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WEIGHTED SHARING OF THREE SETS WITH LEAST CARDINALITIES

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Abstract. We derive some sufficient conditions in terms of three weighted shared sets to make two meromorphic functions identical. Actually to serve our purpose, we try to maintain the cardinalities of the main range sets as small as possible. Consequently, by exhibiting some new analysis in the proof, we obtain three results which extend and improve a number of earlier results. Finally, we pose two relevant open questions for future research.

1. Background and main results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. The notation $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, outside a possible exceptional set of finite linear measure.

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and let us denote

$$E_f(S) = \bigcup_{a\in S} \{z : f(z) - a = 0, \text{ counted according to multiplicities}\},$$

$$\bar{E}_f(S) = \bigcup_{a\in S} \{z : f(z) - a = 0, \text{ counted ignoring multiplicities}\}.$$

If $E_f(S) = E_g(S)$ ($\bar{E}_f(S) = \bar{E}_g(S)$) we say that f and g share the set S CM (IM).

Let us invoke the definition of weighted sharing of sets introduced by Lahiri.

Definition 1.1. [**10**] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or infinity. For any complex number a, we denote by $E_k(a; f)$, the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k + 1$ times if $m > k$. We denote by $E_f(S, k)$ the set $\bigcup_{a\in S} E_k(a; f)$.

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If $E_f(S,k)=E_g(S,k)$, we say that f, g share the set S with weight k and we write it as f, g share (S,k). Clearly $E_f(S)=E_f(S,\infty)$ and $\bar{E}_f(S)=E_f(S,0)$.

When S is a singleton set then Definition 1.1 coincides with the traditional definition of weighted value sharing. If $E_k(a;f)=E_k(a;g)$, we write it as f, g share (a,k).

Let $S=\{a_1,a_2,\ldots,a_r\}$, where $a_i\in\mathbb{C}\ \ \forall i\in\{1,2,\ldots,r\}$. The polynomial

$$P_S(z)=(z-a_1)(z-a_2)\ldots(z-a_r),$$

is called generating polynomial of S. Let $P_S^{'}(z)=(z-b_1)^{t_1}(z-b_2)^{t_2}\ldots(z-b_m)^{t_m}$, where $t_1,t_2,\ldots,t_m$ $(m\leqslant r)$ be non-negative integers and $b_i\in\mathbb{C}\ \ \forall i\in\{1,2,\ldots,m\}$. Then $S^{'}=\{b_1,b_2,\ldots,b_m\}$ will be called derived set of $P_S(z)$, and S is called ground set of $P_S(z)$.

In 1994, regarding sharing of three sets and uniqueness of meromorphic function the following question was asked by Yi [**17**], which is pertinent to the famous question of Gross [**8**].

Question 1.1. [**17**] Can one find three finite sets S_j $(j=1,2,3)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2,3$ must be identical?

Several research articles were published to find the possible answers of the above question. In terms of the following theorem, in 2002, for the least cardinalities of the range sets, Qiu and Fang [**15**] answered Question 1.1.

Theorem 1.1. [**15**] *Let $n\geqslant 3$ be a positive integer, $S^*=\{z: z^n-z^{n-1}-1=0\}$ and f and g be two non–constant meromorphic functions whose poles are of multiplicities at least 2. If $E_f(S^*,\infty)=E_g(S^*,\infty)$, $E_f(\{0\},\infty)=E_g(\{0\},\infty)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$, then $f\equiv g$.*

During last few decades, many authors improved Theorem 1.1 imposing different conditions on deficient values in different directions (see [**1**, **3**, **5**]).

For a non-zero complex number a, we definethe polynomial $P(z)$ by

$$P(z)=\frac{z^3}{3}-\frac{az^2}{2}-c=Q(z)-c,\quad c\neq 0,\ -\frac{a^3}{6}, \tag{1.1}$$

where $Q(z)=\big(\frac{z^3}{3}-\frac{az^2}{2}\big)$.

With respect to this polynomial we now present our main results.

Theorem 1.2. *Let $S=\{z\mid P(z)=0\}$, where $P(z)$ is defined by (1.1). Suppose that f and g be two non-constant meromorphic functions having no simple poles satisfying $E_f(S,4)=E_g(S,4)$, $E_f(\{0\},0)=E_g(\{0\},0)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ then $f\equiv g$.*

The following example shows that the condition of having no simple poles for f and g can not be removed in Theorem 1.2.

