted by $(\boldsymbol{i}, \boldsymbol{F_i})$, where $\boldsymbol{i} = (i_1, i_2) \in I \times I$, is a vector with spatial coordinates of pixel i_1, i_2 (points on the screen with integer coordinates), $\boldsymbol{F_i}$ is a three-dimensional vector, the first coordinate of which is a quantity of red color, the second coordinate is a quantity of green color, while the third is a quantity of blue color, i.e., $(\boldsymbol{F_i^1}, \boldsymbol{F_i^2}, \boldsymbol{F_i^3})$.

When filtering an image, a window (a set of square-shaped pixels) is used, most often denoted by W, the size of which is $n \times n$, where n is an odd number. The essence of image filtering is to replace the noise-generating pixel with a noiseless pixel, which can be achieved by replacing a middle pixel in window W with a pixel that represents the other pixels from window W in the best possible way, i.e., with a pixel that is the closest match in color and spatial distance to all the other pixels in W.

It is essential to choose a robust criterion for selecting the noiseless pixel, which will replace the noisy pixel in a given window W, because the choice of pixels affects the image quality, i.e., the degree of the removed noise.

Key to selecting that criterion will be a broad selection of fuzzy metric $\boldsymbol{c}$. On the set of all pixels in the given window W an order relation will be induced by using fuzzy metric $\boldsymbol{c}$. This order relation will be used to compare pixels $(\boldsymbol{i}, \boldsymbol{F_i})$ ("position", "color") of the image and to choose a pixel that differs at least from all other pixels in the window, i.e. which is the most similar to all other pixels in W (in terms of color and distance). The pixel found by using the algorithm will replace the middle pixel in the given window W. The algorithm is applied to each sliding window.

In the image filtering algorithm, the mapping $\boldsymbol{c}\colon W \times W \to \mathbb{R}$ will be used, defined on the window $W = \{(\boldsymbol{i}, \boldsymbol{F_i}) \mid \boldsymbol{i} \in I \times I\}$, $I = \{0, 1, \dots, n-1\}$, defined by

$$\boldsymbol{c}((\boldsymbol{i}, \boldsymbol{F_i}), (\boldsymbol{j}, \boldsymbol{F_j})) = N_1(N_2(\alpha_1, \boldsymbol{\delta}(\boldsymbol{F_i}, \boldsymbol{F_j})), N_2(\alpha_2, \boldsymbol{\partial}(\boldsymbol{i}, \boldsymbol{j}))) \tag{4.1}$$

where $\boldsymbol{\delta}$ and $\boldsymbol{\partial}$ are fuzzy metrics with respect to a norm N_1.

If N_1 and N_2 are continuous triangular norm and conorm, respectively, where the superdistributivity of N_2 according to N_1 hold and $\boldsymbol{\delta}$, $\boldsymbol{\partial}$ are fuzzy T-metrics with respect to the strict t-norm N_1 and $\alpha_i \in [0, 1)$, $i = 1, 2$, then the function $\boldsymbol{c}$ is the fuzzy T-metric with respect to the norm N_1 which follows from Theorem 3.1.

Fuzzy metric $\boldsymbol{\delta}$ is defined by

(4.2)
$$\boldsymbol{\partial}(\boldsymbol{F_i}, \boldsymbol{F_j}) = (N_3^2(N_4(\kappa_1, \boldsymbol{\delta}_1(\boldsymbol{F_i^1}, \boldsymbol{F_j^1})), N_4(\kappa_2, \boldsymbol{\delta}_2(\boldsymbol{F_i^2}, \boldsymbol{F_j^2})), N_4(\kappa_3, \boldsymbol{\delta}_3(\boldsymbol{F_i^3}, \boldsymbol{F_j^3})),$$

and it is used to measure the similarity between corresponding colors (the equality of colors quantity) between two pixels $\boldsymbol{F_i}$ and $\boldsymbol{F_j}$, i.e., similarity of k-th color $(k = 1, 2, 3)$ is measured by fuzzy metric $\boldsymbol{\delta}_k$. If the conditions of Theorems 3.1 or 3.2 are satisfied, it follows that $\boldsymbol{\delta}$ (so defined) is a fuzzy metric.

