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3-DIAGONAL EQUATION AND PLANARITY OF GRAPHS

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Abstract. In his PhD thesis, S. Krstić used (multi)graphs to solve generalized quadratic quasigroup functional equations. In particular, he showed the fundamental role of Kuratowski Theorem on planarity of graphs in determining properties of general solutions of such equations. As a first step towards generalization of his results to functional equations on ternary quasigroups, we consider generalized 3-diagonal equation $A\Big(B(x,y,z),C(y,u,v),D(z,v,w)\Big) = E(x,u,w)$. This is one of equations with complete graph K_5 as a corresponding graph. General solution of this equation is given, confirming the important role of Kuratowski Theorem in this case as well.

1. Introduction

We consider generalized 3-diagonal functional equation:

$$A(B(x,y,z),C(y,u,v),D(z,v,w)) = E(x,u,w) \tag{D3}$$

where A,B,C,D,E are unknown ternary quasigroups (For the related case of *diagonal algebras* see Płonka [**11**]). Our interest in (D3) leads to graph theory. Namely, in his PhD Thesis [**8**], Krstić used some (multi)graphs (called Krstić graphs in [**6**]) to solve quasigroup functional equations. Let us look at a special case of a generalized quadratic equation Eq (with the Krstić graph $K(Eq)$) such that all operations appearing in Eq are binary and mutually isostrophic. Then (Alimpić [**2**], Krapež [**4**], Krstić [**8**]):

- All quasigroups are isostrophic to the same loop L.
- L is a group iff there are more than two operations in Eq iff tetrahedron K_4 is embeddable in $K(Eq)$.
- L is an Abelian group iff $K(Eq)$ is not planar iff $K_{3,3}$ is embeddable in $K(Eq)$.

We see that the properties of solutions of Eq crucially depend on properties of the Krstić graph $K(Eq)$. In particular, commutativity case evokes the Kuratowski theorem.

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Theorem 1.1 (Kuratowski [**9, 10**]). *Graph is planar if it does not contain a subgraph that is a subdivision of either $K_{3,3}$ or K_5.*

Graphs $K_{3,3}$ and K_5 are given in Figure 1.

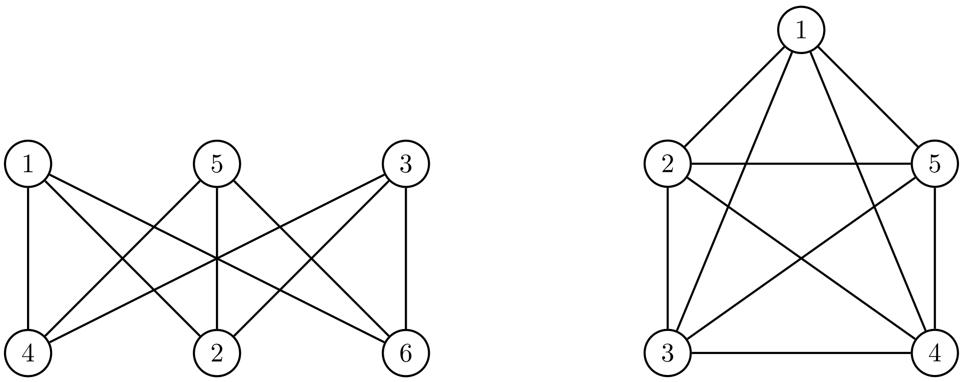

Figure 1. Graphs $K_{3,3}$ and K_5

Krstić considered equations with binary quasigroup operations only. His graphs for such equations are cubic. Consequently, the graph K_5 (which is of degree 4) has no role in his theory. However, if we define graphs of degree 4 for functional equations on ternary quasigroups, the graph corresponding to the equation (D3) is exactly K_5. Therefore, the relationship between (D3) and K_5 becomes the center of our interest. We give a general solution of the equation (D3) showing how it fits with Krstić's method of using graphs to solve quadratic functional equations on binary quasigroups. This opens the possibility to describe solutions of any generalized quadratic functional equation on ternary (and even n–ary) quasigroups. For some results in that direction see also [**13, 14, 15, 16**].

