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HYPERBOLIC ARCTANGENT SUMMATIONS OF PELL AND PELL–LUCAS POLYNOMIALS

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Abstract. The telescopic approach is systematically employed to examine sums for products of hyperbolic arctangent functions whose arguments are Pell and Pell–Lucas polynomials. Numerous summation formulae are established in closed forms. Several identities concerning Fibonacci and Lucas numbers are deduced as particular cases.

1. Introduction and motivation

Fibonacci and Lucas numbers are well–known for their wide applications in number theory and combinatorics (see the monograph by Koshy [**8**] and [**3**, **4**, **6**]). Their polynomial extensions introduced by Horadam and Mahon [**7**] are now called Pell and Pell–Lucas polynomials (see also Koshy [**9**]) that are defined by the following recurrence relations:

$$\mathrm{P}_n(x) = 2x\,\mathrm{P}_{n-1}(x) + \mathrm{P}_{n-2}(x) \quad \text{with} \quad \mathrm{P}_0(x) = 0 \quad \text{and} \quad \mathrm{P}_1(x) = 1,$$
$$\mathrm{Q}_n(x) = 2x\,\mathrm{Q}_{n-1}(x) + \mathrm{Q}_{n-2}(x) \quad \text{with} \quad \mathrm{Q}_0(x) = 2 \quad \text{and} \quad \mathrm{Q}_1(x) = 2x.$$

These polynomials have the expressions in the Binet forms

$$\mathrm{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{Q}_n(x) = \alpha^n + \beta^n$$

where

$$\alpha := \alpha(x) = x + \sqrt{x^2+1} \quad \text{and} \quad \beta := \beta(x) = x - \sqrt{x^2+1}.$$

As particular examples, we have the following well–known numbers:

- Fibonacci number $F_n = \mathrm{P}_n(\frac{1}{2})$:
$$F_n = F_{n-1} + F_{n-2} \quad \text{with} \quad F_0 = 0 \quad \text{and} \quad F_1 = 1.$$
- Lucas number $L_n = \mathrm{Q}_n(\frac{1}{2})$:
$$L_n = L_{n-1} + L_{n-2} \quad \text{with} \quad L_0 = 2 \quad \text{and} \quad L_1 = 1.$$

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- Pell number $P_n = \mathrm{P}_n(1)$:

$$P_n = 2P_{n-1} + P_{n-2} \quad \text{with} \quad P_0 = 0 \quad \text{and} \quad P_1 = 1.$$

- Pell–Lucas number $Q_n = \mathrm{Q}_n(1)$:

$$Q_n = 2Q_{n-1} + Q_{n-2} \quad \text{with} \quad Q_0 = 2 \quad \text{and} \quad Q_1 = 2.$$

The telescoping method has been shown powerful in dealing with finite sums and infinite series (see [**2**, **13**] for example). This approach was employed by Adegoke [**1**] and Melham–Shannon [**13**] to examine finite sums of arctangent function with arguments being Fibonacci and Lucas numbers. Mahon–Horadam [**12**] and the authors [**5**] evaluated a large class of arctant series whose arguments involve both Pell and Pell–Lucas polynomials.

Inspired by the following formula due to Melham and Shannon [**13**]

$$\ln\sqrt{3} = \sum_{k=1}^{\infty} \operatorname{artanh} \frac{1}{F_{2k+2}} \tag{1.1}$$

we shall explore applications of the telescopic approach to the series concerning the products of two hyperbolic arctangent functions whose arguments involve both Pell and Pell–Lucas polynomials. In the next section, five summation formulae about products of artanh function involving the Pell polynomials will be proved. Five analogous formulae will be shown in Section 3 about the Pell–Lucas polynomials. Then in Section 4, we shall establish six closed formulae about cross products of artanh functions with their arguments involving both Pell and Pell–Lucas polynomials. Finally, the paper will end up with Section 5, where more identities will be derived by making use of Cassini–like formulae for Pell and Pell–Lucas polynomials.

Throughout the paper, the following two known formulae about hyperbolic arctangent functions will be crucial:

$$\operatorname{artanh} x + \operatorname{artanh} y = \operatorname{artanh} \frac{x+y}{1+xy}, \tag{1.2}$$

$$\operatorname{artanh} x - \operatorname{artanh} y = \operatorname{artanh} \frac{x-y}{1-xy}. \tag{1.3}$$

When passing from finite sums to infinite series, we shall frequently utilize the following zero limits $\lim_{n\to\infty} \alpha^{-n}(x) = \lim_{n\to\infty} \operatorname{artanh} \alpha^{-n}(x) = 0$ provided that x is a positive real number. In addition, the contents from Section 2 to Section 4 are synchronized with their counterparts appearing in [**5**] so that the reader are facilitated for comparison between these two classes of identities.

2. Identities involving $\mathbf{P}_k(x)$

By combining (1.2) and (1.3) with the Binet form of the Pell polynomial $\mathrm{P}_k(x)$, we can easily derive the following relations:

$$\text{[P1]} \qquad \operatorname{artanh} \frac{1}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}} = 2 \operatorname{artanh} \alpha^{1-2k}.$$

$$\text{[P2]} \qquad \operatorname{artanh} \frac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}} = \operatorname{artanh} \alpha^{1-2k} - \operatorname{artanh} \alpha^{-1-2k}.$$

[P3] $\operatorname{artanh}\dfrac{1}{\mathrm{P}_{2k-1}(x)}=\operatorname{artanh}\alpha^{2-2k}+\operatorname{artanh}\alpha^{-2k}.$

[P4] $\operatorname{artanh}\dfrac{2x}{\mathrm{P}_{2k}(x)}=\operatorname{artanh}\alpha^{2-2k}-\operatorname{artanh}\alpha^{-2-2k}.$

[P5] $\operatorname{artanh}\dfrac{2x^2+1}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}=\operatorname{artanh}\alpha^{3-2k}+\operatorname{artanh}\alpha^{-1-2k}.$

According to the telescoping approach, we have from [P2]

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}=\operatorname{artanh}\alpha^{-1}-\operatorname{artanh}\alpha^{-1-2n}$$

and its limiting form as $n\to\infty$

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}=\operatorname{artanh}\alpha^{-1}$$

as well as two particular examples:

$$x=\frac{1}{2}:\qquad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{1}{\sqrt{5}F_{2k}}=\frac{1}{2}\ln(2+\sqrt{5}),$$

$$x=1:\qquad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{1}{\sqrt{2}P_{2k}}=\frac{1}{2}\ln(1+\sqrt{2}).$$

Analogously, from [P4], we have the summation formula

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}=\operatorname{artanh}\alpha^{-2}+\operatorname{artanh}\alpha^{-4}-\operatorname{artanh}\alpha^{-2n}-\operatorname{artanh}\alpha^{-2-2n}$$

and its limiting form as $n\to\infty$

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}=\operatorname{artanh}\alpha^{-2}+\operatorname{artanh}\alpha^{-4}$$

as well as two infinite series identities:

$$x=\frac{1}{2}:\qquad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{1}{F_{2k}}=\ln\sqrt{3},$$

$$x=1:\qquad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{2}{P_{2k}}=\ln\sqrt{\frac{3}{2}};$$

where the former corresponding to (1.1), anticipated in the introduction.

From [P3], we have the alternating series

$$\sum_{k=2}^{n}(-1)^k\operatorname{artanh}\frac{1}{\mathrm{P}_{2k-1}(x)}=\operatorname{artanh}\alpha^{-2}+(-1)^n\operatorname{artanh}\alpha^{-2n}$$

and its limiting form as $n\to\infty$

$$\sum_{k=2}^{\infty}(-1)^k \operatorname{artanh}\frac{1}{\mathrm{P}_{2k-1}(x)}=\operatorname{artanh}\alpha^{-2}$$

as well as two infinite identities:

$$x=\frac{1}{2}:\qquad \sum_{k=2}^{\infty}(-1)^k \operatorname{artanh}\frac{1}{F_{2k-1}}=\frac{1}{4}\ln 5,$$

$$x=1:\qquad \sum_{k=2}^{\infty}(-1)^k \operatorname{artanh}\frac{1}{P_{2k-1}}=\frac{1}{4}\ln 2.$$

They are simple examples. Now we are going to derive, by examining products of [P1–P5], five further summation formulae involving $\mathrm{P}_k(x)$.