The spatial distance of pixels $\boldsymbol{i}$ and $\boldsymbol{j}$ is measured with fuzzy metric $\boldsymbol{\partial}$ in which there is usually a parameter that affects the sensitivity of fuzzy metric $\boldsymbol{\partial}$.

The UIQI (Universal Image Quality Index), defined by Wang, Bovik [**17**], is employed to compare the quality of images. It assesses image distortion as a combination of three factors: loss of correlation, luminance distortion, and contrast distortion. As the filtered images are in RGB format, the UIQI is calculated for each color separately, resulting in a three-dimensional vector representing the image quality instead of a single value. The metric value for each color falls within the range of -1 to 1, where a value closer to one indicates better image quality. Consequently, a higher quality index for each color signifies improved image processing quality. The UIQI is computed using a sliding window that traverses the entire image from top to bottom, pixel by pixel. The UIQI is calculated for each window using the formula provided in [**17**]. The individual window values are then summed and averaged over the total number of windows.

Image quality could also be examined by the sharpness of the image defined by Narvekar, Karam [**9**]. It is based on the cumulative probability of image blur detection (CPBD). The cumulative probability of blur detection in an image serves as the basis for calculating of this image quality metric.

Sharpness metric CPBD (discretized version) is determined in the following steps. The image is divided into 64×64 blocks, which are classified into two groups: edge block or a non-edge block. If the block is non-edge, it is not processed further. The width of each edge in the block is calculated for every edge block. At each edge, the probability of blur detection is calculated in the following manner:

$$P_{\mathrm{BLUR}} = \mathbb{P}(e_i) = 1 - \exp\Big(-\Big|\frac{w(e_i)}{w_{\mathrm{JNB}}(e_i)}\Big|\Big),$$

where $w_{\mathrm{JNB}}(e_i)$ is the JNB edge width that depends on the local contrast C (of the edge block to which the edge belongs) and $w(e_i)$ is the width of the edge e_i. The *cumulative probability of blur detection* (CPBD) is determined using the formula

$$\mathrm{CPBD} = \mathbb{P}(P_{\mathrm{BLUR}} \leqslant P_{\mathrm{JNB}}) = \sum_{P_{\mathrm{BLUR}}=0}^{P_{\mathrm{JNB}}} \mathbb{P}(P_{\mathrm{BLUR}}).$$

$\mathbb{P}(P_{\mathrm{BLUR}})$ denotes the value of the probability distribution function at a given P_{BLUR}.

5. Experiments

The process of removing noise from an image using fuzzy metrics, was tested on the jpg format image "Plant". The dimensions of the test images are 256×192 pixels (Fig. 1). Fig. 2 shows the image that is contaminated with 10% or 20% salt and pepper noise. For each color, the quality metric UIQI is quantified. Sharpness is calculated as well. The moving window has dimension 5.

The fuzzy metric used to measure pixels colors similarity is denoted by $\boldsymbol{\delta}$. Taking in (4.2) that $\kappa_1 = \kappa_2 = \kappa_3 = 0$, $N_3 = \cdot$ and for N_4 arbitrary t-conorm, $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ fuzzy T-metrics from Example 3.2, we get fuzzy T-metric:

$$\boldsymbol{\delta}(\boldsymbol{F}_i, \boldsymbol{F}_j) = \prod_{l=1}^{3} \frac{(\frac{1}{2}((\boldsymbol{F}_i^l)^p + (\boldsymbol{F}_j^l)^p))^{1/p} + K}{\max\{\boldsymbol{F}_i^l, \boldsymbol{F}_j^l\} + K} \tag{5.1}$$

Fuzzy T-metric that considers spatial distance between pixels is denoted by $\boldsymbol{\delta}$. It is a special case of fuzzy T-metric from Example 3.3:

$$\boldsymbol{\delta}(\boldsymbol{i}, \boldsymbol{j}) = \frac{t}{t + \sqrt[q]{|i_1 - j_1|^q + |i_2 - j_2|^q}}, \quad q \in \mathbb{N}, \tag{5.2}$$

where $\boldsymbol{i} = (i_1, i_2)$, $\boldsymbol{j} = (j_1, j_2)$.

