PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 116 (130) (2024), 127–146 DOI: https://doi.org/10.2298/PIM2430127K

IMPACT OF STRUCTURE VECTOR FIELD ξ-ON POINTWISE SEMI-SLANT CONFORMAL SUBMERSIONS

Meraj Ali Khan, Mohammad Shuaib, Ibrahim Al-Dayel, and Cenap Ozel

ABSTRACT. We introduce the idea of pointwise semi-slant conformal submersions from Sasakian manifolds onto Riemannian manifolds. We discuss the impact of a structure vector field ξ-by considering it horizontally as well as vertically and investigate the necessary and sufficient conditions for distributions to be integrable and totally geodesic. Because the distributions are neither integrable nor totally geodesic when ξ-is vertical, therefore we examine the conditions of integrability and totally geodesicness by changing the role of ξ.

1. Introduction

The theory of submersions and immersions was originally developed and introduced by O'Neill [**21**] and Gray [**12**] to study the geometric properties of the Riemannian manifolds and establish certain Riemannian equations for them. The subject of submersions theory becomes particularly captivating when examining the interplay between differentiable structures in differential geometry.

Riemannian submersions have found extensive applications in both mathematics and physics, notably in theories such as Yang–Mills and Kaluza–Klein [**8**, **16**, **19**, **33**]. In 1976, Watson [**32**] investigated Riemannian submersions from almost Hermitian manifolds to Riemannian manifolds. Building upon this work, Sahin [**24**] explored the geometry and properties of anti-invariant Riemannian submersions onto Riemannian manifolds. Subsequent authors delved further into this area, examining anti-invariant submersions [**3**,**24**], semi-invariant submersions [**25**], slant submersions [**10**,**26**], and semi-slant submersions [**15**,**22**], among other topics. As a generalized case of semi-invariant and semi-slant submersions, Tastan, Sahin, and Yanan [**31**] defined and studied hemi-slant submersions from almost Hermitian manifolds.

2020 *Mathematics Subject Classification:* 53D10; 53C43.
Key words and phrases: Sasakian manifolds, Riemannian submersions, pointwise semi-slant conformal submersions, conformal submersions.
Communicated by Stevan Pilipović.

Lee and Sahin in [**18**] further extended the concept of slant submersions by introducing pointwise slant submersions from almost Hermitian manifolds to Riemannian manifolds. They not only provided examples illustrating this type of submersion but also established characterizations for pointwise slant submersions. Fuglede [**13**] and Ishihara [**17**] introduced the concept of conformal submersion as a generalization of Riemannian submersions and discussed some of their geometric properties. It is worth noting that a conformal submersion with dilation $\lambda = 1$ reduces to a Riemannian submersion. Gudmundsson and Wood [**14**] investigated conformal holomorphic submersions as a generalization of holomorphic submersions, and they established the necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions. Akyol and Sahin later studied and defined conformal anti-invariant submersions [**23**,**27**], conformal semi-invariant submersions [**4**], conformal slant submersions [**2**], and conformal semi-slant submersions [**1**]. Recently, geometric studies have been conducted on conformal hemi-slant submersions [**29**, **30**], conformal bi-slant submersions [**5**], and quasi bi-slant conformal submersions [**6**], accompanied by several decomposition theorems. Furthermore, the notion of pluriharmonicity was extended to almost contact metric manifolds from almost Hermitian manifolds.

The focus of this study lies in investigating pointwise semi-slant conformal submersions from a Sasakian manifold to a Riemannian manifold, where we consider the Reeb vector field ξ in both its vertical and horizontal aspects. The paper is organized as follows: In Section 2, we introduce almost contact manifolds, specifically the Sasakian manifold, which possesses the necessary characteristics for our investigation. In Section 3 we define pointwise semi-slant conformal submersions and present intriguing results by considering the Reeb vector field ξ in its horizontal form. Section 4 delves into the detailed discussion on the integrability and total geodesicity of the distributions, considering the vector field ξ in its vertical aspect.

Note: In this paper, we use the abbreviation PWSSCS for Pointwise semi-slant conformal submersion.

2. Preliminaries

We start with some definitions and conclusions which will be very helpful in our research and will help in exploring the main subject of the paper.

Definition 2.1. [**32**] Let $\Pi\colon (\Theta_1, g_1) \to (\Theta_2, g_2)$ be a smooth map between two Riemannian manifolds having dimensions m_1 and m_2, respectively. Then Π is called horizontally weakly conformal or semi conformal at $x \in \Theta_1$ if either

(i) $\Pi_{*x} = 0$, or
(ii) Π_{*x} maps horizontal space $\mathcal{H}_x = (\ker(\Pi_{*x}))^{\perp}$ conformally onto $T_{\Pi*}(2)$ i.e., Π_{*x} is surjective and there exits a number $\Lambda(x) \neq 0$ such that

$$g_2(\Pi_{*x}X, \Pi_{*x}Y) = \Lambda(x)g(X, Y), \tag{2.1}$$

for any $X, Y \in \mathcal{H}_x$.

Equation (2.1) can be re-written as $(\Pi_* g_2)_x \mid_{\mathcal{H}_x \times \mathcal{H}_x} = \Lambda(x)g(x) \mid_{\mathcal{H}_x \times \mathcal{H}_x}$.

A point x satisfies (i) in the above definition if and only if it is a critical point of Π. A point, satisfying (ii) is called a regular point. At a critical point, Π_{*x} has rank 0; at a regular point, Π_{*x} has rank n and Π defines a submersion. The number $\lambda(x)$ is called the square dilation (of Π at x); it is necessarily non-negative. Its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation of Π at x. The map Π is called horizontally weakly conformal or semi conformal on Θ_1 if it is horizontally weakly conformal at every point of Θ_1. It is clear that if Π has no critical points, then we call it a (horizontally) conformal submersion.

Definition 2.2. [**7**] Let Π be a Riemannian submersions between two Riemannian manifolds. Then Π is called a horizontally conformal submersion, if there is a positive function λ such that

$$g_1(U_1, V_1) = \frac{1}{\lambda^2} g_2(\Pi_* U_1, \Pi_* V_1), \tag{2.2}$$

for any $U_1, V_1 \in \Gamma(\ker \Pi_*)^\perp$. It is obvious that every Riemannian submersions is a particularly horizontally conformal submersion with $\lambda = 1$.

Let $\Pi\colon (\Theta_1, g_1) \to (\Theta_2, g_2)$ be a Riemannian submersion. A vector field X on Θ_1 is called a basic vector field if $X \in \Gamma(\ker \Pi_*)^\perp$ and Π-related with a vector field X on Θ_2 i.e. $\Pi_*(X(q)) = X\Pi(q)$ for $q \in \Theta_1$.

The two formulae of $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ are given by O'Neill as:

$$\mathcal{A}_{E_1} F_1 = \mathcal{H}\nabla_{\mathcal{H}E_1} \mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{H}E_1} \mathcal{H}F_1, \tag{2.3}$$

$$\mathcal{T}_{E_1} F_1 = \mathcal{H}\nabla_{\mathcal{V}E_1} \mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{V}E_1} \mathcal{H}F_1, \tag{2.4}$$

for any $E_1, F_1 \in \Gamma(T\Theta_1)$ and ∇ is the Levi-Civita connection of g_1. Note that a Riemannian submersion $\Pi\colon (\Theta_1, \phi, \xi, \eta, g_1) \to (\Theta_2, g_2)$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. From equations (2.3) and (2.4), we can deduce

$$\nabla_{U_1} V_1 = \mathcal{T}_{U_1} V_1 + \mathcal{V}\nabla_{U_1} V_1, \tag{2.5}$$

$$\nabla_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + \mathcal{H}\nabla_{U_1} X_1, \tag{2.6}$$

$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + \mathcal{V}_1\nabla_{X_1} U_1, \tag{2.7}$$

$$\nabla_{X_1} Y_1 = \mathcal{H}\nabla_{X_1} Y_1 + \mathcal{A}_{X_1} Y_1 \tag{2.8}$$

for any vector fields $U_1, V_1 \in \Gamma(\ker \Pi_*)$ and $X_1, Y_1 \in \Gamma(\ker \Pi_*)^\perp$ [**11**].

It is obvious that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, that is

$$g(\mathcal{A}_X E_1, F_1) = -g(E_1, \mathcal{A}_X F_1), \quad g(\mathcal{T}_V E_1, F_1) = -g(E_1, \mathcal{T}_V F_1), \tag{2.9}$$

for any vector fields $E_1, F_1 \in \Gamma(T\Theta_1)$. For the special case when Π is horizontally conformal submersion, we have

Proposition 2.1. *Let $\Pi\colon (\Theta_1, g_1) \to (\Theta_2, g_2)$ be a horizontally conformal submersion with dilation λ and X, Y be the horizontal vectors, then*

$$A_X Y = \frac{1}{2}\Big\{\mathcal{V}[X, Y] - \lambda^2 g(X, Y) \operatorname{grad}_{\mathcal{V}} \Big(\frac{1}{\lambda^2}\Big)\Big\}$$

measures the obstruction integrability of the horizontal distribution

The second fundamental form of the smooth map Π is provided by the formula

$$(\nabla\Pi_*)(U_1, V_1) = \nabla^{\Pi}_{U_1}\Pi_* V_1 - \Pi_*\nabla_{U_1}V_1, \tag{2.10}$$

and the map will be totally geodesic if $(\nabla\Pi_*)(U_1, V_1) = 0$ for all $U_1, V_1 \in \Gamma(T\Theta_1)$ where ∇ and ∇^{Π} are the Levi-Civita and pullback connections.

LEMMA 2.1. *Let $\Pi\colon \Theta_1 \to \Theta_2$ be a horizontal conformal submersion. Then, we have*

(i) $(\nabla\Pi_*)(X_1, Y_1) = X_1(\ln\lambda)\Pi_*(Y_1) + Y_1(\ln\lambda)\Pi_*(X_1) - g_1(X_1, Y_1)\Pi_*(\operatorname{grad}\ln\lambda)$,
(ii) $(\nabla\Pi_*)(U_1, V_1) = -\Pi_*(\mathcal{T}_{U_1}V_1)$,
(iii) $(\nabla\Pi_*)(X_1, U_1) = -\Pi_*(\nabla_{X_1}U_1) = -\Pi_*(\mathcal{A}_{X_1}U_1)$

for any horizontal vector fields X_1, Y_1 and vertical vector fields U_1, V_1 [**7**].