2. Quasigroups

We state a few basic definitions and results from the theory of quasigroups, mostly to fix terminology and notation. We warn the reader of the nonstandard term *unit* and our notation for *inverse operations* which is not compatible with standard notation (See for example [**12**]).

Definition 2.1. An algebra $(Q; \cdot, \backslash, /)$ is a (*binary*) *quasigroup* if

$$x \cdot (x\backslash y) = y, \qquad (x/y) \cdot y = x, \qquad x\backslash(x \cdot y) = y, \qquad (x \cdot y)/y = x.$$

An algebra $(Q; F, F^{-1}, F^{-2}, F^{-3})$ is a *ternary* or 3-*quasigroup* if

$$F(F^{-1}(x,y,z),y,z) = x, \quad F(x,F^{-2}(x,y,z),z) = y, \quad F(x,y,F^{-3}(x,y,z)) = z,$$

$$F^{-1}(F(x,y,z),y,z) = x, \quad F^{-2}(x,F(x,y,z),z) = y, \quad F^{-3}(x,y,F(x,y,z)) = z.$$

Note that we use the term 'quasigroup' for *algebra* in Definition 2.1, as well as for the component *operation* $\cdot$ or F.

Definition 2.2. A *loop* is a quasigroup with *unit* e:

$$e \cdot x = x \cdot e = x.$$

A 3-*loop* is a 3-quasigroup with *unit* e:

$$F(e,e,x) = F(e,x,e) = F(x,e,e) = x.$$

Theorem 2.1. *An associative quasigroup is a group.*
Associative and commutative quasigroup is an Abelian group.

Let $\cdot$ be a quasigroup operation on Q. Then

$$x \cdot y = z \quad \text{iff} \quad x \backslash z = y \quad \text{iff} \quad z/y = x.$$

Operation $/$ ($\backslash$) is *left (right) inverse operation* (also *right (left) division*) for $\cdot$. They are also quasigroup operations.

The *dual operations* $*, \backslash\!\!\backslash, /\!\!/$ of, respectively, $\cdot, \backslash, /$ are defined by

$$x * y = y \cdot x, \qquad x \backslash\!\!\backslash y = y \backslash x, \qquad x /\!\!/ y = y/x$$

and they are quasigroup operations as well. We say that $\cdot, \backslash, /, *, \backslash\!\!\backslash, /\!\!/$ are *parastrophes* of $\cdot$ (and of each other).

Analogously, for the ternary quasigroup operation F, all operations $F^\pi (\pi \in S_4)$ defined by

$$F^\pi(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = x_{\pi(4)} \;\text{ iff }\; F(x_1, x_2, x_3) = x_4$$

are *parastrophes* of F. Operations F^π are also 3-quasigroup operations.

Definition 2.3. Let $(Q; \cdot, \backslash, /)$ and $(Q'; \circ, \circ\!\!\!\backslash, /\!\!\!\circ)$ be quasigroups. Operations $\cdot$ and $\circ$ (as well as quasigroup algebras $(Q; \cdot, \backslash, /)$ and $(Q'; \circ, \circ\!\!\!\backslash, \circ\!\!\!/)$) are *isotopic* iff there are bijections $\alpha, \lambda, \varrho : Q \to Q'$ such that $\alpha(x \cdot y) = \lambda x \circ \varrho y$.

Similarly, 3-quasigroups F and G (as well as 3-quasigroup algebras $(Q; F, F^{-1}, F^{-2}, F^{-3})$ and $(Q'; G, G^{-1}, G^{-2}, G^{-3})$) are *isotopic* iff there are bijections $\alpha, \lambda, \mu, \varrho : Q \to Q'$ such that $\alpha F(x,y,z) = G(\lambda x, \mu y, \varrho z)$.

It is well known that:

Theorem 2.2. *Being isotopic is an equivalence relation among* (3-)*quasigroup operations.*

Theorem 2.3. *Isotopic groups are isomorphic.*

Theorem 2.4. *Every* (3-)*quasigroup operation is isotopic to a* (3-)*loop operation.*

Definition 2.4. Two quaigroup operations are *isostrophic* if one of them is isotopic to a parastrophe of the other.

Being isostrophic is also an equivalence relation among (3-)quasigroup operations.