FIGURE 1. Plant, 256×192

In the testing we conducted on the "Plant" image, parameters α_1 and α_2 that appear in the function $\boldsymbol{c}$ are actually weights coefficients, i.e., they determine which of the distances, spatial or we prefer, is the one for measuring the difference in pixel

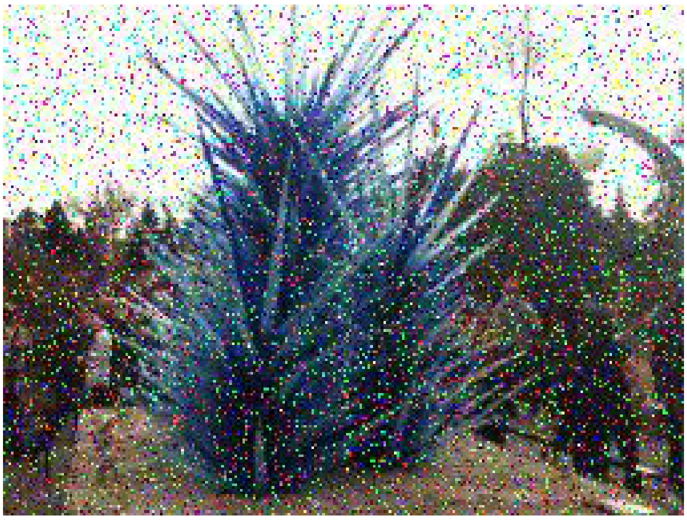

FIGURE 2. Plant, 256×192, 10% salt and pepper

FIGURE 3. Plant, filtered image, 10% salt and pepper, $K = 500$, $t = 1.5$

brightness. In the presented experiment, we took in both cases $\alpha_1 = \alpha_2 = 0$, i.e., $\alpha_1 = \alpha_2 = 1$.

(I) In (4.1), we took probabilistic sum $N_1 = S_P$, N_2 to be arbitrary t-norm and $\alpha_1 = \alpha_2 = 1$, so it is in our experiment fuzzy metric $\boldsymbol{c}$ actually

$$\boldsymbol{c}((\boldsymbol{i}, F_{\boldsymbol{i}}), (\boldsymbol{j}, \boldsymbol{F_j})) = \boldsymbol{\delta}(\boldsymbol{F_i}, \boldsymbol{F_j}) + \boldsymbol{\partial}(\boldsymbol{i}, \boldsymbol{j}) - \boldsymbol{\delta}(\boldsymbol{F_i}, \boldsymbol{F_j}) \cdot \boldsymbol{\partial}(\boldsymbol{i}, \boldsymbol{j}).$$

FIGURE 4. Plant, filtered using a median filter, 10% salt and pepper

Taking the values of the parameters $p = 2$ and $q = 2$, and by varying the value $t \in [0.1, 4.0]$ (with step 0.1) and $K \in [125, 1500]$ (with step 125), in metrics (5.1) and (5.2) for filtered image contaminated with 10% salt and pepper noise, the best values of metric of UIQI for each image color are:

$$[0.362342989, 0.364309676, 0.33569145],$$

for parameter values $t = 1.5$, $K = 500$ (Fig. 3).

It is concluded that the method based on fuzzy metrics generates poorer image quality *vis-á-vis* the quality of the image filtered using a median filter, where the image quality is compared with the metric for UIQI:

$$[0.553537275, 0.581810508, 0.511756397].$$

An image filtered with a median filter (Fig. 4), produces a sharpness of 0.6463, which is much poorer than the value of 0.9864 for the image filtered using a fuzzy filter.

The following table shows the sharpness (Sh) of some images filtered using a fuzzy filter, where $t = 1.5$ and K are fixed takes the given values:

K	125	250	375	500	626	750	875	1000	1125	1250
Sh	0.9838	0.9767	0.9820	0.9864	0.9949	0.9865	0.9940	0.9938	0.9902	0.9881

(II) If we take everything as in the previous testing, except that $\boldsymbol{c}((\boldsymbol{i}, \boldsymbol{F_i}), (\boldsymbol{j}, \boldsymbol{F_j})) = \boldsymbol{\delta}(\boldsymbol{F_i}, \boldsymbol{F_j}) \cdot \boldsymbol{\partial}(\boldsymbol{i}, \boldsymbol{j})$, the best values of metric of UIQI for each color of image are:

$$[0.516638197, 0.529245681, 0.492599165],$$

for parameter values $t = 3.1$, $K = 875$. Those values are slightly different from the values of the metric for UIQI, of the median filtered image:

$$[0.553537275, 0.581810508, 0.511756397].$$

An image filtered with a median filter produces a sharpness of 0.6463, which is far poorer than the value of 0.9914 for the image filtered by a fuzzy filter.