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ, ξ, η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta\circ\phi = 0, \quad \eta(\xi) = 1, \tag{2.11}$$

where I is the identity tensor. An almost contact structure on M is said to be *normal* if the induced almost complex structure J on the product manifold $M\times R$, defined by

$$J\Big(U, \lambda\frac{d}{dt}\Big) = \Big(\phi U - \lambda\xi, \eta(U)\frac{d}{dt}\Big),$$

is integrable, where U is a vector field tangent to M, t is the co-ordinate function on R and λ is a smooth function on $M\times R$. There exists a Riemannian metric g on an almost contact manifold which is compatible with the almost contact structure (ϕ, ξ, η) in such a way that

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.12}$$

from which it can be observed that $\eta(U) = g(U, \xi)$, for any $U, V \in \Gamma(TM)$. Then (ϕ, ξ, η, g)-structure is called an almost contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_U\phi)V = g(U, V)\xi - \eta(V)U \tag{2.13}$$

where ∇ is the Levi-Civita connection of g. For a Sasakian manifold, we can deduce that

$$\nabla_U\xi = -\phi U. \tag{2.14}$$

The covariant derivative of ϕ is defined by

$$(\nabla_X\phi)Y = \nabla_X\phi Y - \phi\nabla_X Y, \tag{2.15}$$

for all vector fields X, Y in M.

Now, we recall the definition of pointwise slant submersion defined by Sepet and Ergut [**28**].

Definition 2.3. Let Π be a Riemannian submersion from almost contact metric manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto Riemannian manifold (Θ_2, g_2). If at each given point $q \in \Theta_1$, the wirtinger angle $\theta(X)$ between ϕX and the space $\ker \Pi_*$ is independent of choice of the non-zero vector field $X \in \Gamma(\ker \Pi_*) - \langle \xi \rangle$, then we say that Π is a pointwise slant submersion. In this case, the angle θ can be regarded as a function on Θ_1, which is called slant function of the pointwise slant submersion.

A pointwise slant submersion called slant submersion if its slant function θ is independent of the choice of the point on Θ_1. Then θ is called the slant angle of the slant submersions.

3. Pointwise semi-slant conformal submersions with horizontal vector field-ξ

This section will review the definition that will enable us to comprehend and investigate the concept of pointwise semi-slant conformal submersions from almost contact metric manifolds by taking the Reeb vector filed ξ horizontal into consideration.

Definition 3.1. Let $\Pi\colon (\Theta_1, \phi, \xi, \eta, g_1) \to (\Theta_2, g_2)$ be a horizontal conformal submersion where $(\Theta_1, \phi, \xi, \eta, g_1)$ is an almost contact metric manifold and (Θ_2, g_2) is a Riemannian manifold. A horizontal conformal submersion Π is called a pointwise semi-slant conformal submersion with $\xi \in \Gamma(\ker \Pi)^\perp$ if there exists a distribution $\mathfrak{D}$ such that $\ker \Pi_* = \mathfrak{D} \oplus \mathfrak{D}^\theta$, $\phi(\mathfrak{D}) = \mathfrak{D}$ and for any given point $q \in \Theta_1$ and $X \in (\mathfrak{D}^\theta)_q$, the angle $\theta = \theta(X)$ between ϕX and space $(\mathfrak{D}^\theta)_q$ is independent of choice of non-zero vector $X \in (\mathfrak{D}^\theta)_q$, where $\mathfrak{D}^\theta$ is the orthogonal complement of $\mathfrak{D}$ in $\ker \Pi_*$. In this case, the angle θ can be regarded as a slant function and called pointwise semi-slant function of submersion.

If we suppose m_1 and m_2 are the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, then we have the following:

(i) If $m_1 = 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then Π is a pointwise slant submersion.
(ii) If $m_1 \neq 0$ and $m_2 = 0$, then Π is a invariant submersion
(iii) If $m_1 \neq 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then Π is a pointwise semi-slant submersion.

Let Π be a PWSSCS from an almost contact metric manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2). Then, for any $W \in (\ker \Pi_*)$, we have

$$W = \mathbb{P}W + \mathbb{Q}W \tag{3.1}$$

where $\mathbb{P}$ and $\mathbb{Q}$ are the projections morphism onto $\mathfrak{D}$ and $\mathfrak{D}^\theta$. Now, for any $W \in (\ker \Pi_*)$, we have

$$\phi W = \psi W + \zeta W \tag{3.2}$$

where $\psi W \in \Gamma(\ker \Pi_*)$ and $\zeta W \in \Gamma(\ker \Pi_*)^\perp$. From (3.1) and (3.2), we have

$$\phi U = \phi(\mathbb{P}W) + \phi(\mathbb{Q}W) = \psi(\mathbb{P}W) + \zeta(\mathbb{P}W) + \psi(QW) + \zeta(\mathbb{Q}W).$$

Since $\phi\mathfrak{D} = \mathfrak{D}$, we have $\zeta(\mathbb{P}W) = 0$, we have $\phi U = \psi(\mathbb{P}W) + \psi(\mathbb{Q}W) + \zeta(\mathbb{Q}W)$.

Now, we have the following decomposition $(\ker \Pi_*)^\perp = \zeta\mathfrak{D}^\theta \oplus \mu$, where μ is the orthogonal complement to $\zeta\mathfrak{D}^\theta$ in $(\ker \Pi_*)^\perp$ such that μ is invariant with respect to ϕ. Now, for any $X \in \Gamma(\ker \Pi_*)^\perp$, we have

$$\phi X = \mathfrak{B}X + \mathfrak{C}X \tag{3.3}$$

where $\mathfrak{B}X \in \Gamma(\ker \Pi_*)$ and $\mathfrak{C}X \in \Gamma(\ker \Pi_*)^\perp$.

LEMMA 3.1. *Let $(\Theta_1, \phi, \xi, \eta, g_1)$ be almost contact metric manifold and (Θ_2, g_2) be a Riemannian manifold. If $\Pi\colon \Theta_1 \to \Theta_2$ is a PWSSCS, then we have*

$$-U = -\psi^2 U + \mathfrak{B}\zeta U, \quad \zeta\psi U + \mathfrak{C}\zeta U = 0, \quad -X = \zeta\mathfrak{B}X + \mathfrak{C}^2 X, \quad \eta(X)\xi = \psi\mathfrak{B}X + \mathfrak{B}\mathfrak{C}X,$$

for any vector field $U \in \Gamma(\ker \Pi_*)$ and $X \in \Gamma(\ker \Pi_*)^\perp$.

PROOF. By considering (2.11), (3.2) and (3.3), the proof of Lemma exists. □

Let us now present some beneficial results that will be used throughout the study since $\Pi\colon \Theta_1 \to \Theta_2$ is a PWSSCS.

LEMMA 3.2. *Let Π be a PWSSCS from an almost contact metric manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2); then $\psi^2 W = (-\cos^2\theta)W$, for any vector fields $W \in \Gamma(\mathfrak{D}^\theta)$.*

LEMMA 3.3. *Let Π be a PWSSCS from an almost contact metric manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2); then*

$$\text{(i)}\ \ g_1(\psi Z, \psi W) = \cos^2\theta g_1(Z, W), \quad \text{(ii)}\ \ g_1(\zeta Z, \zeta W) = \sin^2\theta g_1(Z, W),$$

for any vector fields $Z, W \in \Gamma(\mathfrak{D}^\theta)$.

PROOF. The proof of the preceding Lemmas is identical to the proof of Theorem 2.2 of [**9**]. As a result, we omit the proofs. □

Assuming that $(\Theta_1, \phi, \xi, \eta, g_1)$ is a Sasakian manifold and (Θ_2, g_2) is a Riemannian manifold. The effect of the Sasakian structure on the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of PWSSCS $\Pi\colon (\Theta_1, \phi, \xi, \eta, g_1) \to (\Theta_2, g_2)$ is presently being examined.

LEMMA 3.4. *Let $\Pi\colon \Theta_1 \to \Theta_2$ be PWSSCS with semi-slant function θ where, $(\Theta_1, \phi, \xi, \eta, g_1)$ Sasakian manifold and (Θ_2, g_2) be a Riemannian manifold, then we have*

(i) $\mathcal{A}_X \mathfrak{C}Y + \mathcal{V}\nabla_X \mathfrak{B}Y = \mathfrak{B}\mathcal{H}\nabla_X Y + \psi\mathcal{A}_X Y,$
(ii) $\mathcal{H}\nabla_X \mathfrak{C}Y + \mathcal{A}_X \mathfrak{B}Y = \mathfrak{C}\mathcal{H}\nabla_X Y + \zeta\mathcal{A}_X Y + g_1(X, Y)\xi - \eta(Y)X,$
(iii) $\mathcal{V}\nabla_X \psi V + \mathcal{A}_X \zeta V = \mathfrak{B}\mathcal{A}_X V + \psi\mathcal{V}\nabla_X V,$
(iv) $\mathcal{A}_X \psi V + \mathcal{H}\nabla_X \zeta V = \mathfrak{C}\mathcal{A}_X V + \zeta\mathcal{V}\nabla_X V,$
(v) $\mathcal{V}\nabla_V \mathfrak{B}X + \mathcal{T}_V \mathfrak{C}X = \psi\mathcal{T}_V X + \mathfrak{B}\mathcal{H}\nabla_V X + \eta(X)V,$
(vi) $\mathcal{T}_V \mathfrak{B}X + \mathcal{H}\nabla_V \mathfrak{C}X = \zeta\mathcal{T}_V X + \mathfrak{C}\mathcal{H}\nabla_V X,$
(vii) $\mathcal{V}\nabla_U \psi V + \mathcal{T}_U \zeta V = \psi\mathcal{V}\nabla_U V + \mathfrak{B}\mathcal{T}_U V,$
(viii) $\mathcal{T}_U \psi V + \mathcal{H}\nabla_U \zeta V = \mathfrak{C}\mathcal{T}_U V + \zeta\mathcal{V}\nabla_U V - g_1(U, V)\xi,$

for any vector fields $U, V \in \Gamma(\ker \Pi_*)$ and $X, Y \in \Gamma(\ker \Pi_*)^\perp$.

Proof. By using (2.13), (2.15) and (2.8) (3.3), we get first two relations (i) and (ii). Similarly, by considering (2.13), (2.15) (2.8), (2.5)–(2.8) and (3.2) (3.3), the desired results hold good. □

We will now go through some key conclusions that can be utilised to examine the geometry of PWSSCS $\Pi\colon \Theta_1 \to \Theta_2$. From the direct calculations, we can conclude the following:

(a) $(\nabla_U \psi)V = \mathcal{V}\nabla_U \psi V - \psi\mathcal{V}\nabla_U V$,
(b) $(\nabla_U \zeta)V = \mathcal{H}\nabla_U \zeta V - \zeta\mathcal{V}\nabla_U V$,
(c) $(\nabla_X \mathfrak{B})Y = \mathcal{V}\nabla_X \mathfrak{B}Y - \mathfrak{B}\mathcal{H}\nabla_X Y$,
(d) $(\nabla_X \mathfrak{C})Y = \mathcal{H}\nabla_X \mathfrak{C}Y - \mathcal{H}\nabla_X Y$,

for any vector fields $U, V \in \Gamma(\ker \Pi_*)$ and $X, Y \in \Gamma(\ker \Pi_*)^\perp$.