We denote by Id the identity permutation of Q.

Definition 2.5. Let F be a 3-quasigroup operation on a set Q and f_i $(i = 1, 2, 3)$ three arbitrary fixed elements from Q. We define

$$F_{12}(x, y) = F(x, y, f_3), \quad F_{13}(x, z) = F(x, f_2, z), \; F_{23}(y, z) = F(f_1, y, z)$$
$$F_1(x) = F(x, f_2, f_3), \qquad F_2(y) = F(f_1, y, f_3), \qquad F_3(z) = F(f_1, f_2, z)$$

and $f = F(f_1, f_2, f_3)$. Operations F_{12}, F_{13}, F_{23} are *binary retracts* while F_1, F_2 and F_3 are *unary retracts* of F. Element f is a *nullary retract* of F.

It is easy to see that F_{12}, F_{13}, F_{23} are quasigroup operations and that F_1, F_2, F_3 are bijections. Note also that operations $F_{12}, F_{13}, F_{23}, F_1, F_2, F_3$ and element f depend on the choice of f_1, f_2, f_3, so fixing these elements enables us to simplify the notation we use.

If we apply this to equation (D3), we get five operations A, B, C, D, E, their 15 binary retracts, further 15 unary retracts and the total of 20 elements. However, each of the six object variables appears in (D3) twice, which requires that $b_1 = e_1$, $b_2 = c_1$, $b_3 = d_1$, $c_2 = e_2$, $c_3 = d_2$, $d_3 = e_3$. Further, since $B(x, y, z)$ $(C(y, u, v), D(z, v, w))$ is the first (second, third) subterm of $A(B(x, y, z), C(y, u, v), D(z, v, w))$, we have $a_1 = b$ $(a_2 = c$, $a_3 = d)$. Finally, because of $A(\dots) = E(\dots)$, we have $a = e$. The conclusion is that we have 20 *names* for elements, but only six of them are independent. Not at all coincidentally, we are back to the number of object variables in (D3).

Definition 2.6. If $\{x_i, \dots, x_j\}$ is a subset of $\{x, y, z, u, v, w\}$, then we define $D3[x_i, \dots, x_j]$ to be an equation obtained from D3 when we replace every variable $x_k \in \{x, \dots, w\} \smallsetminus \{x_i, \dots, x_j\}$ by the corresponding element from $\{b_1, b_2, b_3, c_2, c_3, d_3\}$.

Here are several examples:

- $D3[x, y, z]$ is $A(B(x, y, z), C_1 y, D_1 z) = E_1 x$.
- $D3[z, w]$ is $A_{13}(B_3 z, D_{13}(z, w)) = E_3 w$.
- $D3[]$ is $a = e$.

More formally, if we take D3 to be $D3(x, y, z, u, v, w)$, then $D3[x, y, z]$ is $D3(x, y, z, c_2, c_3, d_3)$, $D3[z, w]$ is $D3(b_1, b_2, z, c_2, c_3, w)$ and $D3[]$ is $D3(b_1, b_2, b_3, c_2, c_3, d_3)$.

Definition 2.7. Ternary quasigroup operation F is *reducible* iff there are binary quasigroups A, B such that $F(x, y, z) = A(B(u, v), w)$ for all x, y, z and some u, v, w such that $\{u, v, w\} = \{x, y, z\}$.

3. Functional equations

3-diagonal equation belongs to a class of equations we are interested in. They are:

- functional (there is at least one functional variable in the equation),
- generalized (every functional variable occurrs at most once in the equation),
- quadratic (see Definition 3.1),
- on binary and ternary quasigroups (functional variables are interpreted as either binary or ternary quasigroup operations).

Definition 3.1. Functional equation $s = t$ is *quadratic* iff every object variable occurrs exactly twice in the equation. It is *balanced* iff every object variable from $s = t$ appears exactly once in s and once in t.

Definition 3.2. Functional equation Eq is *reducible* if it contains a ternary quasigroup operation symbol F such that in every solution of Eq, the interpretation of F is a reducible quasigroup operation.