Example 1.1. Let

$$g(z)=\frac{3a}{2}\Big(\frac{e^z+1}{e^{2z}+e^z+1}\Big),\quad f\equiv e^z g$$

and S be as in Theorem 1.2. Note that as $f^2\left(f-\frac{3a}{2}\right) \equiv g^2\left(g-\frac{3a}{2}\right)$, we have $E_f(S,\infty)=E_g(S,\infty)$. Also $E_f(\{0\},\infty)=E_g(\{0\},\infty)$, $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$. Here both f and g have simple poles, but $f\not\equiv g$.

In the direction of least cardinalities, Fang and Xu [**7**] proved a result of three set sharing with conditions on deficient values. In order to get rid of the conditions over deficient values, in 2007, Lü and Xu [**14**] improved a result of Fang [**6**] by taking the sharing of a doubleton set. Further, in 2010, first author [**4**] modified the result of Lü and Xu [**14**] to obtain the following:

THEOREM 1.3. [**4**] *Let $S^{\#}=\{z: z^3-z^2-1=0\}$, f and g be two non-constant meromorphic functions satisfying $E_f(S^{\#},3)=E_g(S^{\#},3)$, $E_f(\{0,\frac{2}{3}\},0)=E_g(\{0,\frac{2}{3}\},0)$, $E_f(\{\infty\},1)=E_g(\{\infty\},1)$, then $f\equiv g$.*

One of the intentions of writing the paper is to generalize Theorem 1.3. In this connection, we mention that the idea of some portion of the proof of Theorem 1.3 was somehow been taken from that of [**14**], which made the analysis of that particular part clumsy. In our theorem, we have adopted a new analysis technique, which has not been used so far to make the proof well organized.

THEOREM 1.4. *Let S be defined as in Theorem 1.2 with $c\neq-\frac{a^3}{12}$ and if for two non-constant meromorphic functions f and g, $E_f(S,3)=E_g(S,3)$, $E_f(\{0,a\},0)=E_g(\{0,a\},0)$ and $E_f(\{\infty\},1)=E_g(\{\infty\},1)$, then $f\equiv g$.*

Next example shows that one can't replace the second set by an arbitrary set rather that derived set of the corresponding polynomial.

EXAMPLE 1.2. Let us consider the set S be defined in Theorem 1.2 and f and g be defined in Example 1.1. Then $E_f(S,\infty)=E_g(S,\infty)$, $E_f\left(\{0,\frac{3a}{2}\},\infty\right)=E_g\left(\{0,\frac{3a}{2}\},\infty\right)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ but $f\not\equiv g$.

The following example shows that, if the set S defined in Theorem 1.2 is replaced by zeros of polynomial of degree 2, similar to (1.1), then conclusion of Theorem 1.4 does not hold in general.

EXAMPLE 1.3. For a complex number a, let us consider the polynomial

$$Q(z)=\frac{z^2}{2}-az-c,\quad c\neq 0,\ -\frac{a^2}{2}.$$

Case 1. For $a\neq 0$, let us assume that $g(z)=\frac{2a}{1+e^z}$, $f\equiv e^z g$. and $S_1=\{z\mid Q(z)=0\}$. As $(f^2-2af)=(g^2-2ag)$, $E_f(S_1,\infty)=E_g(S_1,\infty)$ and also $E_f(\{0,a\},\infty)=E_g(\{0,a\},\infty)$, $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$, but $f\not\equiv g$.

Case 2. When $a=0$, let us assume that $f(z)=\frac{e^z}{(e^z+1)}$, $g=-f$. As $f^2=g^2$, $E_f(S_1,\infty)=E_g(S_1,\infty)$ and also $E_f(\{0\},\infty)=E_g(\{0\},\infty)$, $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ but $f\not\equiv g$.

Now, if we minutely observe the results of [**1, 3, 13**], we see that in each of the results corresponding to the main range sets for $n=4$, the sharing of poles was taken as CM sharing. It will be natural to ponder over the case whether the CM sharing of the poles can further be reduced to finite weight. In this respect, we next introduce the following polynomial

$$\tilde{P}(z) = \frac{z^4}{4} - \frac{az^3}{3} - c, \quad c \neq 0, -\frac{a^4}{12}. \tag{1.2}$$

With respect to the above introduced polynomial, we now present our last theorem.

THEOREM 1.5. *Let* $\tilde{S} = \{z \mid \tilde{P}(z) = 0\}$*, where* $\tilde{P}(z)$ *is defined by (1.2). Suppose that* f *and* g *be two non-constant meromorphic functions having multiple poles satisfying* $E_f(\tilde{S}, 3) = E_g(\tilde{S}, 3)$*,* $E_f(\{0\}, 0) = E_g(\{0\}, 0)$ *and* $E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$ *then* $f \equiv g$*.*

We have significantly reduced the weight of ∞ in Theorem 1.5.