Lemma 3.5. *Let $\Pi : \Theta_1 \to \Theta_2$ be a PWSSCS with semi-slant function θ from Sasakian manifold onto a Riemannian manifolds; then*

(i) $(\nabla_U \psi)V = \mathfrak{B}\mathcal{T}_U V - \mathcal{T}_U \zeta V$,
(ii) $(\nabla_U \zeta)V = \mathfrak{C}\mathcal{T}_U V - \mathcal{T}_U \psi V + g_1(U, V)\xi$,
(iii) $(\nabla_X \mathfrak{B})Y = \psi\mathcal{A}_X Y - \mathcal{A}_X \mathfrak{C}Y$,
(iv) $(\nabla_X \mathfrak{C})Y = \zeta\mathcal{A}_X Y - \mathcal{A}_X \mathfrak{B}Y - \eta(Y)X + g_1(X, Y)\xi$,

for all vector fields $U, V \in \Gamma(\ker \Pi_*)$ and $X, Y \in \Gamma(\ker \Pi_*)^\perp$.

Proof. By using (2.15), (2.5)–(2.8) and formulae (a)–(d) from above, we can obtain the results. □

The tensor fields ψ and ζ, if they are parallel with regard to the Levi-Civita connection ∇ of Θ_1, then we obtain $\mathfrak{B}\mathcal{T}_U V = \mathcal{T}_U \zeta V$, $\mathfrak{C}\mathcal{T}_U V + g_1(U, V)\xi = \mathcal{T}_U \psi V$ for any vector fields $U, V \in \Gamma(T\Theta_1)$.

4. Necessary and sufficient conditions for integrability and totally geodesic

The PWSSCS from Sasakian manifolds onto Riemannian manifolds is discussed in this section. We assume that the Reeb vector field ξ is horizontal and investigate the integrability of both invariant and slant distributions. Aside from this, we likewise look at the important and adequate circumstances for the leaves of distributions to be characterize total geodesic foliation:

Theorem 4.1. *Let $\Pi\colon \Theta_1 \to \Theta_2$ be a PWSSCS with semi-slant function θ such that $\xi \in \Gamma(\ker \Pi_*)^\perp$ where $(\Theta_1, \phi, \xi, \eta, g_1)$ is a Sasakian manifold and (Θ_2, g_2) be a Riemannian manifold. Then the invariant distribution $\mathfrak{D}$ is integrable if and only if $\mathcal{V}\nabla_X \psi W \in \Gamma(\mathfrak{D}^\theta)$, for any vector fields $X, Y \in \Gamma(\mathfrak{D})$ and $W \in \Gamma(\mathfrak{D}^\theta)$.*

Proof. For all vector fields $X, Y \in \Gamma(\mathfrak{D})$, $W \in \Gamma(\mathfrak{D}^\theta)$ and by using (2.11), (2.13) and (2.15), we have $g_1([X, Y], W) = g_1(\nabla_X \phi W, \phi Y) - g_1(\nabla_Y \phi W, \phi X)$. By using (2.5), (2.6) and (3.2), we get

$$g_1([X, Y], W) = g_1(\mathcal{V}\nabla_X \psi W + \mathcal{T}_X \zeta W, \psi Y) - g_1(\mathcal{V}\nabla_Y \psi W + \mathcal{T}_Y \zeta W, \psi X).$$

From this, we get the desired result. □

THEOREM 4.2. *Let* Π *be a PWSSCS with semi-slant function* θ *from Sasakian manifold* $(\Theta_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (Θ_2, g_2) *such that* $\xi \in \Gamma(\ker \Pi_*)^{\perp}$. *Then* $\mathfrak{D}^{\theta}$ *is integrable if and only if*

$$\psi(\mathcal{T}_Z \zeta W - \mathcal{T}_W \zeta Z) = (\mathcal{T}_W \zeta \psi Z + \mathcal{T}_Z \zeta \psi W),$$

for any vector fields $Z, W \in \Gamma(\mathfrak{D}^{\theta})$ *and* $U \in \Gamma(\mathfrak{D})$.

PROOF. By using equation (2.11), (2.13) and (2.15), we may yield

$$g_1([Z,W],U) = g_1(\nabla_Z \phi W, \phi U) - g_1(\nabla_W \phi Z, \phi U),$$

for every vector fields $Z, W \in \Gamma(\mathfrak{D}^{\theta})$ and $U \in \Gamma(\mathfrak{D})$. On using (3.2), we can write

$$\begin{aligned} g_1([Z,W],U) = &-g_1(\nabla_Z \psi W, \phi U) - g_1(\nabla_W \psi Z, \phi U) \\ &+ g_1(\nabla_Z \zeta W, \phi U) - g_1(\nabla_W \zeta Z, \phi U). \end{aligned}$$

By using (2.11) and (2.6) in the third and fourth terms, the above equation can be written as

$$\begin{aligned} g_1([Z,W],U) = &\ g_1(\nabla_Z \phi\psi W, U) + g_1(\nabla_W \phi\psi Z, U) \\ &+ g_1(\mathcal{T}_Z \zeta W, \phi U) - g_1(\mathcal{T}_W \zeta Z, \phi U). \end{aligned} \tag{4.1}$$

Taking into account the fact from (3.2) and Lemma 3.2 in the first term, we have

$$g_1(\nabla_Z \phi\psi W, U) = \sin 2\theta Z(\theta) g_1(W,U) - \cos^2\theta g_1(\nabla_Z W, U) + g_1(\nabla_Z \zeta\psi W, U).$$

Similarly, the second term can be written as:

$$g_1(\nabla_W \phi\psi Z, U) = \sin 2\theta W(\theta) g_1(Z,U) - \cos^2\theta g_1(\nabla_W Z, U) + g_1(\nabla_W \zeta\psi Z, U).$$

By using calculation in the second term, (4.1) can be written as $-g_1(\nabla_W \phi\psi Z, U) = -\cos^2\theta g_1(\nabla_W Z, U) + g_1(\nabla_W \zeta\psi Z, U)$. By using the calculations with (2.6), finally the above equations takes the form

$$\sin^2\theta g_1([Z,W],U) = g_1(\mathcal{T}_W \zeta\psi Z, U) - g_1(\mathcal{T}_Z \zeta\psi W, U) + g_1(\mathcal{T}_Z \zeta W - \mathcal{T}_W \zeta Z, \phi U).$$

From which, we can conclude the result. □

Since $\Pi\colon (\Theta_1, \phi, \xi, \eta, g_1) \to (\Theta_2, g_2)$ be a PWSSCS which ensures the availability of the slant distribution. Following our discussion of the distributions' integrability condition, we will look at the necessary and sufficient conditions that make it possible for distributions leaves to establish a totally geodesic foliation on Θ_1.

THEOREM 4.3. *Let* Π *be PWSSCS with semi-slant function* θ *from Sasakian manifold* $(\Theta_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (Θ_2, g_2) *such that* $\xi \in \Gamma(\ker \Pi_*)^{\perp}$. *Then* $\mathfrak{D}$ *defines totally geodesic foliation on* Θ_1 *if and only if*

$$\mathcal{T}_U \zeta\psi Z = -\psi(\mathcal{T}_U \zeta Z) \quad \text{and} \quad g_1(\mathcal{V}\nabla_U \psi V, \mathfrak{B}X) + g_1(\mathcal{T}_U \psi V, \mathfrak{C}X) = 0,$$

for any vector fields $U, V \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^{\theta})$ *and* $X \in \Gamma(\ker \Pi_*)^{\perp}$.

PROOF. For any vector fields $U, V \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^{\theta})$ and by using orthogonality of V and Z, we get $g_1(\nabla_U V, Z) = -g_1(\nabla_U Z, V)$. Further, in the light of equations (2.11), (2.13), (2.15) and (3.2) (2.6), we get

$$g_1(\nabla_U V, Z) = -g_1(\nabla_U \psi^2 Z, V) + g_1(\nabla_U \zeta\psi Z, V) - g_1(\mathcal{T}_U \zeta Z, \phi V).$$

Since, Π is a PWSSCS with semi-slant function θ, then by using Lemma 3.2 in first term of the above equation, finally this will take the form

$$\sin^2\theta g_1(\nabla_U V, Z) = g_1(\nabla_U \zeta\psi Z, V) - g_1(\mathcal{T}_U \zeta Z, \phi V).$$

From this we can get the first part of the theorem. Now, for any vector fields $U, V \in \Gamma(\mathfrak{D})$ and $X \in \Gamma(\ker \Pi_*)^{\perp}$ with using (2.11), (2.13), (2.15), (2.5) and (3.3), (3.2), we can write $g_1(\nabla_U V, X) = g_1(\mathcal{V}\nabla_U \psi V, \mathfrak{B}X) + g_1(\mathcal{T}_U \psi V, \mathfrak{C}X)$. from which the second part of the theorem holds good. □

The slant distribution is mutually orthogonal to invariant distribution. After discussion geometry of leaves of invariant distribution, it is quite interesting to study the leaves of the slant distribution with geometrical point of view in the following manner.

THEOREM 4.4. *Let $\Pi\colon (\Theta_1, \phi, \xi, \eta, g_1) \to (\Theta_2, g_2)$ be PWSSCS with semi-slant function θ such that $\xi \in \Gamma(\ker \Pi_*)^{\perp}$ where, $(\Theta_1, \phi, \xi, \eta, g_1)$ a Sasakian manifold and (Θ_2, g_2) a Riemannian manifold. Then $\mathfrak{D}^{\theta}$ defines totally geodesic foliation on Θ_1 if and only if $\psi(\mathcal{T}_Z \zeta QW) \in \Gamma(\mathfrak{D}^{\theta})$ and*

$$\begin{aligned}\sin 2\theta X(\theta) g_1(QZ, W) &- g_1(\mathcal{A}_X \zeta\psi \mathbb{Q}Z, W) - \cos^2\theta g_1(\nabla_X \mathbb{Q}Z, W)\\ &= g_1(\mathcal{T}_X \zeta QZ, \psi W) - g_1([Z, X], W) + g_1(X, \operatorname{grad}\ln\lambda) g_1(\zeta QZ, \zeta W)\\ &\quad + g_1(\zeta QZ, \operatorname{grad}\ln\lambda) g_1(X, \zeta W) - g_1(\zeta W, \operatorname{grad}\ln\lambda) g_1(X, \zeta QZ)\\ &\quad - \frac{1}{\lambda^2} g_2(\nabla^{\Pi}_X \Pi_* \zeta \mathbb{Q}Z, \Pi_* \zeta W),\end{aligned}$$

for any vector fields $Z, W \in \Gamma(\mathfrak{D}^{\theta})$, $U \in \Gamma(\mathfrak{D})$ and $X \in \Gamma(\ker \Pi_*)^{\perp}$.