Theorem 2.1.

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\,\mathrm{P}_{2k}(x)}\left\{\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k+1}(x)}+\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}\right\}$$
$$=2\operatorname{artanh}^2\alpha^{-1}-2\operatorname{artanh}^2\alpha^{-1-2n}.$$

When $n\to\infty$, we have the infinite series identity.

Corollary 2.1.

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\,\mathrm{P}_{2k}(x)}\left\{\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k+1}(x)}+\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}\right\}$$
$$=2\operatorname{artanh}^2\alpha^{-1}.$$

Proof of Theorem 2.1. According to [P1], we have

$$\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}+\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k+1}(x)}$$
$$=2\operatorname{artanh}\alpha^{1-2k}+2\operatorname{artanh}\alpha^{-1-2k}.$$

Multiplying this equation by [P2] and then summing the resultant one over k from 1 to n by telescoping, we obtain the desired formula. □

Theorem 2.2.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}\left\{\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\,\mathrm{P}_{2k-2}(x)}+\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\,\mathrm{P}_{2k}(x)}\right\}$$
$$=2\operatorname{artanh}\alpha^{-1}\operatorname{artanh}\alpha^{-3}-2\operatorname{artanh}\alpha^{1-2n}\operatorname{artanh}\alpha^{-1-2n}.$$

Its limiting form as $n\to\infty$ gives rise to the infinite series identity.

Corollary 2.2.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}\left\{\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\,\mathrm{P}_{2k-2}(x)}+\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\,\mathrm{P}_{2k}(x)}\right\}$$
$$=2\operatorname{artanh}\alpha^{-1}\operatorname{artanh}\alpha^{-3}.$$

PROOF OF THEOREM 2.2. By using [P2], we have

$$\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\mathrm{P}_{2k-2}(x)}+\operatorname{artanh}\frac{x}{\sqrt{x^2+1}\mathrm{P}_{2k}(x)}=\operatorname{artanh}\alpha^{3-2k}-\operatorname{artanh}\alpha^{-1-2k}.$$

The proof follows by summing the product of the above equation with [P1] for k from 2 to n. ☐

As special cases, the following two identities about Fibonacci and Pell numbers can be obtained from both Corollary 2.1 and Corollary 2.2:

$$x=\frac{1}{2}:\ \sum_{k=1}^{\infty}\operatorname{artanh}\frac{1}{\sqrt{5}F_{2k}}\left\{\operatorname{artanh}\frac{2}{\sqrt{5}F_{2k-1}}+\operatorname{artanh}\frac{2}{\sqrt{5}F_{2k+1}}\right\}=\frac{1}{2}\ln^2\left(2+\sqrt{5}\right),$$

$$x=1:\ \sum_{k=1}^{\infty}\operatorname{artanh}\frac{1}{\sqrt{2}P_{k}}\operatorname{artanh}\frac{1}{\sqrt{2}P_{k+1}}=\frac{1}{2}\ln^2\left(1+\sqrt{2}\right);$$

where the last series is justified by the bisection series

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{1}{\sqrt{2}P_{2k}}\left\{\operatorname{artanh}\frac{1}{\sqrt{2}P_{2k-1}}+\operatorname{artanh}\frac{1}{\sqrt{2}P_{2k+1}}\right\}.$$

THEOREM 2.3.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}\left\{\operatorname{artanh}\frac{1}{\mathrm{P}_{2k-1}(x)}+\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)}\right\}$$
$$=\left(\operatorname{artanh}\alpha^{-2}+\operatorname{artanh}\alpha^{-4}\right)^2-\left(\operatorname{artanh}\alpha^{-2n}+\operatorname{artanh}\alpha^{-2-2n}\right)^2.$$

Its limiting form as $n\to\infty$ leads to the infinite series identity.

COROLLARY 2.3.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}\left\{\operatorname{artanh}\frac{1}{\mathrm{P}_{2k-1}(x)}+\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-2}+\operatorname{artanh}\alpha^{-4})^2.$$

PROOF OF THEOREM 2.3. In view of [P3], we have

$$\operatorname{artanh}\frac{1}{\mathrm{P}_{2k-1}(x)}+\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)}$$
$$=\operatorname{artanh}\alpha^{2-2k}+2\operatorname{artanh}\alpha^{-2k}+\operatorname{artanh}\alpha^{-2-2k}.$$

Multiplying [P4] with the above equation and then summing over k from 2 to n by telescoping, we get the expected formula. ☐

THEOREM 2.4.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)}\left\{\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}+\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k+2}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-2}+\operatorname{artanh}\alpha^{-4})\times(\operatorname{artanh}\alpha^{-4}+\operatorname{artanh}\alpha^{-6})$$
$$-(\operatorname{artanh}\alpha^{-2n}+\operatorname{artanh}\alpha^{-2-2n})\times(\operatorname{artanh}\alpha^{-2-2n}+\operatorname{artanh}\alpha^{-4-2n}).$$

Its limiting form as $n\to\infty$ results in the infinite series identity.

COROLLARY 2.4.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)}\left\{\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}+\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k+2}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-2}+\operatorname{artanh}\alpha^{-4})\times(\operatorname{artanh}\alpha^{-4}+\operatorname{artanh}\alpha^{-6}).$$

PROOF OF THEOREM 2.4. From [P4], we have

$$\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}+\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k+2}(x)}=\operatorname{artanh}\alpha^{2-2k}-\operatorname{artanh}\alpha^{-2-2k}$$
$$+\operatorname{artanh}\alpha^{-2k}-\operatorname{artanh}\alpha^{-4-2k}.$$

Multiplying this by [P3], we can reformulate the result as

$$\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)}\left\{\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}+\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k+2}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-2k}+\operatorname{artanh}\alpha^{-2-2k})\times(\operatorname{artanh}\alpha^{-2k}+\operatorname{artanh}\alpha^{2-2k})$$
$$-(\operatorname{artanh}\alpha^{-2k}+\operatorname{artanh}\alpha^{-2-2k})\times(\operatorname{artanh}\alpha^{-2-2k}+\operatorname{artanh}\alpha^{-4-2k}).$$

Summing this equation over k from 2 to n by telescoping, we get the desired result. □

The following two identities can be deduced from both Corollary 2.3 and Corollary 2.4:

$$x=\frac{1}{2}:\quad\sum_{k=3}^{\infty}\operatorname{artanh}\frac{1}{F_k}\operatorname{artanh}\frac{1}{F_{k+1}}=\frac{1}{4}\ln^2 3,$$
$$x=1:\quad\sum_{k=2}^{\infty}\operatorname{artanh}\frac{2}{P_{2k}}\left\{\operatorname{artanh}\frac{1}{P_{2k-1}}+\operatorname{artanh}\frac{1}{P_{2k+1}}\right\}=\frac{1}{4}\ln^2\frac{3}{2};$$

where the former follows from the bisection series

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{1}{F_{2k}}\left\{\operatorname{artanh}\frac{1}{F_{2k-1}}+\operatorname{artanh}\frac{1}{F_{2k+1}}\right\}.$$

THEOREM 2.5.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{2x^2+1}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}\left\{\operatorname{artanh}\frac{x}{\mathrm{P}_{2k-2}(x)\sqrt{x^2+1}}+\operatorname{artanh}\frac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\right\}$$
$$=\operatorname{artanh}^2\alpha^{-1}+\operatorname{artanh}^2\alpha^{-3}-\operatorname{artanh}^2\alpha^{1-2n}-\operatorname{artanh}^2\alpha^{-1-2n}.$$

Its limiting form as $n\to\infty$ yields the infinite series identity.