Aczél, Belousov and Hosszú considered in [**1**] important equations of generalized associativity (GA) and generalized bisymmetry (GB)

$$\text{(GA)} \qquad (x \; \mathsf{1} \; y) \; \mathsf{2} \; z = x \; \mathsf{3} \; (y \; \mathsf{4} \; z),$$

$$\text{(GB)} \qquad (x \; \mathbf{1} \; y) \; \mathbf{2} \; (u \; \mathbf{3} \; v) = (x \; \mathbf{4} \; u) \; \mathbf{5} \; (y \; \mathbf{6} \; v) \; .$$

All equations are formulas of the form $t_1 = t_2$. Therefore we can represent them as pairs of trees: tree T_1 for the term t_1 and T_2 for t_2. For equations (GA) and (GB) the pictures of corresponding pairs T_1, T_2 are given in Figures 2 and 3.

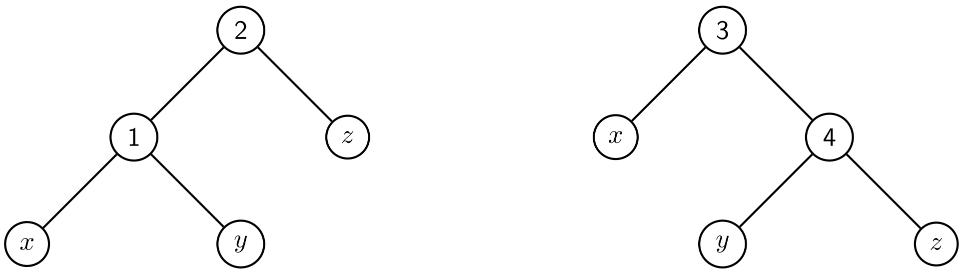

Figure 2. Generalized associativity (GA)

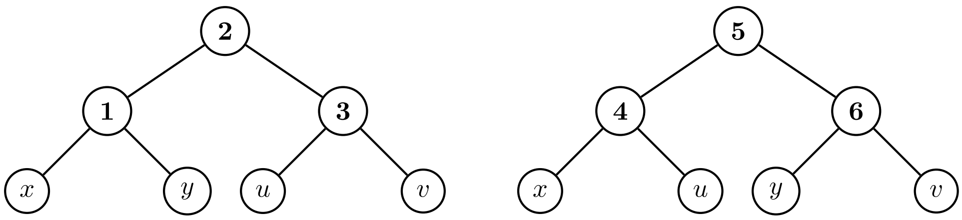

Figure 3. Generalized bisymmetry (GB)

We give a general solution of (GB) as we shall need it later.

Theorem 3.1 (J. Aczél et al. [**1**]). *A general solution of* (GB) *(on a set $Q \neq \emptyset$) is given by*

$$\begin{aligned} x \; \mathbf{1} \; y &= \delta^{-1}(\alpha x + \beta y), & x \; \mathbf{4} \; y &= \psi^{-1}(\alpha x + \chi y), \\ x \; \mathbf{2} \; y &= \delta x + \varphi y, & x \; \mathbf{5} \; y &= \psi x + \varepsilon y, \\ x \; \mathbf{3} \; y &= \varphi^{-1}(\chi x + \gamma y), & x \; \mathbf{6} \; y &= \varepsilon^{-1}(\beta x + \gamma y), \end{aligned}$$

where + is an arbitrary Abelian group on Q, while $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi, \psi$ and χ are arbitrary permutations on Q.

Alimpić in [**2**] gave a formula of a general solution of any generalized balanced quasigroup equation. Krapež in [**3**] generalized her result to generalized balanced equations for quasigroups of arbitrary arity. Generalized quadratic quasigroup equations were defined in Krapež [**4**] and the case where all (binary) quasigroup operations are mutually isostrophic was solved. A general solution for arbitrary quadratic equations for binary quasigroups was given in [**8**] by Krstić.

4. Krstić graphs of functional equations

In order to solve generalized quadratic functional equations with binary quasigroups, Krstić in [**8**] did establish correspondence between such equations and *connected cubic multigraphs*. With some slight modifications (see [**6, 7**]), we have:

Definition 4.1. Let $s = t$ be a generalized quadratic quasigroup functional equation. *Krstić graph* $K(s = t)$ of $s = t$ is a multigraph $(V, E; I)$ given by:

- *Vertices* of $K(s = t)$ are operation symbols from $s = t$.
- *Edges* of $K(s = t)$ are subterms of s and t, including s and t, which are considered to be a single edge. Likewise, any variable (which appears twice in $s = t$) is taken to be a single edge.
- If $F(p, q)$ is a subterm of s or t then the vertex F is *incident* to edges $p, q, F(p, q)$ and no other.