PROOF. Let us consider for any vector fields $Z, W \in \Gamma(\mathfrak{D}^{\theta})$ and $U \in \Gamma(\mathfrak{D})$. In light of (2.11), (2.13), (2.15) with decomposition (3.1) and (3.2), we have

$$\begin{aligned}g_1(\nabla_Z W, U) &= g_1(\nabla_Z \psi \mathbb{P}W, \phi U) + g_1(\nabla_Z \zeta \mathbb{P}W, \phi U)\\ &\quad + g_1(\nabla_Z \psi QW, \phi U) + g_1(\nabla_Z \zeta QW, \phi U).\end{aligned}$$

Considering (2.5), (2.6), from the fact $\mathbb{P}W = 0$ if $W \in \Gamma(\mathfrak{D}^{\theta})$ and since $\mathfrak{D}$ is invariant under ϕ, i.e., $\phi\mathfrak{D} = \mathfrak{D}$, we may yield

$$g_1(\nabla_Z W, U) = -g_1(\nabla_Z \psi^2 QW, U) + g_1(\nabla_Z \zeta QW, \phi U). \tag{4.2}$$

On using Lemma 3.2 in the third term of the above equation, which can be write as $-g_1(\nabla_Z \psi^2 QW, U) = g_1(\nabla_Z (\cos^2\theta) QW, U)$. Then (4.2), will take the form as

$$g_1(\nabla_Z W, U) = g_1(\nabla_Z \zeta QW, \phi U) - 2\sin\theta\cos\theta Z(\theta) g_1(QW, U) + \cos^2\theta g_1(\nabla_Z QW, U).$$

From which the first part of the theorem holds good.

For the second part of theorem, let us suppose for any vector fields $Z, W \in \Gamma(\mathfrak{D}^{\theta})$ and $X \in \Gamma(\ker \Pi_*)^{\perp}$. We start with considering the term $g_1(\nabla_Z W, X)$, and by using equation (2.11), (2.13), (2.5) and (2.15), we have

$$\begin{aligned}g_1(\nabla_Z W, X) = -g([Z, X], W) &- g_1(\nabla_X \psi \mathbb{P}Z, \phi W)\\ &- g_1(\nabla_X \psi QZ, \phi W) - g_1(\nabla_X \zeta QZ, \phi W).\end{aligned}$$

By using (2.11), (2.5), (2.6), and from the fact that $\mathbb{P}Z = 0$ if $Z \in \Gamma(\mathfrak{D}^\theta)$, we have

$$\begin{aligned} g_1(\nabla_Z W, X) = -g([Z, X], W) + g_1(\nabla_X \psi^2 QZ, W) + g_1(\nabla_X \zeta\psi QZ, W) \\ - g_1(\mathcal{A}_X \zeta QZ, \psi W) - g_1(\mathcal{H}\nabla_X \zeta QZ, \zeta W). \end{aligned}$$

Since, Π is a PWSSCS with semi-slant function θ, then with simple steps of calculations, we can write

$$\begin{aligned} g_1(\nabla_Z W, X) = \sin 2\theta X(\theta) g_1(QZ, W) + g_1([Z, X], W) - \cos^2\theta g_1(\nabla_X QZ, W) \\ + g_1(\nabla_X \zeta\psi QZ, W) - g_1(\mathcal{A}_X \zeta QZ, \psi W) - g_1(\mathcal{H}\nabla_X \zeta QZ, \zeta W). \end{aligned}$$

Now, using the conformality of Π from Lemma 2.1 and (2.14), (2.10), we get

$$\begin{aligned} g_1(\nabla_Z W, X) = {} & g_1([Z, X], W) - \cos^2\theta g_1(\nabla_X \mathbb{Q}Z, W) + \sin 2\theta X(\theta) g_1(QZ, W) \\ & + g_1(\nabla_X \zeta\psi QZ, W) - g_1(\mathcal{A}_X \zeta QZ, \psi W) - g_1(X, \operatorname{grad}\ln\lambda) g_1(\zeta QZ, \zeta W) \\ & - g_1(\zeta QZ, \operatorname{grad}\ln\lambda) g_1(X, \zeta W) + g_1(\zeta W, \operatorname{grad}\ln\lambda) g_1(X, \zeta QZ) \\ & + \frac{1}{\lambda^2} g_1(\nabla^\Pi_X \Pi_*(\zeta QZ), \Pi_*(\zeta W)). \qquad \square \end{aligned}$$

Now, we discuss the necessary and sufficient conditions for vertical distributions for $\ker \Pi_*$ is totally geodesic.

Theorem 4.5. *Let us suppose that* Π *be a PWSSCS with semi-slant function* θ *from a Sasakian* $(\Theta_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (Θ_2, g_2) *such that* $\xi \in \Gamma(\ker \Pi_*)^\perp$. *Then* $\ker \Pi_*$ *defines totally geodesic foliation if and only if*

$$\begin{aligned} & \frac{1}{\lambda^2} g_2(\nabla^\Pi_X \Pi_* \zeta \mathbb{Q}U, \Pi_* \zeta V) + g_1(\mathcal{A}_X \psi \mathbb{P}U, \zeta V) - g_1(\mathcal{V}\nabla_X \psi \mathbb{P}U, \psi V) \\ & = \cos^2\theta g_1(\nabla_X QU, V) - \sin 2\theta X(\theta) g_1(\mathbb{Q}U, V) \\ & \quad - g_1(\zeta V, \operatorname{grad}\ln\lambda) g_1(X, \zeta\mathbb{Q}U) + g_1(X, \operatorname{grad}\ln\lambda) g_1(\zeta\mathbb{Q}U, \zeta V) \\ & \quad + g_1(\zeta\mathbb{Q}U, \operatorname{grad}\ln\lambda) g_1(X, \zeta V) + g_1([U, X], V) - g_1(\mathcal{A}_X \zeta\mathbb{Q}U, \psi V), \end{aligned}$$

for any vector fields $U, V \in \Gamma(\ker \Pi_*)$ *and* $X \in \Gamma(\ker \Pi_*)^\perp$.

Proof. From simple steps of calculations with using (2.11), (2.13), (2.15) and decompositions (3.1), (3.2), we can write

$$\begin{aligned} g_1(\nabla_U V, X) = -g_1([U, X], V) - g_1(\nabla_X \psi \mathbb{P}U, \phi V) \\ - g_1(\nabla_X \psi \mathbb{Q}U, \phi V) - g_1(\nabla_X \zeta \mathbb{Q}U, \phi V), \end{aligned} \tag{4.3}$$

for any vector fields $U, V \in \Gamma(\ker \Pi_*)$ and $X \in \Gamma(\ker \Pi_*)^\perp$. In the light of (3.2) and (2.7), second term of above equation become

$$-g_1(\nabla_X \psi \mathbb{P}U, \phi V) = g_1(\mathcal{A}_X \psi \mathbb{P}U, \zeta V) - g_1(\mathcal{V}\nabla_X \psi \mathbb{P}U, \psi V).$$

Similarly, by using (2.11), (2.13) and (2.7), the third term as:

$$-g_1(\nabla_X \psi \mathbb{Q}U, \phi V) = g_1(\nabla_X \psi^2 \mathbb{Q}U, V) + g_1(\nabla_X \zeta\psi \mathbb{Q}U, V).$$

In the last term, taking into account the fact from decomposition (3.2) and equation (2.8), this will take place as

$$-g_1(\nabla_X \zeta \mathbb{Q}U, \phi V) = -g_1(\mathcal{H}\nabla_X \zeta \mathbb{Q}U, \zeta V) - g_1(\mathcal{A}_X \zeta \mathbb{Q}U, V).$$

By using all these facts in (4.3), we get

$$\begin{aligned} g_1(\nabla_U V, X) = &-g_1([U,X],V) + g_1(\mathcal{A}_X\psi\mathbb{P}U,\zeta V) - g_1(\mathcal{V}\nabla_X\psi\mathbb{P}U,\psi V) \\ &+ g_1(\nabla_X\zeta\psi\mathbb{Q}U,V) - g_1(\mathcal{H}\nabla_X\zeta\mathbb{Q}U,\zeta V) - g_1(\mathcal{A}_X\zeta\mathbb{Q}U,\psi V) \\ &+ g_1(\nabla_X\psi^2\mathbb{Q}U,V). \end{aligned}$$

Since, Π is a PWSSCS with semi-slant function θ, using Lemma 3.2 in the fourth term and considering (2.14) and (2.10) in the second last term, above equation finally turns into

$$\begin{aligned} g_1(\nabla_U V, X) = &-g_1([U,X],V) + g_1(\mathcal{A}_X\psi\mathbb{P}U,\zeta V) - g_1(\mathcal{V}\nabla_X\psi\mathbb{P}U,\psi V) \\ &+ 2\sin\theta\cos\theta X(\theta)g_1(\mathbb{Q}U,V) - \cos^2\theta g_1(\nabla_X QU,V) \\ &- g_1(X,\operatorname{grad}\ln\lambda)g_1(\zeta\mathbb{Q}U,\zeta V) - g_1(\zeta\mathbb{Q}U,\operatorname{grad}\ln\lambda)g_1(X,\zeta V) \\ &+ g_1(X,\zeta\mathbb{Q}U)g_1(\zeta V,\operatorname{grad}\ln\lambda) - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_X\Pi_*\zeta QU,\Pi_*\zeta V) \\ &+ g_1(\nabla_X\zeta\psi\mathbb{Q}U,V) - g_1(\mathcal{A}_X\zeta\mathbb{Q}U,\psi V), \end{aligned}$$

from which we can get the result. □

THEOREM 4.6. *Let Π be PWSSCS from a Sasakian manifold $(\Theta_1,\phi,\xi,\eta,g_1)$ onto a Riemannian manifold (Θ_2,g_2) with semi-slant function θ such that $\xi\in\Gamma(\ker\Pi_*)^\perp$. Then the map Π is totally geodesic map if and only if*

(i) $$\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Z\Pi_*\zeta\psi W,\Pi_*X) = g_1(\psi Z,W)\eta(X) - g_1(\mathcal{T}_Z\psi^2 W,X)$$

(ii) $$\begin{aligned} &g_1([Y,U],V) + \sin 2\theta Y(\theta)g_1(U,V) - \cos^2\theta g_1(\nabla_Y U,V) + g_1(\mathcal{A}_Y\zeta\psi U,V) \\ &= -\frac{1}{\lambda^2}g_2(Y(\ln\lambda)\Pi_*\zeta U + \zeta U(\ln\lambda)\Pi_*Y - g_2(Y,\zeta U)\Pi_*(\operatorname{grad}\ln\lambda),\Pi_*\zeta V) \\ &\quad + g_1(\mathcal{A}_Y\zeta U,\psi V) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Y\Pi_*\zeta U,\Pi_*\zeta V) \end{aligned}$$

(iii) $$\begin{aligned} &g_1(\mathcal{A}_X\zeta\mathbb{Q}U,\mathfrak{B}Y) + g_1(\mathfrak{B}X,U)\eta(Y) - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_X\Pi_*\zeta\mathbb{Q}U,\Pi_*\mathfrak{C}Y) \\ &= -\frac{1}{\lambda^2}g_2(X(\ln\lambda)\Pi_*\zeta\mathbb{Q}U + \zeta\mathbb{Q}U(\ln\lambda)\Pi_*X \\ &\quad - g_1(X,\zeta\mathbb{Q}U)\Pi_*(\operatorname{grad}\ln\lambda),\Pi_*\mathfrak{C}Y) + g_1(\nabla_X\zeta\psi\mathbb{Q}U,Y) \\ &\quad - \cos^2\theta g_1(\nabla_X\mathbb{Q}U,Y), \end{aligned}$$

for any $U,V\in\Gamma(\mathfrak{D}^\theta)$, $X,Y\in\Gamma(\ker\Pi_*)^\perp$ and $Z,W\in\Gamma(\mathfrak{D})$, $U_1\in\Gamma(\ker\Pi_*)$.