COROLLARY 2.5.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{2x^2+1}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}\left\{\operatorname{artanh}\frac{x}{\mathrm{P}_{2k-2}(x)\sqrt{x^2+1}}+\operatorname{artanh}\tfrac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\right\}$$
$$=\operatorname{artanh}^2\alpha^{-1}+\operatorname{artanh}^2\alpha^{-3}.$$

Proof of Theorem 2.5. Keeping in mind [P2], we have

$$\operatorname{artanh}\frac{x}{\mathrm{P}_{2k-2}(x)\sqrt{x^2+1}}+\operatorname{artanh}\frac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}=\operatorname{artanh}\alpha^{3-2k}-\operatorname{artanh}\alpha^{-1-2k}.$$

Then the formula follows by multiplying the above equation with [P5] and then summing for k from 2 to n by telescoping. □

As particular cases, two infinite identities are recorded below:

$$x=\frac{1}{2}: \quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{3}{\sqrt{5}F_{2k-1}}\left\{\operatorname{artanh}\frac{1}{\sqrt{5}F_{2k-2}}+\operatorname{artanh}\frac{1}{\sqrt{5}F_{2k}}\right\}$$
$$=\frac{1}{4}\ln^2(2+\sqrt{5})+\frac{1}{4}\ln^2\frac{1+\sqrt{5}}{2},$$
$$x=1: \quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{3}{\sqrt{2}P_{2k-1}}\left\{\operatorname{artanh}\frac{1}{\sqrt{2}P_{2k-2}}+\operatorname{artanh}\frac{1}{\sqrt{2}P_{2k}}\right\}$$
$$=\frac{1}{4}\ln^2(1+\sqrt{2})+\frac{1}{4}\ln^2\frac{1+5\sqrt{2}}{7}.$$

3. Identities involving $\mathbf{Q}_k(x)$

In view of (1.2) and (1.3) as well as the Binet form of the Pell–Lucas polynomial $\mathrm{Q}_k(x)$, we have similarly the following expressions:

[Q1] $$\operatorname{artanh}\frac{2}{\mathrm{Q}_{2k}(x)}=2\operatorname{artanh}\alpha^{-2k}.$$

[Q2] $$\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k-1}(x)}=\operatorname{artanh}\alpha^{2-2k}-\operatorname{artanh}\alpha^{-2k}.$$

[Q3] $$\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}=\operatorname{artanh}\alpha^{1-2k}+\operatorname{artanh}\alpha^{-1-2k}.$$

[Q4] $$\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}=\operatorname{artanh}\alpha^{3-2k}-\operatorname{artanh}\alpha^{-1-2k}.$$

[Q5] $$\operatorname{artanh}\frac{2(2x^2+1)}{\mathrm{Q}_{2k}(x)}=\operatorname{artanh}\alpha^{2-2k}+\operatorname{artanh}\alpha^{-2-2k}.$$

Applying the telescoping method to [Q2], we have the summation formula

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k-1}(x)}=\operatorname{artanh}\alpha^{-2}-\operatorname{artanh}\alpha^{-2n}$$

and its limiting version as $n\to\infty$

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k-1}(x)}=\operatorname{artanh}\alpha^{-2}$$

as well as two particular cases:

$$x = \frac{1}{2}: \qquad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{L_{2k-1}} = \frac{1}{4} \ln 5,$$

$$x = 1: \qquad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{2}{Q_{2k-1}} = \frac{1}{4} \ln 2.$$

Alternatively, we have from [Q4]

$$\sum_{k=2}^{n} \operatorname{artanh} \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} = \operatorname{artanh} \alpha^{-1} + \operatorname{artanh} \alpha^{-3} - \operatorname{artanh} \alpha^{1-2n} - \operatorname{artanh} \alpha^{-1-2n}$$

and its limiting version as $n \to \infty$

$$\sum_{k=2}^{n} \operatorname{artanh} \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} = \operatorname{artanh} \alpha^{-1} + \operatorname{artanh} \alpha^{-3}$$

as well as two infinite series:

$$x = \frac{1}{2}: \qquad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{\sqrt{5}}{L_{2k-1}} = \frac{1}{2} \ln \frac{7+3\sqrt{5}}{2},$$

$$x = 1: \qquad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{4\sqrt{2}}{Q_{2k-1}} = \frac{1}{2} \ln \frac{11+6\sqrt{2}}{7}.$$

Similarly, we have from [Q3] the alternating series

$$\sum_{k=1}^{n} (-1)^{k-1} \operatorname{artanh} \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} = \operatorname{artanh} \alpha^{-1} + (-1)^{n-1} \operatorname{artanh} \alpha^{-1-2n}$$

and its limiting version as $n \to \infty$

$$\sum_{k=1}^{\infty} (-1)^{k-1} \operatorname{artanh} \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} = \operatorname{artanh} \alpha^{-1}$$

as well as two particular series:

$$x = \frac{1}{2}: \qquad \sum_{k=1}^{\infty} (-1)^{k-1} \operatorname{artanh} \frac{\sqrt{5}}{L_{2k}} = \frac{1}{2} \ln(2+\sqrt{5}),$$

$$x = 1: \qquad \sum_{k=1}^{\infty} (-1)^{k-1} \operatorname{artanh} \frac{2\sqrt{2}}{Q_{2k}} = \frac{1}{2} \ln(1+\sqrt{2}).$$

Now, we shall examine, by means of telescoping method, products of [Q1–Q5] and establish further five summation formulae for $\mathrm{Q}_k(x)$.

Theorem 3.1.

$$\sum_{k=2}^{n} \operatorname{artanh} \frac{2x}{\mathrm{Q}_{2k-1}(x)} \left\{ \operatorname{artanh} \frac{2}{\mathrm{Q}_{2k-2}(x)} + \operatorname{artanh} \frac{2}{\mathrm{Q}_{2k}(x)} \right\}$$
$$= 2 \operatorname{artanh}^2 \alpha^{-2} - 2 \operatorname{artanh}^2 \alpha^{-2n}.$$

Its limiting form as $n \to \infty$ brings about the infinite series identity.

Corollary 3.1.

$$\sum_{k=2}^{\infty} \operatorname{artanh} \frac{2x}{Q_{2k-1}(x)} \left\{ \operatorname{artanh} \frac{2}{Q_{2k-2}(x)} + \operatorname{artanh} \frac{2}{Q_{2k}(x)} \right\} = 2 \operatorname{artanh}^2 \alpha^{-2}.$$

Proof of Theorem 3.1. According to [Q1], we have

$$\operatorname{artanh} \frac{2}{Q_{2k-2}(x)} + \operatorname{artanh} \frac{2}{Q_{2k}(x)} = 2 \operatorname{artanh} \alpha^{2-2k} + 2 \operatorname{artanh} \alpha^{-2k}.$$

Multiplying this with [Q2] and then summing over k from 2 to n by telescoping, we find the desired summation formula. □

Theorem 3.2.

$$\sum_{k=2}^{n} \operatorname{artanh} \frac{2}{Q_{2k}(x)} \left\{ \operatorname{artanh} \frac{2x}{Q_{2k-1}(x)} + \operatorname{artanh} \frac{2x}{Q_{2k+1}(x)} \right\} = 2 \operatorname{artanh} \alpha^{-2} \operatorname{artanh} \alpha^{-4} - 2 \operatorname{artanh} \alpha^{-2n} \operatorname{artanh} \alpha^{-2-2n}. \tag{3.1}$$

Its limiting form as $n \to \infty$ gives rise to the infinite series identity.

Corollary 3.2.

$$\sum_{k=2}^{\infty} \operatorname{artanh} \frac{2}{Q_{2k}(x)} \left\{ \operatorname{artanh} \frac{2x}{Q_{2k-1}(x)} + \operatorname{artanh} \frac{2x}{Q_{2k+1}(x)} \right\} = 2 \operatorname{artanh} \alpha^{-2} \operatorname{artanh} \alpha^{-4}.$$

Proof of Theorem 3.2. By means of [Q2], we have

$$\operatorname{artanh} \frac{2x}{Q_{2k-1}(x)} + \operatorname{artanh} \frac{2x}{Q_{2k+1}(x)} = \operatorname{artanh} \alpha^{2-2k} - \operatorname{artanh} \alpha^{-2-2k}.$$

Then the expected summation formula follows by multiplying this with [Q1] and then summing over k from 2 to n by telescoping. □

In particular, we can deduce, from both Corollary 3.1 and Corollary 3.2, the following two further interesting identities:

$$x = \frac{1}{2}: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{L_{2k-1}} \left\{ \operatorname{artanh} \frac{2}{L_{2k-2}} + \operatorname{artanh} \frac{2}{L_{2k}} \right\} = \frac{1}{8} \ln^2 5,$$

$$x = 1: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{2}{Q_k} \operatorname{artanh} \frac{2}{Q_{k+1}} = \frac{1}{8} \ln^2 2;$$

where the latter one is simplified from the bisection series

$$\sum_{k=2}^{\infty} \operatorname{artanh} \frac{2}{Q_{2k-1}} \left\{ \operatorname{artanh} \frac{2}{Q_{2k-2}} + \operatorname{artanh} \frac{2}{Q_{2k}} \right\}.$$

Theorem 3.3.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}\left\{\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-2}(x)}+\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-1}+\operatorname{artanh}\alpha^{-3})^2-(\operatorname{artanh}\alpha^{1-2n}+\operatorname{artanh}\alpha^{-1-2n})^2.$$

When $n\to\infty$, this leads us to the infinite series identity.