The following example shows that the condition of having no simple poles for f and g can not be removed in Theorem 1.5.

EXAMPLE 1.4. Let

$$g(z) = \frac{4a}{3}\left(\frac{1+e^z+e^{2z}}{1+e^z+e^{2z}+e^{3z}}\right), \quad f \equiv e^z g$$

and $\tilde{S}$ be as in Theorem 1.5. As $f^3\left(f - \frac{4a}{3}\right) \equiv g^3\left(g - \frac{4a}{3}\right)$, $E_f(\tilde{S}, \infty) = E_g(\tilde{S}, \infty)$ and also $E_f(\{0\}, \infty) = E_g(\{0\}, \infty)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$. Here both f and g have simple poles but $f \not\equiv g$.

Though for the standard definitions and notations of the value distribution theory we refer to [**9**], we now explain some notations which are used in the paper.

DEFINITION 1.2. [**11**] For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leqslant m)$ ($N(r, a; f \mid\geqslant m)$) the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity. $\overline{N}(r, a; f \mid\leqslant m)$ ($\overline{N}(r, a; f \mid\geqslant m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

DEFINITION 1.3. [**1**, **16**] Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, in the same way we can define $\overline{N}_L(r, 1; g)$

DEFINITION 1.4. [**12**] We denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geqslant 2)$.

DEFINITION 1.5. [**12**] Let f, g share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$, when f, g share $(a, 0)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G and $\tilde{F}, \tilde{G}$ are pairs of non-constant meromorphic functions defined in $\mathbb{C}$ as follows:

$$F = \frac{f^2}{3c}\Big(f - \frac{3a}{2}\Big), \qquad G = \frac{g^2}{3c}\Big(g - \frac{3a}{2}\Big), \tag{2.1}$$
$$\tilde{F} = \frac{f^3}{4c}\Big(f - \frac{4a}{3}\Big), \qquad \tilde{G} = \frac{g^3}{4c}\Big(g - \frac{4a}{3}\Big).$$

Henceforth, we shall denote by H, Φ and V the following three functions

$$H = \Big(\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}\Big) - \Big(\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1}\Big), \tag{2.2}$$

$$\Phi = \Big(\frac{F^{'}}{F-1} - \frac{G^{'}}{G-1}\Big), \tag{2.3}$$

$$V = \frac{F^{'}}{F(F-1)} - \frac{G^{'}}{G(G-1)}. \tag{2.4}$$

Similarly, we can define $\tilde{H}$, $\tilde{\Phi}$ and $\tilde{V}$ by replacing F and G by $\tilde{F}$ and $\tilde{G}$ in (2.2), (2.3) and (2.4).

Lemma 2.1. [**16**] *If F, G be two non-constant meromorphic functions such that they share $(1,1)$ and $H \not\equiv 0$, then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leqslant N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. [**4**] *Let f and g be two non-constant meromorphic functions sharing $(1,m)$, where $1 \leqslant m < \infty$. Then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) - N(r,1;f \mid= 1) + \Big(m - \frac{1}{2}\Big)\overline{N}_*(r,1;f,g) \leqslant \frac{1}{2}[N(r,1;f) + N(r,1;g)].$$

Lemma 2.3. [**4**] *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + \ldots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Lemma 2.4. [**4**] *Let S be as in Theorem 1.2 and F, G be defined by (2.1). If for two non-constant meromorphic functions f and g, $E_f(S,0) = E_g(S,0)$, $E_f(\{0,a\},p) = E_g(\{0,a\},p)$, $E_f(\{\infty\},0) = E_g(\{\infty\},0)$ where $0 \leqslant p < \infty$ and $H \neq 0$ then*

$$N(r,\infty;H) \leqslant \overline{N}(r,0;f \mid\geqslant p+1) + \overline{N}(r,a;f \mid\geqslant p+1) + \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not zeros of $f(f-a)(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

LEMMA 2.5. *Let S be as in Theorem 1.2 and F and G be defined by (2.1). If for two non-constant meromorphic functions f and g, $E_f(S,0)=E_g(S,0)$, $E_f(\{0\},0)=E_g(\{0\},0)$ and $E_f(\{\infty\},0)=E_g(\{\infty\},0)$, $H\not\equiv 0$, then*

$$N(r,\infty;H)\leqslant \overline{N}(r,0;f)+\overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not zeros of $f(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined. Similar results hold for $\tilde{F}$ and $\tilde{G}$.