(III) In (4.1), we took $N_1 = \cdot$, N_2 to be arbitrary t-conorm and $\alpha_1 = \alpha_2 = 0$, so it is in our experiment fuzzy metric $\boldsymbol{c}$ actually $\boldsymbol{c}((\boldsymbol{i}, \boldsymbol{F_i}), (\boldsymbol{j}, \boldsymbol{F_j})) = \boldsymbol{\delta}(\boldsymbol{F_i}, \boldsymbol{F_j}) \cdot \boldsymbol{\partial}(\boldsymbol{i}, \boldsymbol{j})$, Taking the values of the parameters $p = 1$ and $q = 1$, and by varying the value $t \in [0.1, 4.0]$ (with step 0.1) and $K \in [125, 1500]$ (with step 125), in metrics (5.1) and (5.2) for filtered image contaminated with 20% salt and pepper noise (Fig. 5), best values of metric of UIQI for each color of image are

$$[0.350641005, 0.36616748, 0.336773334],$$

for parameter values $t = 0.1$, $K = 250$ (Fig. 6).

It is concluded that the method based on fuzzy metrics generates a somewhat poorer image quality *vis-á-vis* the quality of the image filtered using a median filter (Fig. 7), where the image quality is compared with the metric for UIQI:

$$[0.471080678, 0.519562984, 0.432602436].$$

An image filtered with a median filter, produces a sharpness of 0.6927, which is far poorer than the value of 0.9908 for the image filtered by a fuzzy filter.

FIGURE 5. Plant, 256x192, 20% salt and pepper

(IV) If we take everything as in the previous testing, except varying the value $t \in [10, 990]$ (with step 20) and $K \in [125, 5125]$ (with step 500), in metrics (5.1)

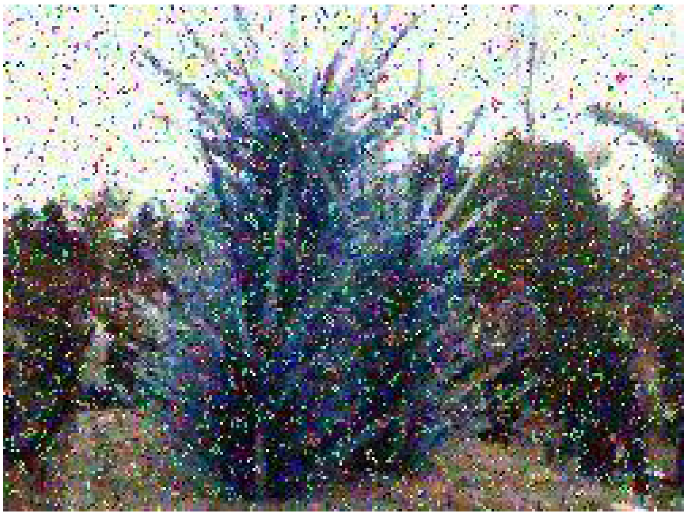

FIGURE 6. Plant, filtered image, 20% salt and pepper, $K = 250$, $t = 0.1$

FIGURE 7. Plant, filtered using a median filter, 20% salt and pepper

and (5.2) for filtered image contaminated with 5% salt and pepper noise (Fig. 8), the best values of metric of UIQI for each color of image are:

$$[0.622099998, 0.637857341, 0.609252374],$$

for parameter values $t = 10$, $K = 2625$ (Fig. 9).

It is concluded that the method based on fuzzy metrics generates a somewhat better image quality compared with the quality of the image filtered using a median filter (Fig. 10), where the image quality is compared with the metric UIQI:

$$[0.571724095, 0.602179063, 0.549128726].$$

This image filtered with a median filter, produces a sharpness of 0.6250, which is far poorer than the value of 0.9828 for the image filtered by a fuzzy filter.