Corollary 3.3.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}\left\{\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-2}(x)}+\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-1}+\operatorname{artanh}\alpha^{-3})^2.$$

Proof of Theorem 3.3. In view of [Q3], we have

$$\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-2}(x)}+\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k+2}(x)}$$
$$=\operatorname{artanh}\alpha^{3-2k}+\operatorname{artanh}\alpha^{-1-2k}+2\operatorname{artanh}\alpha^{1-2k}.$$

Multiplying this by [Q4] and then summing over k from 2 to n, we complete the proof. □

Theorem 3.4.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}\left\{\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}+\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k+1}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-1}+\operatorname{artanh}\alpha^{-3})\times(\operatorname{artanh}\alpha^{-3}+\operatorname{artanh}\alpha^{-5})$$
$$-(\operatorname{artanh}\alpha^{1-2n}+\operatorname{artanh}\alpha^{-1-2n})\times(\operatorname{artanh}\alpha^{-1-2n}+\operatorname{artanh}\alpha^{-3-2n}).$$

Its limiting form as $n\to\infty$ results in the infinite series identity.

Corollary 3.4.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}\left\{\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}+\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k+1}(x)}\right\}$$
$$=(\operatorname{artanh}\alpha^{-1}+\operatorname{artanh}\alpha^{-3})\times(\operatorname{artanh}\alpha^{-3}+\operatorname{artanh}\alpha^{-5}).$$

Proof of Theorem 3.4. Keeping in mind [Q4], we have

$$\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}+\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k+1}(x)}$$
$$=\operatorname{artanh}\alpha^{3-2k}-\operatorname{artanh}\alpha^{-1-2k}+\operatorname{artanh}\alpha^{1-2k}-\operatorname{artanh}\alpha^{-3-2k}.$$

The proof is done by multiplying this by [Q3] and then summing over k from 2 to n. □

Both Corollaries 3.3 and 3.4 contain the following two particular cases.

$$x=\frac{1}{2}: \quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{\sqrt{5}}{L_k}\operatorname{artanh}\frac{\sqrt{5}}{L_{k+1}}=4\ln^2\frac{1+\sqrt{5}}{2},$$

$$x=1: \quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{4\sqrt{2}}{Q_{2k-1}}\left\{\operatorname{artanh}\frac{2\sqrt{2}}{Q_{2k-2}}+\operatorname{artanh}\frac{2\sqrt{2}}{Q_{2k}}\right\}=\ln^2\frac{3+\sqrt{2}}{\sqrt{7}}.$$

Theorem 3.5.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{2(2x^2+1)}{\mathrm{Q}_{2k}(x)}\left\{\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k-1}(x)}+\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k+1}(x)}\right\}$$
$$=\operatorname{artanh}^2\alpha^{-2}+\operatorname{artanh}^2\alpha^{-4}-\operatorname{artanh}^2\alpha^{-2n}-\operatorname{artanh}^2\alpha^{-2-2n}.$$

Its limiting form as $n\to\infty$ yields the infinite series identity:

Corollary 3.5.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{2(2x^2+1)}{\mathrm{Q}_{2k}(x)}\left\{\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k-1}(x)}+\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k+1}(x)}\right\}$$
$$=\operatorname{artanh}^2\alpha^{-2}+\operatorname{artanh}^2\alpha^{-4}.$$

Proof of Theorem 3.5. Considering [Q2], we have

$$\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k-1}(x)}+\operatorname{artanh}\frac{2x}{\mathrm{Q}_{2k+1}(x)}=\operatorname{artanh}\alpha^{2-2k}-\operatorname{artanh}\alpha^{-2-2k}.$$

Multiplying this by [Q5] and then summing over k from 2 to n by telescoping, we arrive at the desired formula in the theorem. □

As special cases, we deduce two further identities from this corollary:

$$x=\frac{1}{2}: \sum_{k=2}^{\infty}\operatorname{artanh}\frac{3}{L_{2k}}\left\{\operatorname{artanh}\frac{1}{L_{2k-1}}+\operatorname{artanh}\frac{1}{L_{2k+1}}\right\}=\frac{1}{16}\ln^2 5+\frac{1}{16}\ln^2\tfrac{9}{5},$$

$$x=1: \sum_{k=2}^{\infty}\operatorname{artanh}\frac{6}{Q_{2k}}\left\{\operatorname{artanh}\frac{2}{Q_{2k-1}}+\operatorname{artanh}\frac{2}{Q_{2k+1}}\right\}=\frac{1}{16}\ln^2 2+\frac{1}{4}\ln^2\tfrac{3\sqrt{2}}{4}.$$

4. Identities involving both $\mathbf{P}_k(x)$ and $\mathbf{Q}_k(x)$

This section will be devoted to cross products between [P1–P5] and [Q1–Q5]. Six summation formulae containing both Pell and Pell–Lucas polynomials will be established via telescoping method.

Firstly, for [P1] and [Q4], by adding their product with respect to k from 2 to n and then using telescoping method, we have the summation formula below.

Theorem 4.1.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}$$
$$=2\operatorname{artanh}\alpha^{-1}\operatorname{artanh}\alpha^{-3}-2\operatorname{artanh}\alpha^{1-2n}\operatorname{artanh}\alpha^{-1-2n}.$$

When $n \to \infty$, this reduces to the infinite series identity.

Corollary 4.1.

$$\sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)} \operatorname{artanh} \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} = 2 \operatorname{artanh} \alpha^{-1} \operatorname{artanh} \alpha^{-3}.$$

Two special cases are highlighted as follows:

$$x = \frac{1}{2}: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{2}{\sqrt{5}F_{2k-1}} \operatorname{artanh} \frac{\sqrt{5}}{L_{2k-1}} = \frac{1}{2} \ln(2+\sqrt{5}) \ln \frac{1+\sqrt{5}}{2},$$

$$x = 1: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{\sqrt{2}P_{2k-1}} \operatorname{artanh} \frac{4\sqrt{2}}{Q_{2k-1}} = \frac{1}{2} \ln(1+\sqrt{2}) \ln \frac{1+5\sqrt{2}}{7}.$$

Secondly, keeping in mind of the product of [P4] and [Q1], we have the following formula.

Theorem 4.2.

$$\sum_{k=2}^{n} \operatorname{artanh} \frac{2x}{\mathrm{P}_{2k}(x)} \operatorname{artanh} \frac{2}{\mathrm{Q}_{2k}(x)} = 2 \operatorname{artanh} \alpha^{-2} \operatorname{artanh} \alpha^{-4} - 2 \operatorname{artanh} \alpha^{-2n} \operatorname{artanh} \alpha^{-2-2n}.$$

Its limiting case as $n \to \infty$ gives rise to the infinite series identity.

Corollary 4.2.

$$\sum_{k=2}^{\infty} \operatorname{artanh} \frac{2x}{\mathrm{P}_{2k}(x)} \operatorname{artanh} \frac{2}{\mathrm{Q}_{2k}(x)} = 2 \operatorname{artanh} \alpha^{-2} \operatorname{artanh} \alpha^{-4}.$$

As particular cases, we record two examples of this corollary:

$$x = \frac{1}{2}: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{F_{2k}} \operatorname{artanh} \frac{2}{L_{2k}} = \frac{1}{4} \ln 5 \ln \frac{3}{\sqrt{5}},$$

$$x = 1: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{2}{P_{2k}} \operatorname{artanh} \frac{2}{Q_{2k}} = \frac{1}{4} \ln 2 \ln \frac{3\sqrt{2}}{4}.$$

Now, making product of [P2] and [Q3], we have the formula below.

Theorem 4.3.

$$\sum_{k=1}^{n} \operatorname{artanh} \frac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}} \operatorname{artanh} \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} = \operatorname{artanh}^2 \alpha^{-1} - \operatorname{artanh}^2 \alpha^{-1-2n}.$$

Letting $n \to \infty$, we get the following infinite series identity.