PROOF. Since $E_f(S,0)=E_g(S,0)$, it follows that F and G share $(1,0)$. We can easily verify that possible poles of H occur at (i) zeros of f, (ii) those poles of f and g whose multiplicities are distinct from the multiplicities of the corresponding poles of g and f respectively, (iii) those 1-points of F and G with different multiplicities, (iv) zeros of $f^{'}$ which are not the zeros of $f(F-1)$, (v) zeros of $g^{'}$ which are not zeros of $g(G-1)$. Since H has only simple poles, the lemma follows from above. □

LEMMA 2.6. [**2**] *Let S is defined as in Theorem 1.2 and F and G be given by (2.1). If for two non-constant meromorphic functions f and g, $E_f(S,m)=E_g(S,m)$, where $0\leqslant m<\infty$, then*

$$\text{(i)}\quad \overline{N}_L(r,1;F)\leqslant \frac{1}{m+1}\big(\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\otimes}(r,0;f^{'})\big)+S(r,f),$$

$$\text{(ii)}\quad \overline{N}_L(r,1;G)\leqslant \frac{1}{m+1}\big(\overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_{\otimes}(r,0;g^{'})\big)+S(r,g),$$

where $N_{\otimes}(r,0;f^{'})=N(r,0;f'\mid f\neq 0,w_1,w_2,w_3)$ and w_1,w_2,w_3 be the roots of the equation $P(z)=0$, $N_{\otimes}(r,0;g')$ is defined similarly to $N_{\otimes}(r,0;f')$. Similar results hold for $\tilde{F}$ and $\tilde{G}$.

LEMMA 2.7. [**4**] *Let f and g be two non-constant meromorphic functions and F and G be given by (2.1) such that $E_f(S,m)=E_g(S,m)$, $E_f(\{0,a\},p)=E_g(\{0,a\},p)$, $E_f(\{\infty\},k)=E_g(\{\infty\},k)$, $0\leqslant p,k<\infty$ and $\Phi\not\equiv 0$. Then*

$$(2.5)\quad (2p+1)\big\{\overline{N}(r,0;f\mid\geqslant p+1)+\overline{N}(r,a;f\mid\geqslant p+1)\big\}$$
$$\leqslant \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g).$$

Similarly, if $E_f(S,m)=E_g(S,m)$, $E_f(\{0\},p)=E_g(\{0\},p)$, $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ and $\Phi\not\equiv 0$, then

$$(2p+1)\overline{N}(r,0;f\mid\geqslant p+1)\leqslant \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g).$$

LEMMA 2.8. [**3**] *Let S be defined as in Theorem 1.2 and F and G be given by (2.1) and $V\not\equiv 0$. If for any two non-constant meromprphic functions f and g, $E_f(S,m)=E_g(S,m)$, $E_f(\{0\},0)=E_g(\{0\},0)$ and $E_f(\{\infty\},k)=E_f(\{\infty\},k)$, where $0\leqslant k<\infty$, then the poles of F and G are the zeros of V and*

$$(3k+2)\overline{N}(r,\infty;f\mid\geqslant k+1)=(3k+2)\overline{N}(r,\infty;g\mid\geqslant k+1)$$
$$\leqslant \overline{N}_*(r,0;f,g)+\overline{N}\Big(r,\frac{3a}{2};f\Big)+\overline{N}\Big(r,\frac{3a}{2};g\Big)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).$$

LEMMA 2.9. [**4**] *Let f and g be two nonconstant meromorphic functions and $\tilde{F}$ and $\tilde{G}$ be given by* (2.1) *and $\tilde{S}$ be as in Theorem* 1.5 *such that $E_f(\tilde{S},m)=E_g(\tilde{S},m)$, $E_f(\{0\},p)=E_g(\{0\},p)$, $E_f(\{\infty\},k)=E_g(\{\infty\},k)$, $0\leqslant p,k<\infty$ and $\tilde{\Phi}\neq 0$. Then*

$$(3p+2)\left\{\overline{N}(r,0;f\mid\geqslant p+1)\right\}\leqslant \overline{N}_*(r,1;\tilde{F},\tilde{G})+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g).$$

LEMMA 2.10. [**3**] *Let $\tilde{F}$ and $\tilde{G}$ be given by* (2.1) *and $\tilde{V}\neq 0$. If $E_f(\tilde{S},m)=E_g(\tilde{S},m)$, $E_f(\{0\},0)=E_g(\{0\},0)$ and $E_f(\{\infty\},k)=E_f(\{\infty\},k)$, where $0\leqslant k<\infty$, then poles of $\tilde{F}$ and $\tilde{G}$ are the zeros of $\tilde{V}$ and*

$$\begin{aligned}&(4k+3)\overline{N}(r,\infty;f\mid\geqslant k+1)=(4k+3)\overline{N}(r,\infty;g\mid\geqslant k+1)\\ &\leqslant \overline{N}_*(r,0;f,g)+\overline{N}\Big(r,\frac{4a}{3};f\Big)+\overline{N}\Big(r,\frac{4a}{3};g\Big)+\overline{N}_*(r,1;\tilde{F};\tilde{G})+S(r,f)+S(r,g).\end{aligned}$$

3. Proofs of the theorems

PROOF OF THEOREM 1.4. Let F and G be given by (2.1). Since $E_f(S,3)=E_g(S,3)$, from (2.1) it follows that F and G share $(1,3)$. Suppose $H\not\equiv 0$.