PROOF. Let us consider $g_2((\nabla\Pi_*)(Z,W),\Pi_*(X))$, for any $Z,W\in\Gamma(\mathfrak{D})$ and $X\in\Gamma(\ker\Pi_*)^\perp$. By using (2.5), (2.6), (2.10), (2.11) and (3.2) with definition 2.2, we get

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(Z,W),\Pi_*(X)) = g_1(\mathcal{T}_Z\psi^2W,X) - g_1(\mathcal{H}\nabla_Z\zeta\psi W,X) + g_1(\psi Z,W)\eta(X).$$

Since Π is a PWSSCS, by using definition 2.2, the second term of above equation can be turn into: $\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(Z,\zeta\psi W),\Pi_*X) - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Z\Pi_*\zeta\psi W,\Pi_*X)$. By Using this in above equation, we may have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(Z,W),\Pi_*(X)) = g_1(\mathcal{T}_Z\psi^2 W,X)&\\ + \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(Z,\zeta\psi W),\Pi_*(X),\Pi_*X)&\\ - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Z\Pi_*\zeta\psi W,\Pi_*(X)) + g_1(\psi Z,W)\eta(X).&\end{aligned}$$

Finally using the conformality of Π with Lemma 3.3, we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(Z,W),\Pi_*(X)) = g_1(\mathcal{T}_Z\psi^2 W,X) - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Z\Pi_*\zeta\psi W,\Pi_*(X))&\\ + g_1(\psi Z,W)\eta(X),&\end{aligned}$$

which is part (i). For part (ii), take into consideration $g_2((\nabla\Pi_*)(U,V),\Pi_*(Y))$, for any $U,V\in\Gamma(\mathfrak{D}^\theta)$ and $Y\in\Gamma(\ker\Pi_*)^\perp$. From (2.10) with definition 2.2, we can write $g_2((\nabla\Pi_*)(U,V),\Pi_*(Y)) = -\lambda^2 g_1(\nabla_U V,Y)$. In the light of (2.11), (2.13), (3.3) and (3.2), we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(U,V),\Pi_*(Y)) = -g_1([Y,U],V) - g_1(\nabla_Y\psi^2 U,V) - g_1(\nabla_Y\zeta\psi U,V)&\\ + g_1(\nabla_Y\zeta U,\psi V) + g_1(\nabla_Y\zeta U,\zeta V).&\end{aligned}$$

Taking into account the fact from (2.6) with Lemma 3.3, we may have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(U,V),\Pi_*(Y)) = -g_1([Y,U],V) - g_1(\nabla_Y(-\cos^2\theta)U,V)&\\ - g_1(\mathcal{A}_Y\zeta\psi U,V) + g_1(\mathcal{A}_Y\zeta U,\psi V) + g_1(\mathcal{H}\nabla_Y\zeta U,\zeta V).&\end{aligned}\tag{4.4}$$

Since Π is a PWSSCS from a Sasakian manifold Θ_1, the second term of (4.4) turn as $g_1(\nabla_Y(\cos^2\theta)U,V) = \sin 2\theta Y(\theta)g_1(U,V) - \cos^2\theta g_1(\nabla_Y U,V)$ where the last term turns as: $g_1(\mathcal{H}\nabla_U\zeta\psi V,Y) = \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Y\Pi_*\zeta U,\Pi_*\zeta V) - \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(Y,\zeta U),\Pi_*\zeta V)$ by using (2.10) and definition 2.2. With all these facts using in (4.4), we can write

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(U,V),\Pi_*(Y)) = -g_1([Y,U],V) - \sin 2\theta Y(\theta)g_1(U,V)&\\ + \cos^2\theta g_1(\nabla_Y U,V) - g_1(\mathcal{A}_Y\zeta\psi U,V) + g_1(\mathcal{A}_Y\zeta U,\psi V)&\\ + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Y\Pi_*\zeta U,\Pi_*\zeta V) - \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(Y,\zeta U),\Pi_*\zeta V)&\end{aligned}$$

Finally, by using Lemma 2.1, the above equations take the form

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(U,V),\Pi_*(Y)) = -g_1([Y,U],V) - \sin 2\theta Y(\theta)g_1(U,V)\\ &+ \cos^2\theta g_1(\nabla_Y U,V) - g_1(\mathcal{A}_Y\zeta\psi U,V) + g_1(\mathcal{A}_Y\zeta U,\psi V) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_Y\Pi_*\zeta U,\Pi_*\zeta V)\\ &- \frac{1}{\lambda^2}g_2(Y(\ln\lambda)\Pi_*\zeta U + \zeta U(\ln\lambda)\Pi_*Y - g_2(Y,\zeta U)\Pi_*(\operatorname{grad}\ln\lambda),\Pi_*\zeta V).\end{aligned}$$

This is the proof of part (ii). For (iii) part, by using (2.11), (2.13), (3.1), (3.2) and consider Lemma 3.3, we can write

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(X,U_1),\Pi_*Y) = -g_1(\mathcal{A}_X\psi\mathbb{P}U,\mathfrak{C}Y) - g_1(\mathcal{V}\nabla_X\psi\mathbb{P}U,\mathfrak{B}Y)$$
$$+ g_1(\phi\nabla_X\psi\mathbb{Q}U,Y) - g_1(\mathcal{H}\nabla_X\zeta\mathbb{Q}U,\mathfrak{C}Y) - g_1(\mathcal{A}_X\zeta\mathbb{Q}U,\mathfrak{B}Y) - g_1(\mathfrak{B}X,U)\eta(Y).$$

In the light of equations (2.11), (2.13), (2.10) and (2.2), we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(X,U_1),\Pi_*Y) &= -g_1(\mathcal{A}_X\psi\mathbb{P}U,\mathfrak{C}Y) - g_1(\mathcal{V}\nabla_X\psi\mathbb{P}U,\mathfrak{B}Y)\\ &- g_1(\mathcal{A}_X\zeta\mathbb{Q}U,\mathfrak{B}Y) - g_1(\mathfrak{B}X,U)\eta(Y) + g_1(\nabla_X\phi\psi\mathbb{Q}U,Y)\\ &- \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(X,\zeta\mathbb{Q}U),\Pi_*\mathfrak{C}Y) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_X\Pi_*\zeta\mathbb{Q}U,\Pi_*\mathfrak{C}Y).\end{aligned} \tag{4.5}$$

Since Π is a PWSSCS from Sasakian manifold onto Riemannian manifold, by using Lemma 2.1, we arrive at equation (4.5), we have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(X,U_1),\Pi_*Y) &= -g_1(\mathcal{A}_X\psi\mathbb{P}U,\mathfrak{C}Y) - g_1(\mathcal{V}\nabla_X\psi\mathbb{P}U,\mathfrak{B}Y)\\ &- g_1(\mathcal{A}_X\zeta\mathbb{Q}U,\mathfrak{B}Y) - g_1(\mathfrak{B}X,U)\eta(Y) + g_1(\nabla_X\zeta\psi\mathbb{Q}U,Y)\\ &+ \sin 2\theta X(\theta)g_1(\mathbb{Q}U,Y) - \cos^2\theta g_1(\nabla_X\mathbb{Q}U,Y)\\ &- \frac{1}{\lambda^2}g_2(X(\ln\lambda)\Pi_*\zeta\mathbb{Q}U,Y + \zeta\mathbb{Q}U(\ln\lambda)\Pi_*X\\ &- g_1(X,\zeta\mathbb{Q}U)\Pi_*(\operatorname{grad}\ln\lambda),\Pi_*\mathfrak{C}Y) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_X\Pi_*\zeta\mathbb{Q}U,\Pi_*\mathfrak{C}Y).\end{aligned}$$

from which we can get part (iii) of the theorem. □

5. Pointwise semi-slant conformal submersions with vertical vector field-ξ

This section will review the definition and results that will enable us to comprehend and investigate the concept of pointwise semi-slant conformal submersions from almost contact metric manifolds by taking the Reeb vector filed ξ vertical into consideration.

Definition 5.1. Let $\Pi\colon(\Theta_1,\phi,\xi,\eta,g_1)\to(\Theta_2,g_2)$ be a horizontal conformal submersion where $(\Theta_1,\phi,\xi,\eta,g_1)$ is an almost contact metric manifold and (Θ_2,g_2) is a Riemannian manifold. A horizontal conformal submersion Π is called a pointwise semi-slant conformal submersion with $\xi\in\Gamma(\ker\Pi*)$, if there exists a distribution $\mathfrak{D}$ such that $\ker\Pi_* = \mathfrak{D}\oplus\mathfrak{D}^\theta\oplus\langle\xi\rangle$, $\phi(\mathfrak{D}) = \mathfrak{D}$ and for any given point $q\in\Theta_1$ and $X\in(\mathfrak{D}^\theta)_q$, the angle $\theta=\theta(X)$ between ϕX and space $(\mathfrak{D}^\theta)_q$ is independent of choice of non-zero vector $X\in(\mathfrak{D}^\theta)_q$, where $\mathfrak{D}^\theta$ is the orthogonal complement of $\mathfrak{D}$ in $\ker\Pi_*$. In this case, the angle θ can be regarded as a slant function and called pointwise semi-slant function of submersion.

Let Π be a PWSSCS from an almost contact metric manifold $(\Theta_1,\phi,\xi,\eta,g_1)$ onto a Riemannian manifold (Θ_2,g_2) with vertical ξ. Then, for any $W\in(\ker\Pi_*)$, we have $W=\mathbb{P}W+\mathbb{Q}W+\eta(W)\xi$ where $\mathbb{P}$ and $\mathbb{Q}$ are the projections morphism onto

$\mathfrak{D}$ and $\mathfrak{D}^\theta$. Let us now present some beneficial results that will be used throughout the study since $\Pi\colon \Theta_1 \to \Theta_2$ is a PWSSCS.

LEMMA 5.1. *Let Π be a PWSSCS from an almost contact metric manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2); then*

$$\psi^2 U = -\cos^2\theta(I - \eta\otimes\xi)U,$$

for any vector field $U \in \Gamma(\ker\Pi_*)$.

LEMMA 5.2. *Let Π be a PWSSCS with vertical ξ from an almost contact metric manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2); then*

(i) $g_1(\psi Z, \psi W) = \cos^2\theta\{g_1(Z, W) - \eta(Z)\eta(W)\}$,

(ii) $g_1(\zeta Z, \zeta W) = \sin^2\theta\{g_1(Z, W) - \eta(Z)\eta(W)\}$,

for any vector fields $Z, W \in \Gamma(\ker\Pi_*)$.