Corollary 4.3.

$$\sum_{k=1}^{\infty} \operatorname{artanh} \frac{x}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}} \operatorname{artanh} \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} = \operatorname{artanh}^2 \alpha^{-1}.$$

Two particular cases for $x = 1/2$ and $x = 1$ read as follows:

$$x = \frac{1}{2}: \qquad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{1}{\sqrt{5}F_{2k}} \operatorname{artanh} \frac{\sqrt{5}}{L_{2k}} = \frac{1}{4} \ln^2(2+\sqrt{5}),$$

$$x = 1: \qquad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{1}{\sqrt{2}P_{2k}} \operatorname{artanh} \frac{2\sqrt{2}}{Q_{2k}} = \frac{1}{4} \ln^2(1+\sqrt{2}).$$

Analogously, the product of [P3] and [Q2] leads to the following formula.

Theorem 4.4.

$$\sum_{k=2}^{n} \operatorname{artanh} \frac{1}{\mathrm{P}_{2k-1}(x)} \operatorname{artanh} \frac{2x}{\mathrm{Q}_{2k-1}(x)} = \operatorname{artanh}^2 \alpha^{-2} - \operatorname{artanh}^2 \alpha^{-2n}.$$

By letting $n \to \infty$, we find the infinite series identity below.

Corollary 4.4.

$$\sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{\mathrm{P}_{2k-1}(x)} \operatorname{artanh} \frac{2x}{\mathrm{Q}_{2k-1}(x)} = \operatorname{artanh}^2 \alpha^{-2}.$$

Two particular cases are given below as examples:

$$x = \frac{1}{2}: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{F_{2k-1}} \operatorname{artanh} \frac{1}{L_{2k-1}} = \frac{1}{16} \ln^2 5,$$

$$x = 1: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{P_{2k-1}} \operatorname{artanh} \frac{2}{Q_{2k-1}} = \frac{1}{16} \ln^2 2.$$

By multiplying [P4] and [Q5], we have the following formula.

Theorem 4.5.

$$\sum_{k=2}^{n} \operatorname{artanh} \frac{2x}{\mathrm{P}_{2k}(x)} \operatorname{artanh} \frac{2(2x^2+1)}{\mathrm{Q}_{2k}(x)}$$
$$= \operatorname{artanh}^2 \alpha^{-2} + \operatorname{artanh}^2 \alpha^{-4} - \operatorname{artanh}^2 \alpha^{-2n} - \operatorname{artanh}^2 \alpha^{-2-2n}.$$

Its limiting case as $n \to \infty$ results in the infinite series identity.

Corollary 4.5.

$$\sum_{k=2}^{\infty} \operatorname{artanh} \frac{2x}{\mathrm{P}_{2k}(x)} \operatorname{artanh} \frac{2(2x^2+1)}{\mathrm{Q}_{2k}(x)} = \operatorname{artanh}^2 \alpha^{-2} + \operatorname{artanh}^2 \alpha^{-4}.$$

This identity contains the two infinite series below as particular cases:

$$x = \frac{1}{2}: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{1}{F_{2k}} \operatorname{artanh} \frac{3}{L_{2k}} = \frac{1}{16} \ln^2 5 + \frac{1}{4} \ln^2 \frac{3}{\sqrt{5}},$$

$$x = 1: \quad \sum_{k=2}^{\infty} \operatorname{artanh} \frac{2}{P_{2k}} \operatorname{artanh} \frac{6}{Q_{2k}} = \frac{1}{16} \ln^2 2 + \frac{1}{4} \ln^2 \frac{3\sqrt{2}}{4}.$$

Alternatively, the product of [P5] and [Q4] yields the summation formula below.

Theorem 4.6.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}\operatorname{artanh}\frac{2x^2+1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}$$
$$=\operatorname{artanh}^2\alpha^{-1}+\operatorname{artanh}^2\alpha^{-3}-\operatorname{artanh}^2\alpha^{1-2n}-\operatorname{artanh}^2\alpha^{-1-2n}.$$

Its limiting form as $n\to\infty$ becomes the infinite series identity.

Corollary 4.6.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}\operatorname{artanh}\frac{2x^2+1}{\sqrt{x^2+1}\,\mathrm{P}_{2k-1}(x)}=\operatorname{artanh}^2\alpha^{-1}+\operatorname{artanh}^2\alpha^{-3}.$$

This identity implies two further summation formulae:

$$x=\frac{1}{2}:\quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{\sqrt{5}}{L_{2k-1}}\operatorname{artanh}\frac{3}{\sqrt{5}F_{2k-1}}=\frac{1}{4}\ln^2(2+\sqrt{5})+\frac{1}{4}\ln^2\frac{1+\sqrt{5}}{2},$$
$$x=1:\quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{4\sqrt{2}}{Q_{2k-1}}\operatorname{artanh}\frac{3}{\sqrt{2}P_{2k-1}}=\frac{1}{4}\ln^2(1+\sqrt{2})+\frac{1}{4}\ln^2\frac{1+5\sqrt{2}}{7}.$$

Finally, we point out that more similar identities can be obtained by using the telescope method. For example, considering the following two formulae

$$\operatorname{artanh}\frac{x(4x^2+3)}{\sqrt{x^2+1}\,\mathrm{P}_{2k}(x)}=\operatorname{artanh}\alpha^{3-2k}-\operatorname{artanh}\alpha^{-3-2k},$$
$$\operatorname{artanh}\frac{2\sqrt{x^2+1}(4x^2+1)}{\mathrm{Q}_{2k}(x)}=\operatorname{artanh}\alpha^{3-2k}+\operatorname{artanh}\alpha^{-3-2k};$$

we can derive the identity

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{x(4x^2+3)}{\sqrt{x^2+1}\,\mathrm{P}_{2k}(x)}\operatorname{artanh}\frac{2\sqrt{x^2+1}(4x^2+1)}{\mathrm{Q}_{2k}(x)}$$
$$=\operatorname{artanh}^2\alpha^{-1}+\operatorname{artanh}^2\alpha^{-3}+\operatorname{artanh}^2\alpha^{-5}$$
$$-\operatorname{artanh}^2\alpha^{1-2n}-\operatorname{artanh}^2\alpha^{-1-2n}-\operatorname{artanh}^2\alpha^{-3-2n},$$

and the corresponding limiting formula

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{x(4x^2+3)}{\sqrt{x^2+1}\,\mathrm{P}_{2k}(x)}\operatorname{artanh}\frac{2\sqrt{x^2+1}(4x^2+1)}{\mathrm{Q}_{2k}(x)}$$
$$=\operatorname{artanh}^2\alpha^{-1}+\operatorname{artanh}^2\alpha^{-3}+\operatorname{artanh}^2\alpha^{-5}.$$

Letting $x=\frac{1}{2}$ and $x=1$, we obtain the following two infinite series:

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{4}{\sqrt{5}F_{2k}}\operatorname{artanh}\frac{2\sqrt{5}}{L_{2k}}=\frac{1}{4}\left\{\ln^2(2+\sqrt{5})+\ln^2\frac{1+\sqrt{5}}{2}+\ln^2\frac{2+5\sqrt{5}}{11}\right\},$$
$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{7}{\sqrt{2}P_{2k}}\operatorname{artanh}\frac{10\sqrt{2}}{Q_{2k}}=\frac{1}{4}\left\{\ln^2(1+\sqrt{2})+\ln^2\frac{1+5\sqrt{2}}{7}+\ln^2\frac{1+29\sqrt{2}}{41}\right\}.$$

5. Identities involving squares of $\mathrm{P}_k(x)$ and $\mathrm{Q}_k(x)$

Recall the Cassini formula for Fibonacci numbers $F_{n+1}F_{n-1} = (-1)^n + F_n^2$. There exist analogous ones for Pell and Pell–Lucas polynomials. They will be employed in this section to evaluate, in closed forms, four classes of sums about products of two hyperbolic arctangent functions.