If possible $\Phi=0$. By (2.3), we obtain $(F-1)=c(G-1)$, where from the definition of H we get, $H\equiv 0$, which is a contradiction. Hence $\Phi\neq 0$.

Using Lemma 2.1, Lemma 2.2 for $m=3$, Lemma 2.3, Lemma 2.4, Lemma 2.6 for $m=3$, Lemma 2.7 for $p=0$ we get

$$\begin{aligned}&4\{T(r,f)+T(r,g)\}\\
&\quad\leqslant \overline{N}(r,0;f)+\overline{N}(r,a;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;F)+\overline{N}(r,0;g)+\overline{N}(r,a;g)\\
&\qquad+\overline{N}(r,\infty;g)+\overline{N}(r,1;G)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g)\\
&\quad\leqslant N(r,1;F\mid=1)+2(\overline{N}(r,0;f)+\overline{N}(r,a;f))+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&\qquad+\frac{3}{2}\{N(r,1;F)+N(r,1;G)\}-\Big(3-\frac{1}{2}\Big)\overline{N}_*(r,1;F,G)\\
&\qquad-N_0(r,0;f^{'})-N_0(r,0;g^{'})+S(r,f)+S(r,g)\\
&\quad\leqslant 3(\overline{N}(r,0;f)+\overline{N}(r,a;f))+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f\mid\geqslant 2)\\
&\qquad+\frac{3}{2}\{T(r,f)+T(r,g)\}-\frac{3}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\quad\leqslant N_2(r,\infty;f)+N_2(r,\infty;g)+2\overline{N}(r,\infty;f\mid\geqslant 2)+\frac{3}{2}\{T(r,f)+T(r,g)\}\\
&\qquad+\frac{3}{2}\{\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\}+S(r,f)+S(r,g)\\
&\quad\leqslant\frac{15}{4}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction. Therefore, $H\equiv 0$.

As $c\neq 0$, for two constants A $(\neq 0)$, B; from (2.2) we get

$$\frac{1}{F-1}\equiv\frac{A}{G-1}+B\implies\frac{1}{P(f)}\equiv\frac{A}{P(g)}+\frac{B}{c},$$

wherefrom, since $\overline{E}_f(\{\infty\}) = \overline{E}_g(\{\infty\})$, it follows $B = 0$. Hence $E_f(\{\infty\}) = E_g(\{\infty\})$ and $P(g) \equiv AP(f)$. From above we have $T(r, f) = T(r, g) + O(1)$ and so $S(r, f) = S(r, g)$.

Suppose that $A \neq 1$. As $A \neq 0$, we can write

$$Q(g) \equiv A\Big\{Q(f) + \frac{c(1-A)}{A}\Big\}. \tag{3.1}$$

Next let us consider the following polynomial

$$\phi(z) = Q(z) + \frac{c(1-A)}{A}. \tag{3.2}$$

Claim. All the factors of $\phi(f)$ are simple. Let us suppose that a is a multiple zero of $\phi(z)$ as from (3.2), it is evident that 0 can not be a multiple zero of $\phi(z)$. From (3.2) we have

$$Q(a) = -\frac{c(1-A)}{A}. \tag{3.3}$$

So in view of (3.2) we have $\phi(z) = Q(z) - Q(a) = \frac{1}{6}(z-a)^2(2z+a)$. From (3.1) we have $Q(g) \equiv A\{\phi(f)\}$. i.e.,

$$g^2\Big(g - \frac{3a}{2}\Big) \equiv \frac{A}{2}(f-a)^2(2f+a). \tag{3.4}$$

From (3.4) and the fact $\overline{E}_f(\{0, a\}) = \overline{E}_g(\{0, a\})$, we know the a-points of f will only correspond to the 0-point of g and that $E_f(\{a\}) = E_g(\{0\})$. So the 0-points of f will correspond to the a-points of g. Let z_0 be a point such that, $f(z_0) = 0$ and $g(z_0) = a$. Then (3.4) gives $A = -1$. Using this in (3.3) we get, $c = -\frac{a^3}{12}$, a contradiction to the hypothesis of Theorem 1.4. Hence we must have 0 is an e.v.P. of f and a is an e.v.P. of g. Also we know f, g share (∞, ∞), $E_f(\{a\}) = E_g(\{0\})$ and $E_f(\{-\frac{a}{2}\}) = E_g(\{\frac{3a}{2}\})$.