PROOF. The proof of the preceding Lemmas is identical to the proof of Theorem (2.2) of [**9**]. As a result, we omit the proofs. □

The topic of the integrability of slant and invariant distributions will now be discussed. It is quite interesting to investigate the effect of the Reeb vector field ξ on the geometry of distributions if we take it as vertical, as we have discussed in the previous section with the assumption that ξ is a horizontal vector field. With an invariant distribution, we begin.

THEOREM 5.1. *Let Π be a PWSSCS from Sasakian manifold onto a Riemannian manifold and θ is a semi-slant function with vertical ξ. Then the invariant distribution $\mathfrak{D}$ is not integrable.*

PROOF. By consideration $g_1([U, V], \xi)$ for $U, V \in \Gamma(\mathfrak{D})$ with using (2.14), we get $g_1(\nabla_U V - \nabla_V U, \xi) = 2g_1(\phi U, V) \neq 0$. From the last equation, we can conclude that $g_1([U, V], \xi) \neq 0$. Hence, the invariant distribution $\mathfrak{D}$ is not integrable. □

THEOREM 5.2. *Let Π be a PWSSCS from Sasakian manifold onto a Riemannian manifold with vertical ξ and θ is a semi-slant function. Then the slant distribution $\mathfrak{D}^\theta$ is not integrable.*

REMARK 5.1. For the duration of the investigation, we took the Reeb vector field ξ to be vertical. It is evident from the above conclusion that distributions $\mathfrak{D}$ and $\mathfrak{D}^\theta$ are not integrable. If we can determine the integrability necessities of distributions $D \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta \oplus \langle\xi\rangle$, we can resolve this issue.

THEOREM 5.3. *Let $\Pi\colon (\Theta_1, \phi, \xi, \eta, g_1) \to (\Theta_2, g_2)$ be a PWSSCS with ξ vertical, where $(\Theta_1, \phi, \xi, \eta, g_1)$ is a Sasakian manifold and (Θ_2, g_2) is a Riemannian manifold and θ is a slant function. Then the invariant distribution $\mathfrak{D} \oplus \langle\xi\rangle$ is integrable if and only if $g_1(\mathcal{T}_X \zeta W, \psi Y) + g_1(\mathcal{T}_Y \zeta W, \psi X) = 0$, for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $W \in \Gamma(\mathfrak{D}^\theta)$.*

PROOF. Considering the vector fields $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$, $W \in \Gamma(\mathfrak{D}^\theta)$ and by using equations (2.11), (2.13) and (2.15), we have

$$g_1([X, Y], W) = -g_1(\nabla_X \phi W, \phi Y) + g_1(\nabla_Y \phi W, \phi X).$$

By using (2.5), (2.6) (3.2), the above equation in the right-hand side, we can deduce

$$g_1([X,Y],W) = -g_1(\mathcal{T}_X\zeta W, \psi Y) - g_1(\mathcal{T}_Y\zeta W, \psi X).$$

The integrability condition of $\mathfrak{D} \oplus \langle\xi\rangle$ with ξ vertical is similar to the proof of Theorem 4.1, where ξ is horizontal, as can be seen from the computation above. □

In view of the above theorem, we are going to examine the integrability condition for $\mathfrak{D}^\theta \oplus \langle\xi\rangle$.

Corollary 5.1. *Let* Π *be PWSSCS with semi-slant function* θ *and vertical* ξ *from Sasakian manifold* $(\Theta_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (Θ_2, g_2). *Then* $\mathfrak{D}^\theta \oplus \langle\xi\rangle$ *is integrable if and only if* $\psi(\mathcal{T}_Z\zeta W - \mathcal{T}_W\zeta Z) = (\mathcal{T}_W\zeta\psi Z + \mathcal{T}_Z\zeta\psi W)$, *for any vector fields* $Z, W \in \Gamma(\mathfrak{D}^\theta \oplus \langle\xi\rangle)$ *and* $U \in \Gamma(\mathfrak{D})$.

The proof of Corollary 5.1 holds good if we consider the Theorem 4.2 with Reeb vector field ξ to be vertical.

Since the distribution leaves are essential to the geometry of PWSSCS from the Sasakian manifold, studying them will be important. To accomplish this, we are figuring out under what conditions distributions define totally geodesic foliation on Θ_1.

Theorem 5.4. *Let* Π *be PWSSCS with semi-slant function* θ *and vertical* ξ, *from Sasakian manifold* $(\Theta_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (Θ_2, g_2). *Then the invariant distribution* $\mathfrak{D}$ *does not define totally geodesic foliation on* Θ_1.

Same result is true for slant distribution $\mathfrak{D}^\theta$.

Theorem 5.5. *Let* Π *be PWSSCS with semi-slant function* θ *and vertical* ξ, *from Sasakian manifold* $(\Theta_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (Θ_2, g_2). *Then the slant distribution* $\mathfrak{D}^\theta$ *does not define totally geodesic foliation on* Θ_1.

The invariant distribution $\mathfrak{D}$ and slant distribution $\mathfrak{D}^\theta$ does not define totally geodesic foliation because it assumes a vertical Reeb vector field ξ. Here we investigate the geometry of the leaves of the distributions $\mathfrak{D} \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta \oplus \langle\xi\rangle$ to address this problem.

Theorem 5.6. *Let* Π *be PWSSCS with semi-slant function* θ *and vertical* ξ, *from Sasakian manifold* $(\Theta_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (Θ_2, g_2). *Then the invariant* $\mathfrak{D} \oplus \langle\xi\rangle$ *is totally geodesic on* Θ_1 *if and only if*

$$g_1(\nabla_U\zeta\psi Z, V) + \eta(V)g_1(U, \psi Z) = g_1(\mathcal{T}_U\zeta Z, \phi V) + \eta(\psi Z)g_1(U,V),$$
$$g_1(\mathcal{V}\nabla_U\psi V, \mathfrak{B}X) + g_1(\mathcal{T}_U\psi V, \mathfrak{C}X) = 0,$$

for any vector fields $U, V \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$, $Z \in \Gamma(\mathfrak{D}^\theta)$ *and* $X \in \Gamma(\ker \Pi_*)^\perp$.

Proof. For any vector fields $U, V \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$ with using (2.11), (2.13), (2.15) and (3.2), we have

$$\begin{aligned} g_1(\nabla_U V, Z) = &-g_1(\nabla_U\psi^2 Z, V) + g_1(\nabla_U\zeta\psi Z, V) \\ &- g_1(\mathcal{T}_U\zeta Z, \phi V) - \eta(V)g_1(U, \psi Z) + \eta(\psi Z)g_1(U,V). \end{aligned}$$

Since, Π is a PWSSCS with semi-slant function θ, then by using Lemma 3.2 in the first term of above equation, finally this will takes the form

$$\sin^2\theta g_1(\nabla_U V, Z) = g_1(\nabla_U \zeta\psi Z, V) - g_1(\mathcal{T}_U \zeta Z, \phi V) - \eta(V) g_1(U, \psi Z) + \eta(\psi Z) g_1(U, V).$$

From this we can get the first part of the theorem. Now, we consider $g_1(\nabla_U V, X)$ for any vector fields $U, V \in \Gamma(\mathfrak{D})$ and $X \in \Gamma(\ker \Pi_*)^\perp$. On using equation (2.11), (2.13), (2.15) and (3.3), (3.2), this term will take the form as $g_1(\nabla_U V, X) = g_1(\nabla_U \psi V, \mathfrak{B}X + \mathfrak{C}X)$. Finally, considering equation (2.5), we can write

$$g_1(\nabla_U V, X) = g_1(\mathcal{V}\nabla_U \psi V, \mathfrak{B}X) + g_1(\mathcal{T}_U \psi V, \mathfrak{C}X).$$

from which the second part of the theorem holds good. □

The above theorem makes it easy to obtain Theorem 4.3, if we consider the Reeb vector field ξ to be horizontal.

The slant and invariant distributions are mutually orthogonal. Given that the vector field is now vertical, it is highly interesting to examine the geometry of the leaves of the slant distribution $\mathfrak{D}^\theta \oplus \langle\xi\rangle$ from a geometric point of view after discussing the geometry of the leaves of the invariant distribution.

COROLLARY 5.2. *Let $\Pi\colon \Theta_1 \to \Theta_2$ be PWSSCS with semi-slant function θ and vertical vector field ξ where, $(\Theta_1, \phi, \xi, \eta, g_1)$ a Sasakian manifold and (Θ_2, g_2) a Riemannian manifold. Then $\mathfrak{D}^\theta \oplus \langle\xi\rangle$ is defines totally geodesic foliation on Θ_1 if and only if $\psi(\mathcal{V}\nabla_Z \psi \mathbb{P}W + \mathcal{T}_Z \zeta \mathbb{P}W + \mathcal{T}_Z \zeta \mathbb{Q}W) \in \Gamma(\mathfrak{D}^\theta)$ and*

$$\begin{aligned} -g_1(\mathcal{A}_X \zeta\psi\mathbb{Q}Z, W) &+ \cos^2\theta g_1(\nabla_X \mathbb{Q}Z, W) + \sin 2\theta X(\theta) g_1(\mathbb{Q}Z, W) \\ &= -g_1([Z, X], W) - g_1(X, \operatorname{grad}\ln\lambda) g_1(\zeta QZ, \zeta W) \\ &- g_1(\zeta QZ, \operatorname{grad}\ln\lambda) g_1(X, \zeta W) + g_1(\zeta W, \operatorname{grad}\ln\lambda) g_1(\zeta QZ, X) \\ &+ g_1(\mathfrak{B}X, W)\eta(\psi\mathbb{Q}Z) + g_1(\mathcal{A}_X \zeta QZ, \psi W) \\ &+ g_1(\mathfrak{B}X, Z)\eta(W) + \frac{1}{\lambda^2} g_2(\nabla^\Pi_X \Pi_* \zeta\mathbb{Q}Z, \Omega_* \zeta W), \end{aligned}$$

for any vector fields $Z, W \in \Gamma(\mathfrak{D}^\theta \oplus \langle\xi\rangle), U \in \Gamma(\mathfrak{D})$ and $X \in \Gamma(\ker \Pi_*)^\perp$.

REMARK 5.2. We can obtain the proof of the above corollary using the same computation and procedures as in Theorem 4.4 by considering the Reeb vector field ξ to be vertical, i.e., Corollary 5.2 holds true if we take into account that the Reeb vector field ξ is vertical in Theorem 4.4.

6. Pluriharmonicity

In this section, we extended the concept of ϕ-pluriharmonicity from almost Hermitian manifolds to almost contact metric manifold which was once studied and defined by Ohnita [**20**]. Let us suppose that Π be a PWSSCS from Sasakian manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2). Then PWSSCS is ϕ-pluriharmonic, $\mathfrak{D}$-ϕ-pluriharmonic, $\mathfrak{D}^\theta$-ϕ-pluriharmonic, $(\mathfrak{D} - \mathfrak{D}^\theta)$-$\phi$ pluriharmonic, $\ker \Pi_*$-ϕ-pluriharmonic, $(\ker \Pi_*)^\perp$-ϕ-pluriharmonic and $((\ker \Pi_*)^\perp - \ker \Pi_*)$-$\phi$-pluriharmonic if $(\nabla\Pi_*)(W, Z) + (\nabla\Pi_*)(\phi W, \phi Z) = 0$, for any $W, Z \in \Gamma(\mathfrak{D})$, for any $W, Z \in \Gamma(\mathfrak{D}^\theta)$, for any $W \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^\theta)$, for any $W, Z \in \Gamma(\ker \Pi_*)$, for any $W, Z \in \Gamma(\ker \Pi_*)^\perp$ and for any $W \in \Gamma(\ker \Pi_*)^\perp$, $Z \in \Gamma(\ker \Pi_*)$.