The following table lists the tested cases when the fuzzy filter produces better image quality than the metric for UIQI obtained from the median filter.

Filter	UIQI	sharpness
FF ($K = 125,\ t = 10$)	[0.578553195, 0.592197474, 0.563651099]	0.9830
FF ($K = 1125,\ t = 10$)	[0.598554328, 0.61427433, 0.584435289]	0.9787
FF ($K = 1625,\ t = 10$)	[0.611160023, 0.627391739, 0.599013258]	0.9752
FF ($K = 1625,\ t = 230$)	[0.557179733, 0.570101444, 0.54043719]	0.9825
FF ($K = 2125,\ t = 10$)	[0.618727277, 0.635052967, 0.606933406]	0.9769
FF ($K = 2625,\ t = 10$)	[0.622099998, 0.637857341, 0.609252374]	0.9828
FF ($K = 2625,\ t = 230$)	[0.584454339, 0.599542426, 0.569519108]	0.9887
FF ($K = 3125,\ t = 10$)	[0.621980924, 0.637125136, 0.609103084]	0.9784
FF ($K = 3125,\ t = 230$)	[0.592817441, 0.607752109, 0.578057758]	0.9836
FF ($K = 3625,\ t = 10$)	[0.619479154, 0.634581478, 0.606633379]	0.9795
FF ($K = 3625,\ t = 230$)	[0.599290257, 0.615183591, 0.585093335]	0.9861
FF ($K = 4125,\ t = 10$)	[0.617337832, 0.632742564, 0.605218738]	0.9779
FF ($K = 4125,\ t = 230$)	[0.605061417, 0.620948146, 0.591745631]	0.9885
FF ($K = 4625,\ t = 10$)	[0.613914944, 0.629144181, 0.601726559]	0.9762
FF ($K = 4625,\ t = 230$)	[0.610191481, 0.626162077, 0.59762365]	0.9886
VMF	[0.571724095, 0.602179063, 0.549128726]	0.6250

FIGURE 8. Plant, 256x192, 5% salt and pepper

FIGURE 9. Plant, filtered image, 5% salt and pepper, $K = 2625$, $t = 10$

FIGURE 10. Plant, filtered using a median filter, 5% salt and pepper

Paper [**13**] presents an image (contaminated with 10% salt and pepper noise) using a fuzzy-metric-based $\boldsymbol{c}$, with parameters $t = 2.6$, $K = 768$ and window size 3. The metric values for UIQI for each color for the filtered image by applying the fuzzy filter are equal to $[0.5257, 0.5702, 0.5662]$. The metric values of UIQI for each color for the filtered image using a median filter with window size 3 are equal to $[0.5033, 0.5649, 0.5447]$. By comparing the index of metric UIQI for corresponding colors (respectively, red, green, blue), it can be concluded that all indices of images filtered using the method proposed in this paper are greater than the corresponding indices of images filtered using the median filter. As those indices are closer to one, it can be concluded that the image quality is superior. The process of removing noise from an image using fuzzy metrics, was tested on the jpg format "Baboon" image. The test image is from USCSIPi Database Source USC-SIPI Image Database. http://sipi.usc.edu/database/, University of Southern California, Signal and Image Processing Institute.

Articles [**14**, **16**], save for the aforementioned cited papers, are also papers that discussed image filtering using a fuzzy-metric-based filter.

6. Conclusion

In this article, the pseudo-linear combination fuzzy metric is defined and it is shown that with certain assumptions, it is also fuzzy metric and that on the corresponding set it has the structure of a complete fuzzy metric space. Concrete examples were used to examine the quality of the filter and its removal sum from the image, which is based on the thus obtained fuzzy metric. The conclusion is that the UIQI is inferior on most of the examined images but that the filtered image's sharpness is superior to the one processed using VMF (vector median filter).

Similarly, this concept can be used in image segmentation, such as [**10**, **11**, **15**]. Account should be taken of pseudo-linear combinations of fuzzy metrics appearing at the pixel descriptor as a characteristic that carries information about the observed pixel and its environment i metrics in the FCM algorithm.

There are numerous possibilities for applying this algorithm in different fields, given that different norms can be selected depending on the problem observed.

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