Case 1. Let $-\frac{a}{2}$ and $\frac{3a}{2}$ are not e.v.P.'s of f and g respectively. We note that there exists an entire function $\sigma_1(z)$ such that,

$$\frac{g}{f-a} = e^{\sigma_1(z)}. \tag{3.5}$$

As $f \not\equiv 0$ and $g \not\equiv a$, from (3.5) we get that, $e^{\sigma_1(z)} \neq -1$, $\forall z$. Now from (3.4) it follows that, there exists a complex number z_2 with, $f(z_2) = -\frac{a}{2}$ and $g(z_2) = \frac{3a}{2}$. From (3.5) we have,

$$e^{\sigma_1(z_2)} = \frac{g(z_2)}{f(z_2) - a} = -1,$$

a contradiction to the fact that $e^{\sigma_1(z)} \neq -1$, $\forall z$.

Case 2. Let $-\frac{a}{2}$ is an e.v.P. of f and $\frac{3a}{2}$ is an e.v.P. of g. In this case there exists an entire function $\sigma_2(z)$ such that,

$$\frac{f}{g-a} = e^{\sigma_2(z)}. \tag{3.6}$$

As $f \not\equiv -\frac{a}{2}$ and $g \not\equiv \frac{3a}{2}$, from (3.6) we get that, $e^{\sigma_2(z)} \neq -1$, $\forall z$. Now from (3.4) it is clear that there exists a complex number z_3 such that, $f(z_3) = a$ and $g(z_3) = 0$. From (3.6) we have,

$$e^{\sigma_2(z_3)} = \frac{f(z_3)}{g(z_3) - a} = -1,$$

a contradiction to the fact that $e^{\sigma_2(z)} \neq -1$, $\forall z$.

Hence our claim is established and so we get, factors of $\phi(f)$ are simple say $(f - \alpha_i)$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, 3$. From (3.1) we can write

$$\frac{g^2}{3}\Big(g - \frac{3a}{2}\Big) \equiv A(f - \alpha_1)(f - \alpha_2)(f - \alpha_3). \tag{3.7}$$

As f, g share $\{0, a\}$, from the above equation it is clear that 0 is an e.v.P. of g and α_i-points ($i = 1, 2, 3$) of f are $\frac{3a}{2}$-points of g. Now, by the second fundamental theorem we have,

$$\begin{aligned}4T(r, f) &\leqslant \sum_{i=1}^{3} \overline{N}(r, \alpha_i; f) + \overline{N}(r, a; f) + \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leqslant \overline{N}\Big(r, \frac{3a}{2}; g\Big) + \overline{N}(r, a; g) + \overline{N}(r, \infty; f) + S(r, f) \leqslant 3T(r, f) + S(r, f),\end{aligned}$$

a contradiction. Hence $A = 1$. So, we get $P(f) \equiv P(g)$. i.e.,

$$\frac{f^2}{3}\Big(f - \frac{3a}{2}\Big) \equiv \frac{g^2}{3}\Big(g - \frac{3a}{2}\Big). \tag{3.8}$$

From (3.8), it is clear that, f, g share 0, $\frac{3a}{2}$ and ∞ CM.

We now wish to prove $f \equiv g$. On the contrary, suppose that $f \not\equiv g$. Consider $h = \frac{f}{g}$ to be a constant. Then from (3.8), it follows that $h \neq 1$, $h^2 \neq 1$, $h^3 \neq 1$ and $g \equiv \frac{3a}{2}\frac{(h^2-1)}{(h^3-1)}$, a constant, which is impossible.

Next, let h be non-constant. Then

$$g \equiv \frac{3a}{2}\Big(\frac{h+1}{h^2+h+1}\Big) \quad \text{and} \quad f \equiv \frac{3a}{2}\Big(\frac{h(h+1)}{h^2+h+1}\Big).$$

In view of the hypothesis of the theorem we know f and g share $(\{0, a\}, 0)$ and from (3.8) we have just deduced f, g share $(0, \infty)$, we can say that f and g share $(a, 0)$. Next observe that,

$$f - a \equiv \frac{a}{2}\frac{(h-1)(h+2)}{h^2+h+1} \quad \text{and} \quad g - a \equiv -\frac{a}{2}\frac{(h-1)(2h+1)}{h^2+h+1}.$$