The process of turning two trees T_1, T_2 of an equation Eq into its Krstić graph $K(Eq)$ is ilustrated by the example of (GB). In this case, T_1 and T_2 are given in Figure 3. If we connect identical variables we get the graph in Figure 4. Turning 'variables' into edges, we get the graph $K_{3,3}$ from Figure 1.

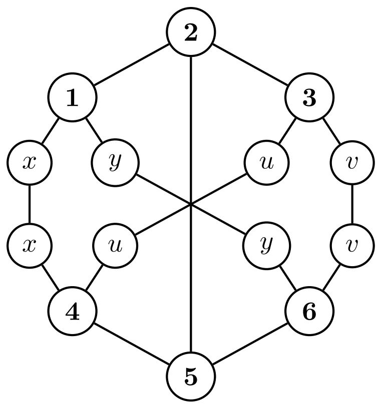

Figure 4. Trees of terms of (GB) 'glued' together

Going back to the general case, we state the basic result about Krstić graphs.

THEOREM 4.1 (Krstić [**8**]). *Operations F and G from generalized quadratic equation Eq are mutually isostrophic iff they are 3-connected in $K(Eq)$.*

There are many similar results connecting Eq and $K(Eq)$ (see [**8, 6, 7**]) but they are beyond the scope of this article.

Here we extend the definition of the Krstić graph $K(s=t)$ allowing ternary quasigroup operations in $s=t$ and consequently vertices of degree four in $K(s=t)$.

- If $F(p,q,r)$ is a subterm of s or t then the vertex F is incident to edges $p,q,r,F(p,q,r)$ and no other.

5. Generalized 3-diagonal equation

In this section we give the proof of the main result of this paper.

THEOREM 5.1. *General solution of* (D3) (*on a set* $Q \neq \emptyset$) *is given by*

$$F(x,y,z) = \alpha_F(\lambda_F x + \mu_F y + \varrho_F z) \quad (F \in \{A,B,C,D,E\}).$$

where $+$ *is an arbitrary Abelian group* (*with unit* 0) *on* Q, *while* α_F, λ_F, μ_F, ϱ_F *are arbitrary permutations of* Q *such that*

$$\begin{gathered}\alpha_A = \alpha_E = \mathrm{Id},\\ \begin{aligned} \lambda_A\alpha_B &= \mathrm{Id}, & \mu_B y + \lambda_C y &= 0, & \lambda_B &= \lambda_E\\ \mu_A\alpha_C &= \mathrm{Id}, & \varrho_B z + \lambda_D z &= 0, & \mu_C &= \mu_E\\ \varrho_A\alpha_D &= \mathrm{Id}, & \varrho_C v + \mu_D v &= 0, & \varrho_D &= \varrho_E.\end{aligned}\end{gathered}$$

PROOF. We show first that the above formulas actually define a solution of (D3).

$$\begin{aligned}
&A(B(x,y,z),C(y,u,v),D(z,v,w))\\
&\qquad = \alpha_A(\lambda_A B(x,y,z) + \mu_A C(y,u,v) + \varrho_A D(z,v,w))\\
&\qquad = \lambda_A\alpha_B(\lambda_B x + \mu_B y + \varrho_B z) + \mu_A\alpha_C(\lambda_C y + \mu_C u + \varrho_C v)\\
&\qquad\quad + \varrho_A\alpha_D(\lambda_D z + \mu_D v + \varrho_D w)\\
&\qquad = \lambda_B x + \mu_B y + \varrho_B z + \lambda_C y + \mu_C u + \varrho_C v + \lambda_D z + \mu_D v + \varrho_D w\\
&\qquad = \lambda_B x + \mu_B y + \lambda_C y + \varrho_B z + \lambda_D z + \mu_C u + \varrho_C v + \mu_D v + \varrho_D w\\
&\qquad = \lambda_B x + (\mu_B y + \lambda_C y) + (\varrho_B z + \lambda_D z) + \mu_C u + (\varrho_C v + \mu_D v) + \varrho_D w\\
&\qquad = \lambda_B x + 0 + 0 + \mu_C u + 0 + \varrho_D w\\
&\qquad = \lambda_E x + \mu_E u + \varrho_E w\\
&\qquad = \alpha_E(\lambda_E x + \mu_E u + \varrho_E w)\\
&\qquad = E(x,u,w).
\end{aligned}$$