THEOREM 6.1. *Let Π be a PWSSCS from Sasakian manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2) with semi-slant function θ and ξ horizontal. Suppose that Π is $\mathfrak{D}^\theta$-ϕ-pluriharmonic. Then $\mathfrak{D}^\theta$ defines totally geodesic foliation on Θ_1 if and only if*

$$\begin{aligned}\nabla^{\Pi}_{\phi X_1}\Pi_*\phi Y_1 + \nabla^{\Pi}_{\zeta X_1}\Pi_*\zeta Y_1 &= \Pi_*(\mathcal{H}\nabla_{\psi X_1}\zeta Y_1 + \mathcal{A}_{\zeta X_1}\psi Y_1 + \mathcal{T}_{\psi X_1}\psi^2\mathbb{P}\psi Y_1 + \mathcal{H}\nabla_{\psi X_1}\zeta\psi\mathbb{P}\psi Y_1)\\ &\quad + \Pi_*(\mathcal{T}_{\psi X_1}\zeta^2\mathbb{Q}\psi Y_1 + \mathcal{H}\nabla_{\psi X_1}\zeta\psi\mathbb{Q}\psi Y_1 + \mathcal{T}_{\psi X_1}\psi\zeta\mathbb{Q}\psi Y_1)\\ &\quad - \cos^2\theta\Pi_*(\nabla_{\psi X_1}\mathbb{Q}\psi Y_1),\end{aligned}$$

for any $X_1, Y_1 \in \Gamma(\mathfrak{D}^\theta)$.

PROOF. For any $X_1, Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and using the pluriharmonicity of ϕ with equation (2.10), we get

$$\Pi_*\nabla_{X_1}Y_1 = \nabla^{\Pi}_{\phi X_1}\Pi_*\phi Y_1 - \Pi_*\nabla_{\phi X_1}\phi Y_1. \tag{6.1}$$

The second term in the right-hand side of the above equation with using equation (3.2), takes the form as $\Pi_*\nabla_{\psi X_1}\psi Y_1 + \Pi_*\nabla_{\psi X_1}\zeta Y_1 + \Pi_*\nabla_{\zeta X_1}\psi Y_1 + \Pi_*\nabla_{\psi X_1}\zeta Y_1$. Now, equation (6.1) can be written as

$$\begin{aligned}\Pi_*\nabla_{X_1}Y_1 = \nabla^{\Pi}_{\phi X_1}\Pi_*\phi Y_1 - \Pi_*\nabla_{\psi X_1}\psi Y_1 - \Pi_*\nabla_{\psi X_1}\zeta Y_1\\ - \Pi_*\nabla_{\zeta X_1}\psi Y_1 - \Pi_*\nabla_{\psi X_1}\zeta Y_1.\end{aligned}$$

Taking account the fact that Π is PWSSCS with using equations (2.6), (2.7), (2.10) and (3.1), we have

$$\begin{aligned}\Pi_*\nabla_{X_1}Y_1 &= -\Pi_*(\mathcal{T}_{\psi X_1}\zeta Y_1 + \mathcal{H}\nabla_{\psi X_1}\zeta Y_1 + \mathcal{A}_{\zeta X_1}\psi Y_1 + \mathcal{V}\nabla_{\zeta X_1}\psi Y_1)\\ &\quad + \{\zeta X_1(\ln\lambda)\Pi_*\zeta Y_1 + \zeta Y_1(\ln\lambda)\Pi_*\zeta X_1 - g_1(\zeta X_1, \zeta Y_1)\Pi_*(\operatorname{grad}\ln\lambda)\}\\ &\quad - \nabla^{\Pi}_{\phi X_1}\Pi_*\phi Y_1 - \nabla^{\Pi}_{\zeta X_1}\Pi_*\zeta Y_1 + \Pi_*(\phi\nabla_{\psi X_1}\phi(\mathbb{P}\psi Y_1 + \mathbb{Q}\psi Y_1)).\end{aligned}$$

In the last term in the right-hand side of the above equation with Lemma 3.2 and equations (2.6) and (2.7), we may have

$$\begin{aligned}\Pi_*\nabla_{X_1}Y_1 &= \{\zeta X_1(\ln\lambda)\Pi_*\zeta Y_1 + \zeta Y_1(\ln\lambda)\Pi_*\zeta X_1 - g_1(\zeta X_1, \zeta Y_1)\Pi_*(\operatorname{grad}\ln\lambda)\}\\ &\quad + \Pi_*(\mathcal{T}_{\psi X_1}\psi^2\mathbb{P}\psi Y_1 + \mathcal{V}\nabla_{\psi X_1}\psi^2\mathbb{P}\psi Y_1 + \mathcal{T}_{\psi X_1}\zeta\psi\mathbb{P}\psi Y_1 + \mathcal{H}\nabla_{\psi X_1}\zeta\psi\mathbb{P}\psi Y_1)\\ &\quad + \sin 2\theta\psi X_1(\theta)\Pi_*(\mathbb{Q}\psi Y_1) - \cos^2\theta\Pi_*(\nabla_{\psi X_1}\mathbb{Q}\psi Y_1) + \Pi_*(\mathcal{T}_{\psi X_1}\zeta\psi\mathbb{Q}\psi Y_1\\ &\quad + \mathcal{H}\nabla_{\psi X_1}\zeta\psi\mathbb{Q}\psi Y_1) + \Pi_*(\mathcal{T}_{\psi X_1}\psi\zeta\mathbb{Q}\psi Y_1 + \mathcal{V}\nabla_{\psi X_1}\psi\zeta\mathbb{Q}\psi Y_1 + \mathcal{T}_{\psi X_1}\zeta^2\mathbb{Q}\psi Y_1\\ &\quad + \mathcal{V}\nabla_{\psi X_1}\zeta^2\mathbb{Q}\psi Y_1) + \Pi_*(\mathcal{T}_{\psi X_1}\zeta Y_1 + \mathcal{H}\nabla_{\psi X_1}\zeta Y_1 + \mathcal{A}_{\zeta X_1}\psi Y_1 + \mathcal{V}\nabla_{\zeta X_1}\psi Y_1)\\ &\quad - \nabla^{\Pi}_{\phi X_1}\Pi_*\phi Y_1 - \nabla^{\Pi}_{\zeta X_1}\Pi_*\zeta Y_1.\end{aligned}$$

□

THEOREM 6.2. *Let Π be a PWSSCS from the Sasakian manifold $(\Theta_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (Θ_2, g_2) with semi-slant function θ and ξ horizontal.*

Suppose that Π *is* $((\ker\Pi_*)^\perp - \ker\Pi_*)$-$\phi$-*pluriharmonic. Then the horizontal distribution* $(\ker\Pi_*)^\perp$ *defines totally geodesic foliation on* Θ_1 *if and only if*

$$\begin{aligned}-\cos^2\theta(\zeta\mathcal{V}\nabla_{\mathfrak{C}X}\mathbb{Q}U + \mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\mathbb{Q}U) + \nabla^{\Pi}_{\mathfrak{C}X}\Pi_*\zeta\psi\mathbb{P}U - \Pi_*\{\sin 2\theta\mathfrak{C}X(\theta)\zeta\mathbb{Q}U\}\\ = \mathfrak{C}X(\ln\lambda)\Pi_*\zeta\psi\mathbb{P}U + \zeta\psi\mathbb{P}U(\ln\lambda)\Pi_*\mathfrak{C}X - g_1(\mathfrak{C}X,\zeta\psi\mathbb{P}U)\Pi_*(\operatorname{grad}\ln\lambda)\\ + \mathfrak{C}X(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}U + \zeta\psi\mathbb{Q}U(\ln\lambda)\Pi_*\mathfrak{C}X - g_1(\mathfrak{C}X,\zeta\psi\mathbb{Q}U)\Pi_*(\operatorname{grad}\ln\lambda)\\ + \Pi_*\{\mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\psi^2\mathbb{P}U + \zeta\mathcal{V}\nabla_{\mathfrak{C}X}\psi^2\mathbb{P}U + \zeta\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{P}U + \zeta\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{Q}U\}\\ + \Pi_*\{\mathcal{A}_XU + \mathcal{H}\nabla_{\mathfrak{B}X}\psi U + \mathfrak{C}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\zeta U + \zeta\mathcal{V}\nabla_{\mathfrak{B}X}\mathfrak{B}\zeta U\} + \nabla^{\Pi}_{\phi X}\Pi_*\zeta U\\ + \Pi_*\{g_1(\zeta\mathfrak{B}X,\zeta U)\xi + \eta(\zeta U)\zeta\mathfrak{B}X + g_1(\mathfrak{B}\mathfrak{C}X,\psi U)\xi\} - \nabla^{\Pi}_{\mathfrak{C}X}\Pi_*\zeta\psi\mathbb{Q}U.\end{aligned}$$

for any $X\in\Gamma(\ker\Pi_*)^\perp$ *and* $U\in\Gamma(\ker\Pi_*)$.