From the above two expressions, we see -2 and $-\frac{1}{2}$ are e.v.P.s of h. Hence h omits four values 0, ∞, -2 and $-\frac{1}{2}$, which is a contradiction to Nevanlinna four value theorem. Therefore, $f \equiv g$. □

Proof of Theorem 1.2. Let F and G be given by (2.1). Since $E_f(S, 4) = E_g(S, 4)$, from (2.1) we know F and G share $(1, 4)$. Suppose $H \not\equiv 0$. Clearly by the same arguments as used in the proof of Theorem 1.4, $\Phi \neq 0$. Again, if possible let $V \equiv 0$. Then from (2.4) we get $\big(1 - \frac{1}{F}\big) \equiv c\big(1 - \frac{1}{G}\big)$.

As F and G share poles, we get $c = 1$ i.e. $H \equiv 0$, which is a contradiction.

Next by Lemma 2.1, Lemma 2.2 for $m=4$, Lemma 2.3, Lemma 2.5, Lemma 2.7 for $p=0$, and Lemma 2.8 for $k=1$ we have

$$\begin{aligned}
&3\{T(r,f)+T(r,g)\}\\
&\leqslant \overline{N}(r,0;f)+\overline{N}(r,1;F)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,1;G)\\
&\quad+\overline{N}(r,\infty;g)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g)\\
&\leqslant N(r,1;F\mid=1)+2\overline{N}(r,0;f)+2\overline{N}(r,\infty;f\mid\geqslant 2)+\frac{3}{2}\{T(r,f)+T(r,g)\}\\
&\quad-\Big(4-\frac{1}{2}\Big)\overline{N}_*(r,1;F,G)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f\mid\geqslant 2)+\frac{3}{2}\{T(r,f)+T(r,g)\}\\
&\quad-\Big(4-\frac{3}{2}\Big)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant \frac{1}{2}\overline{N}_*(r,1;F,G)+\frac{2}{5}\Big\{\overline{N}(r,0;f)+\overline{N}\Big(r,\frac{3a}{2};f\Big)+\overline{N}\Big(r,\frac{3a}{2};g\Big)+\overline{N}_*(r,1;F,G)\Big\}\\
&\quad+\frac{3}{2}\big\{T(r,f)+T(r,g)\big\}+S(r,f)+S(r,g)\\
&\leqslant \frac{13}{10}\overline{N}_*(r,1;F,G)+\Big(\frac{3}{2}+\frac{2}{5}\Big)\big\{T(r,f)+T(r,g)\big\}+S(r,f)+S(r,g)\\
&\leqslant \Big(\frac{13}{10}\times\frac{2}{5}+\frac{19}{10}\Big)\big\{T(r,f)+T(r,g)\big\}+S(r,f)+S(r,g)),
\end{aligned}$$

which is a contradiction. Hence $H\equiv 0$.

For two constants $A\neq 0$ and B from (2.2) we get $\frac{1}{F-1}\equiv\frac{A}{G-1}+B$. As F and G share poles, we have $B=0$. Hence $P(g)\equiv AP(f)$. We know that f and g share 0.

Case 1. Suppose 0 is not an e.v.P. of f and g, then it follows that $c(1-A)=0$. As $c\neq 0$, we have $A=1$. Thus, we get (3.8).

Case 2. Suppose 0 is an e.v.P. of f and g and $A\neq 1$. Now proceeding in the same way we obtain (3.1) and then proceed to show that the right hand expression of (3.1) has all simple factors in terms of f. Supposing the contrary, we can reach up to (3.4). From (3.4) we can conclude that

$$\overline{N}(r,a;f)+\overline{N}\Big(r,-\frac{a}{2};f\Big)\leqslant\overline{N}\Big(r,\frac{3a}{2};g\Big).$$

so in view of the second fundamental theorem we have

$$\begin{aligned}
2T(r,f)&\leqslant\overline{N}(r,0;f)+\overline{N}(r,a;f)+\overline{N}\Big(r,-\frac{a}{2};f\Big)+\overline{N}(r,\infty;f)+S(r,f)\\
&\leqslant\overline{N}\Big(r,\frac{3a}{2};g\Big)+\overline{N}(r,\infty;f)+S(r,f)\leqslant\frac{3}{2}T(r,f)+S(r,f),
\end{aligned}$$

a contradiction. Hence right hand expression of (3.1) have all simple factors in terms of f and we get (3.7). Then by the same argument as done in the proof of Theorem 1.4, we can get a contradiction and so $A=1$. So, we also get (3.8).

Now we will prove $f \equiv g$. On the contrary, let us assume that $f \not\equiv g$. Let $h \equiv \frac{f}{g}$. First suppose that, h is constant. Then with the help of the same arguments which is used in Theorem 1.4 to deal the situation, we get a contradiction.