To prove the converse, assume that the quintuple (A,B,C,D,E) of ternary quasigroups on Q is a paticular solution of (D3).

1) Let $b_1, b_2, b_3, c_2, c_3, d_3$ be a choice of six arbitrary elements from Q. We define:

- $c_1 = b_2$, $d_1 = b_3$, $d_2 = c_3$, $e_1 = b_1$, $e_2 = c_2$, $e_3 = d_3$

- $f = F(f_1, f_2, f_3)$ where $F \in \{A, B, C, D, E\}$ and f is an appropriate constant corresponding to F while $a_1 = b$, $a_2 = c$, $a_3 = d$.

As noted previously, it follows that $a = e$.

2) Direct consequences $D3[x], D3[u]$ and $D3[w]$ are

$$A_1B_1(x) = E_1(x), \quad A_2C_2(u) = E_2(u), \quad A_3D_3(w) = E_3(w),$$

or simply $A_1B_1 = E_1$, $A_2C_2 = E_2$ and $A_3D_3 = E_3$.

3) $D3[x, y, w]$ is: $A\big(B_{12}(x, y), C_1y, D_3w\big) = E_{13}(x, w)$. Let $z = B_{12}(x, y)$; then $x = B_{12}^{-1}(z, y)$ and consequently $A(z, C_1y, D_3w) = E_{13}(B_{12}^{-1}(z, y), w)$. This proves that 3-quasigroup A is reducible.

We further have: $E_{13}(x, w) = A_{13}(B_1x, D_3w)$ and $A_{12}(B_{12}(x, y), C_1y) = E_1x$ or, equivalently: $B_{12}^{-1}(z, y) = E_1^{-1}A_{12}(z, C_1y)$. It follows that

$$\begin{aligned} A(z, C_1y, D_3w) = E_{13}(B_{12}^{-1}(z, y), w) &= A_{13}(B_1E_1^{-1}A_{12}(z, C_1y), D_3w) \\ &= A_{13}(A_1^{-1}A_{12}(z, C_1y), D_3w) \end{aligned}$$

i.e.,

$$A(z, u, v) = A_{13}(A_1^{-1}A_{12}(z, u), v). \tag{5.1}$$

Therefore, it is not that operation A is just reducible, it may be expressed in terms of its own reducts only.

4) Analogously, we can reduce operations B, C, D, E.

$$\begin{aligned} B(x, y, z) &= B_{12}(B_1^{-1}B_{13}(x, z), y) \ \text{ follows from } D3[x, y, z]. \\ C(x, y, z) &= C_{12}(x, C_2^{-1}C_{23}(y, z)) \ \text{ follows from } D3[y, u, v]. \\ D(x, y, z) &= D_{13}(D_1^{-1}D_{12}(x, y), z) \ \text{ follows from } D3[z, v, w]. \\ E(x, y, z) &= E_{13}(E_1^{-1}E_{12}(x, y), z) \ \text{ follows from } D3[x, u, w]. \end{aligned}$$

5) If we replace all terms of the form $F(x, y, z)$, $F \in \{A, B, C, D, E\}$ in (D3) by their values suggested by (5.1) and formulas from 4), we get the equation

$$A_{13}(A_1^{-1}A_{12}(p, q), r) = E_{13}(E_1^{-1}E_{12}(x, u), w) \tag{rD3}$$

where

$$\begin{aligned} p &= B_{12}(B_1^{-1}B_{13}(x, z), y), \\ q &= C_{12}(y, C_2^{-1}C_{23}(u, v)), \\ r &= D_{13}(D_1^{-1}D_{12}(y, v), w). \end{aligned}$$

Equation ($rD3$) is irreducible, quadratic, generalized and contains only binary quasigroups. Krstić graph $K(rD3)$ is shown in Figure 5. From this picture it becomes obvious that $K_{3,3}$ is a subgraph of $K(rD3)$ and that it corresponds to the equation $rD3[x, z, u, v]$.