5.1. By making use of the following Cassini–like formulae (cf. Koshy [**9**, §14.7])

$$\mathrm{P}_{k-1}(x) + \mathrm{P}_{k+1}(x) = \mathrm{Q}_k(x),$$
$$\mathrm{P}_{k-1}(x)\,\mathrm{P}_{k+1}(x) = \mathrm{P}_k^2(x) + (-1)^k;$$

it is not difficult to deduce from (1.2) and (1.3)

$$\operatorname{artanh}\frac{\mathrm{Q}_k(x)}{\mathrm{P}_k^2(x)+(-1)^k+1} = \operatorname{artanh}\frac{1}{\mathrm{P}_{k-1}(x)} + \operatorname{artanh}\frac{1}{\mathrm{P}_{k+1}(x)}, \tag{5.1}$$

$$\operatorname{artanh}\frac{2x\,\mathrm{P}_k(x)}{\mathrm{P}_k^2(x)+(-1)^k-1} = \operatorname{artanh}\frac{1}{\mathrm{P}_{k-1}(x)} - \operatorname{artanh}\frac{1}{\mathrm{P}_{k+1}(x)}. \tag{5.2}$$

They were directly utilized by Melham–Shannon [**13**] to establish

$$\sum_{k=1}^{n}(-1)^{k-1}\operatorname{artanh}\frac{\mathrm{Q}_{2k+3}(x)}{\mathrm{P}_{2k+3}^2(x)} = \operatorname{artanh}\frac{1}{\mathrm{P}_4(x)} + (-1)^{n-1}\operatorname{artanh}\frac{1}{\mathrm{P}_{2n+4}(x)},$$
$$\sum_{k=1}^{n}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k+2}(x)} = \operatorname{artanh}\frac{1}{\mathrm{P}_3(x)} - \operatorname{artanh}\frac{1}{\mathrm{P}_{2n+3}(x)}.$$

Instead, we shall utilize both (5.1) and (5.2) to derive further summation formulae concerning products of two hyperbolic arctangent functions.

§5.1A. . Replacing k by $2k$, we can rewrite (5.1) and (5.2) as

$$\operatorname{artanh}\frac{1}{\mathrm{P}_{2k-1}(x)} + \operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)} = \operatorname{artanh}\frac{\mathrm{Q}_{2k}(x)}{\mathrm{P}_{2k}^2(x)+2},$$
$$\operatorname{artanh}\frac{1}{\mathrm{P}_{2k-1}(x)} - \operatorname{artanh}\frac{1}{\mathrm{P}_{2k+1}(x)} = \operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)}.$$

Their multiplication gives rise to

$$\sum_{k=2}^{n+1}\operatorname{artanh}\frac{\mathrm{Q}_{2k}(x)}{\mathrm{P}_{2k}^2(x)+2}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k}(x)} = \sum_{k=2}^{n+1}\Big\{\operatorname{artanh}^2\tfrac{1}{\mathrm{P}_{2k-1}(x)} - \operatorname{artanh}^2\tfrac{1}{\mathrm{P}_{2k+1}(x)}\Big\}.$$

Then summing this equation for k from 2 to $n+1$ by telescoping and replacing k by $k+1$, we find the following formula.

THEOREM 5.1.

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{\mathrm{Q}_{2k+2}(x)}{\mathrm{P}_{2k+2}^2(x)+2}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k+2}(x)} = \operatorname{artanh}^2\frac{1}{4x^2+1} - \operatorname{artanh}^2\frac{1}{\mathrm{P}_{2n+3}(x)}.$$

Its limiting case as $n \to \infty$ yields the infinite series identity.

Corollary 5.1.

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{\mathrm{Q}_{2k+2}(x)}{\mathrm{P}_{2k+2}^2(x)+2}\operatorname{artanh}\frac{2x}{\mathrm{P}_{2k+2}(x)}=\operatorname{artanh}^2\frac{1}{4x^2+1}.$$

By specifying particular values for x, we can derive, from the above corollary, the following two infinite series identities:

$$x=\frac{1}{2}:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{L_{2k+2}}{F_{2k+2}^2+2}\operatorname{artanh}\frac{1}{F_{2k+2}}=\frac{1}{4}\ln^2 3,$$
$$x=1:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{Q_{2k+2}}{P_{2k+2}^2+2}\operatorname{artanh}\frac{2}{P_{2k+2}}=\frac{1}{4}\ln^2\frac{3}{2}.$$

§5.1B. . Alternatively, (5.1) and (5.2) can be restated under the replacement k by $2k+1$ as

$$\operatorname{artanh}\frac{1}{\mathrm{P}_{2k}(x)}+\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+2}(x)}=\operatorname{artanh}\frac{\mathrm{Q}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)},$$
$$\operatorname{artanh}\frac{1}{\mathrm{P}_{2k}(x)}-\operatorname{artanh}\frac{1}{\mathrm{P}_{2k+2}(x)}=\operatorname{artanh}\frac{2x\,\mathrm{P}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)-2}.$$

First summing their product for k from 2 to n by telescoping, we get the theorem below.

Theorem 5.2.

$$\sum_{k=2}^{n}\operatorname{artanh}\frac{\mathrm{Q}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)}\operatorname{artanh}\frac{2x\,\mathrm{P}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)-2}=\operatorname{artanh}^2\frac{1}{\mathrm{P}_4(x)}-\operatorname{artanh}^2\frac{1}{\mathrm{P}_{2n+2}(x)}.$$

Now letting $n\to\infty$ in this theorem, we deduce the infinite series identity.

Corollary 5.2.

$$\sum_{k=2}^{\infty}\operatorname{artanh}\frac{\mathrm{Q}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)}\operatorname{artanh}\frac{2x\,\mathrm{P}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)-2}=\operatorname{artanh}^2\frac{1}{4x(2x^2+1)}.$$

We record two interesting formulae by choosing particular values of x:

$$x=\frac{1}{2}:\quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{L_{2k+1}}{F_{2k+1}^2}\operatorname{artanh}\frac{F_{2k+1}}{F_{2k+1}^2-2}=\frac{1}{4}\ln^2 2,$$
$$x=1:\quad \sum_{k=2}^{\infty}\operatorname{artanh}\frac{Q_{2k+1}}{P_{2k+1}^2}\operatorname{artanh}\frac{2P_{2k+1}}{P_{2k+1}^2-2}=\frac{1}{4}\ln^2\frac{13}{11}.$$

We remark that when $x\neq\frac{1}{2}$, the formulae displayed in Theorem 5.2 and Corollary 5.2 can slightly be modified, by adding the respective initial terms corresponding to $k=1$, as

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{\mathrm{Q}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)}\operatorname{artanh}\frac{2x\,\mathrm{P}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)-2}=\operatorname{artanh}^2\frac{1}{\mathrm{P}_2(x)}-\operatorname{artanh}^2\frac{1}{\mathrm{P}_{2n+2}(x)},$$

$$\sum_{k=1}^{\infty} \operatorname{artanh} \frac{\mathrm{Q}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)} \operatorname{artanh} \frac{2x\,\mathrm{P}_{2k+1}(x)}{\mathrm{P}_{2k+1}^2(x)-2} = \operatorname{artanh}^2 \frac{1}{2x};$$

where for $x = 1$, the latter becomes the infinite series identity

$$\sum_{k=1}^{\infty} \operatorname{artanh} \frac{Q_{2k+1}}{P_{2k+1}^2} \operatorname{artanh} \frac{2P_{2k+1}}{P_{2k+1}^2-2} = \frac{1}{4}\ln^2 3.$$

5.2. Analogously, there are also two Cassini–like formulae (cf. Koshy [**9**, §14.7]) for Pell–Lucas polynomials

$$\mathrm{Q}_{k-1}(x) + \mathrm{Q}_{k+1}(x) = 4(x^2+1)\,\mathrm{P}_k(x),$$
$$\mathrm{Q}_{k-1}(x)\,\mathrm{Q}_{k+1}(x) = \mathrm{Q}_k^2(x) + 4(-1)^{k-1}(x^2+1).$$

In view of (1.2) and (1.3), we have the addition and difference formulae

$$\operatorname{artanh} \frac{1}{\mathrm{Q}_{k-1}(x)} + \operatorname{artanh} \frac{1}{\mathrm{Q}_{k+1}(x)} = \operatorname{artanh} \frac{4(x^2+1)\,\mathrm{P}_k(x)}{\mathrm{Q}_k^2(x) - 4(-1)^k(x^2+1)+1}, \tag{5.3}$$

$$\operatorname{artanh} \frac{1}{\mathrm{Q}_{k-1}(x)} - \operatorname{artanh} \frac{1}{\mathrm{Q}_{k+1}(x)} = \operatorname{artanh} \frac{2x\,\mathrm{Q}_k(x)}{\mathrm{Q}_k^2(x) - 4(-1)^k(x^2+1)-1}. \tag{5.4}$$

The two relations can now be employed, by telescoping, to evaluate finite sums for both a single hyperbolic arctangent function and products of two hyperbolic arctangent functions.