Next let h be non-constant. Then from (3.8) we get

$$g \equiv \frac{3a}{2}\left(\frac{h+1}{h^2+h+1}\right).$$

Let α_1 and α_2 be the zeros of the polynomial (z^2+z+1). By the Second Fundamental Theorem and by the definition of g defined above we obtain,

$$\begin{aligned} 2T(r,h) &\leqslant \sum_{i=1}^{2} \overline{N}(r,\alpha_i;h) + \overline{N}(r,0;h) + \overline{N}(r.\infty;h) + S(r,h) \\ &\leqslant \overline{N}(r,\infty;g) + S(r,h) \leqslant T(r,h) + S(r,h), \end{aligned}$$

which is a contradiction. Hence $f \equiv g$. □

Proof of Theorem 1.5. Let $\tilde{F}$ and $\tilde{G}$ be given by (2.1). Since $E_f(\tilde{S},3) = E_g(\tilde{S},3)$, from (2.1) it follows that $\tilde{F}$ and $\tilde{G}$ share $(1,3)$. Suppose $\tilde{H} \not\equiv 0$. By the arguments of Theorem 1.4, we have $\tilde{\Phi} \neq 0$ and $\tilde{V} \neq 0$. Using Lemma 2.1, Lemma 2.2 for $m=3$, Lemma 2.3, Lemma 2.5 for $\tilde{F}$ and $\tilde{G}$, Lemma 2.6 for $m=3$, Lemma 2.9 for $p=0$ and Lemma 2.10 for $k=1$ we get

$$\begin{aligned} &4\{T(r,f)+T(r,g)\} \\ &\leqslant \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;\tilde{F}) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;\tilde{G}) \\ &\quad - N_0(r,0;f') - N_0(r,0;g') + S(r,f) + S(r,g) \\ &\leqslant N(r,1;F \mid= 1) + 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty,f \mid\geqslant 2) + 2\{T(r,f)+T(r,g)\} \\ &\quad - \left(3-\frac{1}{2}\right)\overline{N}_*(r,1;\tilde{F},\tilde{G}) - N_0(r,0;f') - N_0(r,0;g') + S(r,f) + S(r,g) \\ &\leqslant 3\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f \mid\geqslant 2) + 2\{T(r,f)+T(r,g)\} \\ &\quad - \frac{3}{2}\overline{N}_*(r,1;\tilde{F},\tilde{G}) + S(r,f) + S(r,g) \\ &\leqslant \frac{3}{2}\{\overline{N}_*(r,1;\tilde{F},\tilde{G}) + \overline{N}(r,\infty;g \mid\geqslant 2)\} \\ &\quad + \frac{3}{7}\{\overline{N}(r,0;f) + T(r,f) + T(r,g) + \overline{N}_*(r,1;\tilde{F},\tilde{G})\} + 2\{T(r,f)+T(r,g)\} \\ &\quad - \frac{3}{2}\overline{N}_*(r,1;\tilde{F},\tilde{G}) + S(r,f) + S(r,g) \\ &\leqslant \frac{17}{7}\{T(r,f)+T(r,g)\} + \left(\frac{3}{2}+\frac{3}{14}\right)\overline{N}(r,\infty;f \mid\geqslant 2) \\ &\quad + \left(\frac{3}{14}+1\right)\overline{N}_*(r,1;\tilde{F},\tilde{G}) + S(r,f) + S(r,g) \\ &\leqslant \frac{97}{28}\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g), \end{aligned}$$

a contradiction. Hence $\tilde{H} \equiv 0$.

Now by the similar arguments as in Theorem 1.2, we can say that $f \equiv g$. $\square$

4. Two open questions

In view of Example 1.2, we can see that in Theorem 1.4, instead of taking derived set of corresponding polynomial, an arbitrary set cannot be chosen. At this juncture, the following question arises;

Question 4.1. Does there exist any polynomial for which the derived set can be replaced by an arbitrary set, so that the conclusion of Theorem 1.4 remains unaltered?

Next, for two non-zero complex numbers a and b, let us consider the following polynomial

$$R(z) = \frac{z^3}{3} - \frac{(a+b)z^2}{2} + (ab)z - c,$$

where c is a constant such that $R(z)$ does not have any multiple zero. Note that $R'(z) = (z-a)(z-b)$, hence derived set of $R(z)$ is $\{a, b\}$. At this point, second question appears;

Question 4.2. Can we replace first two sets of Theorem 1.2, by $S = \{z \mid R(z) = 0\}$ and by $\{a, b\}$?

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