Note. While building $K(rD3)$, we replaced *absolutely correct* subgraph

$$\rangle\!- F_{ik} - F_i^{-1} - F_{ij} -\!\langle$$

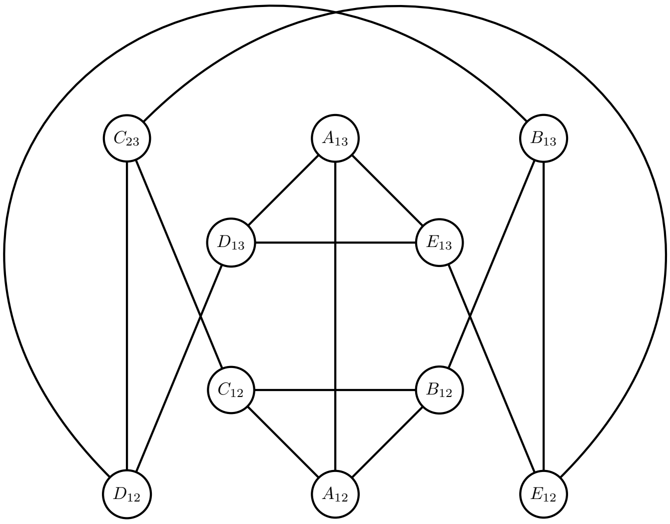

FIGURE 5. Graph $K(rD3)$

by the *relatively correct* subgraph

$$\rangle\!-F_{ik}-F_{ij}-\!\langle$$

Since our reasoning is 'up to isostrophy', it follows that relatively correct is correct enough.

6) Therefore, let us consider $rD3[x,z,u,v]$ i.e.,

$$A_{13}(A_1^{-1}A_{12}(B_{13}(x,z),C_{23}(u,v)),D_{12}(z,v))=E_{12}(x,u).$$

If we introduce new variables y,w by $y=B_{13}(x,z)$ and $w=C_{23}(u,v)$, we may solve for x,u to get $x=B_{13}^{-1}(y,z)$ and $u=C_{23}^{-1}(w,v)$. Replacement of x and u in $E_{12}(x,u)$ yields

$$A_{13}(A_1^{-1}A_{12}(y,w),D_{12}(z,v))=E_{12}(B_{13}^{-1}(y,z),C_{23}^{-1}(w,v))$$

which is a special case of (GB).

In Theorem 3.1, the formulas of general solution of (GB) were given. In particular: $x\,\mathbf{2}\,y=\delta x+\varphi y$ where $+$ is an arbitrary Abelian group while δ and φ are arbitrary permutations of Q. In our case $\mathbf{2}$ is A_{13} and if we choose $\delta=A_1$ and $\varphi=A_3$ we end up with $A_{13}(x,z)=A_1x+A_3z$.

7) $A_1B_1x+a=A_1B_1x+A_3a_3=A_{13}(B_1x,a_3)=A(B_1x,a_2,a_3)=A_1B_1x$ i.e., $a=A(a_1,a_2,a_3)$ is the unit of the Abelian group $+$. We shall use yet another name 0 for a, so that $0=a=e$.

8) $D3[v]$ is $A_{23}(C_3v,D_2v)=e=0$. Therefore

$$0=A(a_1,C_3v,D_2v)=A_{13}(A_1^{-1}A_{12}(a_1,C_3v),D_2v)=A_1A_1^{-1}A_2C_3v+A_3D_2v$$

and finally $-A_3D_2v = A_2C_3v$.

Analogously, we can prove $A_1B_2y + A_2C_1y = 0$ and $A_1B_3z + A_3D_1z = 0$.