§5.2A. . Replacing k by $2k$, we can rewrite (5.3) and (5.4) as

$$\operatorname{artanh} \frac{4(x^2+1)\,\mathrm{P}_{2k}(x)}{\mathrm{Q}_{2k}^2(x) - 4x^2 - 3} = \operatorname{artanh} \frac{1}{\mathrm{Q}_{2k-1}(x)} + \operatorname{artanh} \frac{1}{\mathrm{Q}_{2k+1}(x)}, \tag{5.5}$$

$$\operatorname{artanh} \frac{2x\,\mathrm{Q}_{2k}(x)}{\mathrm{Q}_{2k}^2(x) - 4x^2 - 5} = \operatorname{artanh} \frac{1}{\mathrm{Q}_{2k-1}(x)} - \operatorname{artanh} \frac{1}{\mathrm{Q}_{2k+1}(x)}. \tag{5.6}$$

Summing for k from 2 to $n+1$ by telescoping, and then replacing k by $k+1$, we get from (5.5) the formula bellow.

THEOREM 5.3.

$$\sum_{k=1}^{n} (-1)^k \operatorname{artanh} \frac{4(x^2+1)\,\mathrm{P}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 3} = (-1)^n \operatorname{artanh} \frac{1}{\mathrm{Q}_{2n+3}(x)} - \operatorname{artanh} \frac{1}{\mathrm{Q}_3(x)}.$$

When $n \to \infty$, the above theorem gives rise to the infinite series evaluation below.

COROLLARY 5.3.

$$\sum_{k=1}^{\infty} (-1)^k \operatorname{artanh} \frac{4(x^2+1)\,\mathrm{P}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 3} = -\operatorname{artanh} \frac{1}{2x(4x^2+3)}.$$

For this corollary, we have two particular cases bellow:

$$x = \frac{1}{2}: \quad \sum_{k=1}^{\infty} (-1)^{k-1} \operatorname{artanh} \frac{5F_{2k+2}}{L_{2k+2}^2 - 4} = \frac{1}{2}\ln\frac{5}{3},$$

$$x = 1: \quad \sum_{k=1}^{\infty} (-1)^{k-1} \operatorname{artanh} \frac{8P_{2k+2}}{Q_{2k+2}^2 - 7} = \frac{1}{2} \ln \frac{15}{13}.$$

Analogously, summing for k from 2 to $n+1$ by telescoping and then taking replacement $k \to k+1$, we get from (5.6) the series bellow.

Theorem 5.4.

$$\sum_{k=1}^{n} \operatorname{artanh} \frac{2x\,\mathrm{Q}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 5} = \operatorname{artanh} \frac{1}{\mathrm{Q}_3(x)} - \operatorname{artanh} \frac{1}{\mathrm{Q}_{2n+3}(x)}.$$

When $n \to \infty$, the above theorem gives rise to the infinite series evaluation below.

Corollary 5.4.

$$\sum_{k=1}^{\infty} \operatorname{artanh} \frac{2x\,\mathrm{Q}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 5} = \operatorname{artanh} \frac{1}{2x(4x^2+3)}.$$

For particular values of x, we deduce the following two formulae:

$$x = \frac{1}{2}: \quad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{F_{2k+2}}{L_{2k+2}^2 - 6} = \frac{1}{2} \ln \frac{5}{3},$$

$$x = 1: \quad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{2Q_{2k+2}}{Q_{2k+2}^2 - 9} = \frac{1}{2} \ln \frac{15}{13}.$$

Furthermore, summing the product of (5.5) and (5.6) for k from 2 to $n+1$ and then replacing k by $k+1$, we find the formula in the following theorem.

Theorem 5.5.

$$\sum_{k=1}^{n} \operatorname{artanh} \frac{4(x^2+1)\,\mathrm{P}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 3} \operatorname{artanh} \frac{2x\,\mathrm{Q}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 5} = \operatorname{artanh}^2 \frac{1}{2x(4x^2+3)} - \operatorname{artanh}^2 \frac{1}{\mathrm{Q}_{2n+3}(x)}.$$

When $n \to \infty$, the above theorem gives rise to the infinite series evaluation below.

Corollary 5.5.

$$\sum_{k=1}^{\infty} \operatorname{artanh} \frac{4(x^2+1)\,\mathrm{P}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 3} \operatorname{artanh} \frac{2x\,\mathrm{Q}_{2k+2}(x)}{\mathrm{Q}_{2k+2}^2(x) - 4x^2 - 5} = \operatorname{artanh}^2 \frac{1}{2x(4x^2+3)}.$$

As applications, it yields the following two infinite series identities:

$$x = \frac{1}{2}: \quad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{5F_{2k+2}}{L_{2k+2}^2 - 4} \operatorname{artanh} \frac{L_{2k+2}}{L_{2k+2}^2 - 6} = \frac{1}{4} \ln^2 \frac{5}{3},$$

$$x = 1: \quad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{8P_{2k+2}}{Q_{2k+2}^2 - 7} \operatorname{artanh} \frac{2Q_{2k+2}}{Q_{2k+2}^2 - 9} = \frac{1}{4} \ln^2 \frac{15}{13}.$$

§5.2B. . Under the replacement k by $2k-1$, the two equalities in (5.3) and (5.4) become

$$\operatorname{artanh}\frac{1}{\mathrm{Q}_{2k-2}(x)}+\operatorname{artanh}\frac{1}{\mathrm{Q}_{2k}(x)}=\operatorname{artanh}\frac{4(x^2+1)\,\mathrm{P}_{2k-1}(x)}{\mathrm{Q}^2_{2k-1}(x)+4x^2+5}, \tag{5.7}$$

$$\operatorname{artanh}\frac{1}{\mathrm{Q}_{2k-2}(x)}-\operatorname{artanh}\frac{1}{\mathrm{Q}_{2k}(x)}=\operatorname{artanh}\frac{2x\,\mathrm{Q}_{2k-1}(x)}{\mathrm{Q}^2_{2k-1}(x)+4x^2+3}. \tag{5.8}$$

Firstly, summing for k from 1 to n by telescoping, we get from (5.7) the following formula.

Theorem 5.6.

$$\sum_{k=1}^{n}(-1)^{k-1}\operatorname{artanh}\frac{4(x^2+1)\,\mathrm{P}_{2k-1}(x)}{\mathrm{Q}^2_{2k-1}(x)+4x^2+5}=\operatorname{artanh}\frac{1}{\mathrm{Q}_0(x)}+(-1)^{n-1}\operatorname{artanh}\frac{1}{\mathrm{Q}_{2n}(x)}.$$

Its limiting case as $n\to\infty$ yields a remarkable series whose sum is independent of x.

Corollary 5.6 (Independent of x).

$$\sum_{k=1}^{\infty}(-1)^{k-1}\operatorname{artanh}\frac{4(x^2+1)\,\mathrm{P}_{2k-1}(x)}{\mathrm{Q}^2_{2k-1}(x)+4x^2+5}=\frac{1}{2}\ln 3.$$

The following two infinite series identities correspond, respectively, to $x=\frac{1}{2}$ and $x=1$:

$$x=\frac{1}{2}:\quad \sum_{k=1}^{\infty}(-1)^{k-1}\operatorname{artanh}\frac{5F_{2k-1}}{L^2_{2k-1}+6}=\frac{1}{2}\ln 3,$$

$$x=1:\quad \sum_{k=1}^{\infty}(-1)^{k-1}\operatorname{artanh}\frac{8P_{2k-1}}{Q^2_{2k-1}+9}=\frac{1}{2}\ln 3.$$

Secondly, summing for k from 1 to n, we have from (5.7) another formula.

Theorem 5.7.

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{2x\,\mathrm{Q}_{2k-1}(x)}{\mathrm{Q}^2_{2k-1}(x)+4x^2+3}=\operatorname{artanh}\frac{1}{\mathrm{Q}_0(x)}-\operatorname{artanh}\frac{1}{\mathrm{Q}_{2n}(x)}.$$

Its limiting case as $n\to\infty$ yields also a sum whose value is independent of x.