Proof. For any $X\in\Gamma(\ker\Pi_*)^\perp$, $U\in\Gamma(\ker\Pi_*)$ and using (2.10), (3.1), (3.2), (2.5) with considering the fact of pluriharminicity of ϕ, we can write

$$\begin{aligned}\Pi_*(\nabla_{\mathfrak{C}X}\zeta U) = -\Pi_*\nabla_XU + \nabla^{\Pi}_{\phi X}\Pi_*\phi U + \mathcal{H}\nabla_{\mathfrak{B}X}\psi U\\ - \Pi_*(\nabla_{\mathfrak{B}X}\zeta U - \Pi_*(\mathcal{T}_{\mathfrak{B}X}\psi U + \nabla_{\mathfrak{C}X}\psi U).\end{aligned} \tag{6.2}$$

The second last term of the above equation, by using the equations (2.11) and (2.12) turns into:

$$\Pi_*(\nabla_{\mathfrak{B}X}\zeta U) = \Pi_*(\phi\nabla_{\mathfrak{B}X}\phi\zeta U) + g_1(\zeta\mathfrak{B}X,\zeta U)\Pi_*\xi + \eta(\zeta U)\Pi_*\zeta\mathfrak{B}X$$

whereas, the last term as:

$$-\Pi_*(\phi\nabla_{\mathfrak{C}X}\psi U) = \Pi_*(\phi\nabla_{\mathfrak{C}X}\phi\psi U) + g_1(\mathfrak{B}\mathfrak{C}X,\psi U)\Pi_*\xi.$$

By using these facts (6.2) reduces to

$$\begin{aligned}\Pi_*(\nabla_{\mathfrak{C}X}\zeta U) = -\Pi_*\nabla_XU + \nabla^{\Pi}_{\phi X}\Pi_*\phi U - \Pi_*(\mathcal{T}_{\mathfrak{B}X}\psi U + \mathcal{H}\nabla_{\mathfrak{B}X}\psi U) + \Pi_*(\phi\nabla_{\mathfrak{C}X}\phi\psi U)\\ + g_1(\mathfrak{B}\mathfrak{C}X,\psi U)\Pi_*\xi + \Pi_*(\phi\nabla_{\mathfrak{B}X}\phi\zeta U) + g_1(\zeta\mathfrak{B}X,\zeta U)\Pi_*\xi + \eta(\zeta U)\Pi_*\zeta\mathfrak{B}X.\end{aligned}$$

Now, by using equation (3.1), (3.2), (3.3), (2.10) with Lemma 3.2, we can write

$$\begin{aligned}\Pi_*(\nabla_{\mathfrak{C}X}\zeta U) = \Pi_*(\mathfrak{A}_XU + \mathcal{V}\nabla_XU - \mathcal{T}_{\mathfrak{B}X}\psi U + \mathcal{H}\nabla_{\mathfrak{B}X}\psi U + \zeta\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{Q}U)\\ + \Pi_*\{g_1(\zeta\mathfrak{B}X,\zeta U)\xi + \eta(\zeta U)\zeta\mathfrak{B}X + g_1(\mathfrak{B}\mathfrak{C}X,\psi U)\xi\} + \nabla^{\Pi}_{\phi X}\Pi_*\phi U\\ + \Pi_*\{\mathfrak{B}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\zeta U + \mathfrak{C}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\zeta U + \psi\mathcal{V}\nabla_{\mathfrak{B}X}\mathfrak{B}\zeta U + \zeta\mathcal{V}\nabla_{\mathfrak{B}X}\mathfrak{B}\zeta U\}\\ + \Pi_*\{\mathfrak{B}\mathcal{A}_{\mathfrak{C}X}\psi^2\mathbb{P}U + \mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\psi^2\mathbb{P}U + \psi\mathcal{V}\nabla_{\mathfrak{C}X}\psi^2\mathbb{P}U + \zeta\mathcal{V}\nabla_{\mathfrak{C}X}\psi^2\mathbb{P}U\}\\ + \Pi_*\{\psi\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{P}U + \zeta\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{P}U + \mathfrak{B}\mathcal{H}\nabla_{\mathfrak{C}X}\zeta\psi\mathbb{P}U + \psi\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{Q}U\}\\ + \nabla^{\Pi}_{\mathfrak{C}X}\Pi_*\zeta\psi\mathbb{P}U + (\nabla\Pi_*)(\mathfrak{C}X,\zeta\psi\mathbb{P}U) + \nabla^{\Pi}_{\mathfrak{C}X}\Pi_*\zeta\psi\mathbb{Q}U + (\nabla\Pi_*)(\mathfrak{C}X,\zeta\psi\mathbb{Q}U)\\ + \Pi_*\{\sin 2\theta\mathfrak{C}X(\theta)\zeta\mathbb{Q}U - \cos^2\theta\phi\nabla_{\mathfrak{C}X}\mathbb{Q}U\} + \nabla^{\Pi}_{\mathfrak{C}X}\Pi_*\zeta\psi\mathbb{P}U.\end{aligned}$$

Since Π is a PWSSCS, then by using Lemma 2.1, the above equation finally turn into

$$\begin{aligned}\Pi_*(\nabla_{\mathfrak{C}X}\zeta U) = \cos^2\theta(\zeta\mathcal{V}\nabla_{\mathfrak{C}X}\mathbb{Q}U + \mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\mathbb{Q}U) - \nabla^{\Pi}_{\mathfrak{C}X}\Pi_*\zeta\psi\mathbb{P}U + \Pi_*\{\sin 2\theta\mathfrak{C}X(\theta)\zeta\mathbb{Q}U\}\\ + \mathfrak{C}X(\ln\lambda)\Pi_*\zeta\psi\mathbb{P}U + \zeta\psi\mathbb{P}U(\ln\lambda)\Pi_*\mathfrak{C}X - g_1(\mathfrak{C}X,\zeta\psi\mathbb{P}U)\Pi_*(\operatorname{grad}\ln\lambda)\\ + \mathfrak{C}X(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}U + \zeta\psi\mathbb{Q}U(\ln\lambda)\Pi_*\mathfrak{C}X - g_1(\mathfrak{C}X,\zeta\psi\mathbb{Q}U)\Pi_*(\operatorname{grad}\ln\lambda)\end{aligned}$$

$$+ \Pi_*\{\mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\psi^2\mathbb{P}U + \zeta\mathcal{V}\nabla_{\mathfrak{C}X}\psi^2\mathbb{P}U + \zeta\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{P}U + \zeta\mathcal{A}_{\mathfrak{C}X}\zeta\psi\mathbb{Q}U\}$$
$$+ \Pi_*\{\mathcal{A}_X U + \mathcal{H}\nabla_{\mathfrak{B}X}\psi U + \mathfrak{C}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\zeta U + \zeta\mathcal{V}\nabla_{\mathfrak{B}X}\mathfrak{B}\zeta U\} + \nabla^{\Pi}_{\phi X}\Pi_*\zeta U$$
$$+ \Pi_*\{g_1(\zeta\mathfrak{B}X, \zeta U)\xi + \eta(\zeta U)\zeta\mathfrak{B}X + g_1(\mathfrak{B}\mathfrak{C}X, \psi U)\xi\} - \nabla^{\Pi}_{\mathfrak{C}X}\Pi_*\zeta\psi\mathbb{Q}U.$$

From which we can get the desired result. □

References

1. M. A. Akyol, *Conformal semi-slant submersions*, Int. J. Geom. Methods Mod. Phys. **14**(7) (2017), 1750114.
2. M. A. Akyol, B. Şahin, *Conformal slant submersions*, Hacet. J. Math. Stat. **48**(1) (2019), 28–44.
3. ______, *Conformal anti-invariant submersions from almost Hermitian manifolds*, Turk. J. Math. **40** (2016), 43–70.
4. ______, *Conformal semi-invariant submersions*, Commun. Contemp. Math. **19** (2017), 1650011.
5. I. Al-Dayel, T. Fatima, S. Deshmukh, M. Shuaib, *A note on conformal bi-slant submersion from Kenmotsu manifold*, J. Geom. Phys. **190** (2023), 104864.
6. I. Al-Dayel, M. Shuaib S. Deshmukh, T. Fatima, *ϕ-pluriharmonicity in quasi bi-slant conformal $\xi^{\perp}$-submersions: a comprehensive study*, AIMS Math. **8**(9) (2023), 21746–21768.
7. P. Baird, J. C. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, Lond. Math. Soc. Monogr. **29**, Oxford University Press, The Clarendon Press, Oxford, 2003.
8. J.-P. Bourguignon, H. B. Lawson Jr, *Stability and isolation phenomena for Yang Mills fields*, Comm. Math. Phys. **79**(2) (1981), 189–230.
9. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J. **42**(1) (2000), 125–138.
10. I. K. Erken, C. Murathan, *On slant Riemannian submersions for cosymplectic manifolds*, Bull. Korean Math. Soc. **51**(6) (2014), 1749–1771.
11. M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian Submersions and Related Topics*, World Scientific, River Edge, NJ, 2004.
12. A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16** (1967), 715–737.
13. B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier **28** (1978), 107–144.
14. S. Gudmundsson, J. C. Wood, *Harmonic morphisms between almost Hermitian manifolds*, Boll. Unione Mat. Ital., VII. Ser., B **11**(2, supl.) (1997), 185–197.
15. Y. Gunduzalp, *Semi-slant submersions from almost product Riemannian manifolds*, Demonstr. Math. **49**(3) (2016), 345–356.
16. S. Ianu¸s, M. Vişinescu, *Kaluza-Klein theory with scalar fields and generalised Hopf manifolds*, Classical Quantum Gravity **4**(5) (1987), 1317–1325.
17. T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. **19** (1979), 215–229.
18. T. W. Lee, B. Şahin, *Pointwise slant submersions*, Bull. Korean Math. Soc. **51**(4) (2014), 1115–1126.
19. M. T. Mustafa, *Applications of harmonic morphisms to gravity*, J. Math. Phys. **41** (2000), 6918–6929.
20. Y. Ohnita, *On pluriharmonicity of stable harmonic maps*, J. Lond. Math. Soc. **2** (1987), 563–568.
21. B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. **13** (1966), 459–469.
22. K. S. Park, R. Prasad, *Semi-slant submersions*, Bull. Korean Math. Soc. **50**(3) (2013), 951–962.
23. R. Prasad, S. Kumar, *Conformal anti-invariant submersions from nearly Kaehler manifolds*, Palest. J. Math. **8**(2) (2019).

24. B. Şahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds*, Cent. Eur. J. Math. **3** (2010), 437–447.
25. ———, *Semi-invariant Riemannian submersions from almost Hermitian manifolds*, Can. Math. Bull. **56** (2013), 173–183.
26. ———, *Slant submersions from almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. **1** (2011), 93–105.
27. B. Şahin, M. A. Akyol, *Conformal anti-invariant submersion from almost Hermitian manifolds*, Turk. J. Math. **40** (2016), 43–70
28. S. A. Sepet, M. A. Ergut, *Pointwise slant submersions from cosymplectic manifolds*, Turk. J. Math. **40** (2016), 582–593.
29. M. Shuaib, T. Fatima, *A note on conformal hemi-slant submersions*, Afr. Mat. **34**(4) (2023).
30. S. Kumar, *Conformal hemi-slant submersions from almost hermitian manifolds*, Commun. Korean Math. Soc. **35**(3) (2020), 999–1018.
31. H. M. Tastan, B. Şahin, Ş. Yanan, *Hemi-slant submersions*, Mediterr. J. Math. **13**(4) (2016), 2171–2184.
32. B. Watson, *Almost Hermitian submersions*, J. Differ. Geom. **11**(1) (1976), 147–165.
33. ———, *G, G'-Riemannian submersions and nonlinear gauge field equations of general relativity*, In: T. Rassias (ed.), *Global Analysis – Analysis on Manifolds, Dedicated to Marston Morse*, Teubner-Texte Math. **57** (1983), 324–349.

(Received 17 02 2024)

Department of Mathematics and Statistics
College of Science
Imam Mohammad Ibn Saud Islamic University (IMSIU)
Riyadh
Saudi Arabia
mskhan@imamu.edu.sa

Department of Mathematics
Lovely Professional University
Phagwara
Punajb
India
shuaibyousuf6@gmail.com

Department of Mathematics and Statistics
College of Science
Imam Mohammad Ibn Saud Islamic University (IMSIU)
Riyadh
Saudi Arabia
iaaldayel@imamu.edu.sa

Department of Mathematics
College of Science
King Abdulaziz University
Jeddah
Saudi Arabia
cenap.ozel@gmail.com