9) $D3[x, v]$ yields

$$\begin{aligned} A_1B_1x = E_1x = A(B_1x, C_3v, D_2v) &= A_{13}(A_1^{-1}A_{12}(B_1x, C_3v), D_2v) \\ &= A_{12}(B_1x, C_3v) + A_3D_2v. \end{aligned}$$

Therefore

$$A_{12}(B_1x, C_3v) = A_1B_1x - A_3D_2v = A_1B_1x + A_2C_3v \ \text{ i.e., } \ A_{12}(y, z) = A_1y + A_2z.$$

10) From Theorem 3.1 we know that quasigroup operations E_{12}, B_{13}, C_{23} and D_{12} are also isostrophic to $+$, but we have to determine the exact form of these isostrophies. $E_{12}(x, u) = A_{12}(B_1x, C_2u) = A_1B_1x + A_2C_2u = E_1x + E_2u$. $D3[x, z]$ is $A_{13}(B_{13}(x, z), D_1z) = E_1x$ and consequently $A_1B_{13}(x, z) + A_3D_1z = E_1x$. Finally $B_{13}(x, z) = A_1^{-1}(A_1B_1x + A_1B_3z)$.

Analogously we can prove the relations

$$C_{23}(u, v) = A_2^{-1}(A_2C_2u + A_2C_3v) \ \text{ and } \ D_{12}(z, v) = A_3^{-1}(A_3D_1z + A_3D_2v).$$

11) Remaining binary retracts are also isostrophic to $+$.

$$\begin{aligned} B_{12}(x, y) &= A_1^{-1}(A_1B_1x + A_1B_2y) && \text{(Follows from } D3[x, y]). \\ C_{12}(y, u) &= A_2^{-1}(A_2C_1y + A_2C_3u) && \text{(Follows from } D3[y, u]). \\ D_{13}(z, w) &= A_3^{-1}(A_3D_1z + A_3D_3w) && \text{(Follows from } D3[z, w]). \\ E_{13}(x, w) &= E_1x + E_3w && \text{(Follows from } D3[x, w]). \end{aligned}$$

12) From (5.1) we get

$$A(z, u, v) = A_{13}(A_1^{-1}A_{12}(z, u), v) = A_{12}(z, u) + A_3v = A_1z + A_2u + A_3v.$$

Similarily, from relations in 4), it follows that:

$$\begin{aligned} B(x, y, z) &= A_1^{-1}(A_1B_1x + A_1B_2y + A_1B_3z), \\ C(y, u, v) &= A_2^{-1}(A_2C_1y + A_2C_2u + A_2C_3v), \\ D(z, v, w) &= A_3^{-1}(A_3D_1z + A_3D_2v + A_3D_3w), \\ E(x, u, w) &= E_1x + E_2u + E_3w \end{aligned}$$

13) Let us define

$$\begin{array}{llll} \alpha_A = \mathrm{Id}, & \lambda_A = A_1, & \mu_A = A_2, & \varrho_A = A_3, \\ \alpha_B = A_1^{-1}, & \lambda_B = A_1B_1, & \mu_B = A_1B_2, & \varrho_B = A_1B_3, \\ \alpha_C = A_2^{-1}, & \lambda_C = A_2C_1, & \mu_C = A_2C_2, & \varrho_C = A_2C_3, \\ \alpha_D = A_3^{-1}, & \lambda_D = A_3D_1, & \mu_D = A_3D_2, & \varrho_D = A_3D_3, \\ \alpha_E = \mathrm{Id}, & \lambda_E = E_1, & \mu_E = E_2, & \varrho_E = E_3. \end{array}$$

We now have: $\alpha_A = \alpha_E = \mathrm{Id}$. Also: $\lambda_A\alpha_B = \mathrm{Id}$, $\mu_A\alpha_C = \mathrm{Id}$ and $\varrho_A\alpha_D = \mathrm{Id}$.
From 2), it follows that: $\lambda_B = \lambda_E$, $\mu_C = \mu_E$ and $\varrho_D = \varrho_E$.
From 8), we infer: $\mu_By + \lambda_Cy = 0$, $\varrho_Bz + \lambda_Dz = 0$ and $\varrho_Cv + \mu_Dv = 0$.

14) Finally, we can express all five quasigroup operations using common formula $F(x,y,z)=\alpha_F(\lambda_F x+\mu_F y+\varrho_F z)$. $\square$

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