Corollary 5.7 (Independent of x).

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{2x\,\mathrm{Q}_{2k-1}(x)}{\mathrm{Q}^2_{2k-1}(x)+4x^2+3}=\frac{1}{2}\ln 3.$$

For special cases, we have the two infinite series identities:

$$x=\frac{1}{2}:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{L_{2k-1}}{L^2_{2k-1}+4}=\frac{1}{2}\ln 3,$$

$$x=1:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{2Q_{2k-1}}{Q_{2k-1}^2+7}=\frac{1}{2}\ln 3.$$

Multiplying (5.7) and (5.8) and then summing for k from 1 to n, we have the formula below.

THEOREM 5.8.

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{4(x^2+1)\,\mathrm{P}_{2k-1}(x)}{\mathrm{Q}_{2k-1}^2(x)+4x^2+5}\operatorname{artanh}\frac{2x\,\mathrm{Q}_{2k-1}(x)}{\mathrm{Q}_{2k-1}^2(x)+4x^2+3}$$
$$=\operatorname{artanh}^2\frac{1}{2}-\operatorname{artanh}^2\frac{1}{\mathrm{Q}_{2n}(x)}.$$

Its limiting case as $n\to\infty$ yields the following series whose value is independent of x.

COROLLARY 5.8 (Independent of x).

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{4(x^2+1)\,\mathrm{P}_{2k-1}(x)}{\mathrm{Q}_{2k-1}^2(x)+4x^2+5}\operatorname{artanh}\frac{2x\,\mathrm{Q}_{2k-1}(x)}{\mathrm{Q}_{2k-1}^2(x)+4x^2+3}=\frac{1}{4}\ln^2 3.$$

In particular, two infinite series identities are given as follows:

$$x=\frac{1}{2}:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{5F_{2k-1}}{L_{2k-1}^2+6}\operatorname{artanh}\frac{L_{2k-1}}{L_{2k-1}^2+4}=\frac{1}{4}\ln^2 3,$$
$$x=1:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{8P_{2k-1}}{Q_{2k-1}^2+9}\operatorname{artanh}\frac{2Q_{2k-1}}{Q_{2k-1}^2+7}=\frac{1}{4}\ln^2 3.$$

5.3. By combining the Cassini–like formula (cf. Koshy [**9**, §14.10])

$$\mathrm{P}_k^2(x)-\mathrm{P}_{k+\lambda}(x)\,\mathrm{P}_{k-\lambda}(x)=(-1)^{k+\lambda}\,\mathrm{P}_\lambda^2(x) \tag{5.9}$$

with (1.2) and (1.3), we can show the following two identities

(5.10)

$$\operatorname{artanh}\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+\lambda}(x)}+\operatorname{artanh}\frac{\mathrm{P}_{k-\lambda}(x)}{\mathrm{P}_k(x)}=\begin{cases}\operatorname{artanh}\dfrac{2\,\mathrm{P}_k^2(x)+(-1)^k\,\mathrm{P}_\lambda^2(x)}{\mathrm{P}_{2k}(x)\,\mathrm{P}_\lambda(x)}, & \lambda \text{ odd};\\ \operatorname{artanh}\dfrac{2\,\mathrm{P}_k^2(x)-(-1)^k\,\mathrm{P}_\lambda^2(x)}{\mathrm{Q}_\lambda(x)\,\mathrm{P}_k^2(x)}, & \lambda \text{ even};\end{cases}$$

(5.11)

$$\operatorname{artanh}\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+\lambda}(x)}-\operatorname{artanh}\frac{\mathrm{P}_{k-\lambda}(x)}{\mathrm{P}_k(x)}=\begin{cases}\operatorname{artanh}\dfrac{(-1)^k\,\mathrm{P}_\lambda(x)}{\mathrm{P}_{2k}(x)}, & \lambda \text{ even};\\ \operatorname{artanh}\dfrac{(-1)^{k+1}\,\mathrm{P}_\lambda^2(x)}{\mathrm{Q}_\lambda(x)\,\mathrm{P}_k^2(x)}, & \lambda \text{ odd}.\end{cases}$$

By means of (5.11), Melham–Shannon [**13**] obtained directly

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{(-1)^{k-1}}{2x\,\mathrm{P}_k^2(x)}=\operatorname{artanh}\frac{\mathrm{P}_n(x)}{\mathrm{P}_{n+1}(x)},\ \text{with } x\neq\frac{1}{2}.$$

Similarly, letting $\lambda = 1$ in (5.10), we have another one

$$\sum_{k=1}^{n}(-1)^k \operatorname{artanh}\frac{2\,\mathrm{P}_{k+2}^2(x)+(-1)^k}{\mathrm{P}_{2k+4}(x)} = (-1)^n \operatorname{artanh}\frac{\mathrm{P}_{n+2}(x)}{\mathrm{P}_{n+3}(x)} - \operatorname{artanh}\frac{2x}{4x^2+1}.$$

Now we are going to examine sums corresponding to the products of (5.10) and (5.11).

§5.3A. . When $\lambda = 1$, rewriting (5.10) and (5.11) as

$$\operatorname{artanh}\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+1}(x)} + \operatorname{artanh}\frac{\mathrm{P}_{k-1}(x)}{\mathrm{P}_k(x)} = \operatorname{artanh}\frac{2\,\mathrm{P}_k^2(x)+(-1)^k}{\mathrm{P}_{2k}(x)},$$
$$\operatorname{artanh}\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+1}(x)} - \operatorname{artanh}\frac{\mathrm{P}_{k-1}(x)}{\mathrm{P}_k(x)} = \operatorname{artanh}\frac{(-1)^{k+1}}{2x\,\mathrm{P}_k^2(x)},$$

and then summing their product for k from 3 to n by telescoping, we find, after having made replacement $k \to k+2$ and $n \to n+2$, the summation formula.

Theorem 5.9.

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{(-1)^{k+1}}{2x\,\mathrm{P}_{k+2}^2(x)}\operatorname{artanh}\frac{2\,\mathrm{P}_{k+2}^2(x)+(-1)^k}{\mathrm{P}_{2k+4}(x)} = \operatorname{artanh}^2\frac{\mathrm{P}_{n+2}(x)}{\mathrm{P}_{n+3}(x)} - \operatorname{artanh}^2\frac{2x}{4x^2+1}.$$

Its limiting case as $n \to \infty$ results in the infinite series identity.

Corollary 5.9.

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{(-1)^{k+1}}{2x\,\mathrm{P}_{k+2}^2(x)}\operatorname{artanh}\frac{2\,\mathrm{P}_{k+2}^2(x)+(-1)^k}{\mathrm{P}_{2k+4}(x)} = \operatorname{artanh}^2(\sqrt{x^2+1}-x) - \operatorname{artanh}^2\frac{2x}{4x^2+1}.$$

Two particular cases are recorded bellow:

$$x=\frac{1}{2}: \quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{(-1)^{k+1}}{F_{k+2}^2}\operatorname{artanh}\frac{2F_{k+2}^2+(-1)^k}{F_{2k+4}} = \frac{1}{4}\ln^2(2+\sqrt{5}) - \frac{1}{4}\ln^2 3,$$
$$x=1: \quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{(-1)^{k+1}}{2P_{k+2}^2}\operatorname{artanh}\frac{2P_{k+2}^2+(-1)^k}{P_{2k+4}} = \frac{1}{4}\ln^2(1+\sqrt{2}) - \frac{1}{4}\ln^2\frac{7}{3}.$$

§5.3B. . Analogously for $\lambda = 2$, both (5.10) and (5.11) become

$$\operatorname{artanh}\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+2}(x)} + \operatorname{artanh}\frac{\mathrm{P}_{k-2}(x)}{\mathrm{P}_k(x)} = \operatorname{artanh}\frac{\mathrm{P}_k^2(x)-2(-1)^k x^2}{(2x^2+1)\,\mathrm{P}_k^2(x)},$$
$$\operatorname{artanh}\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+2}(x)} - \operatorname{artanh}\frac{\mathrm{P}_{k-2}(x)}{\mathrm{P}_k(x)} = \operatorname{artanh}\frac{2(-1)^k x}{\mathrm{P}_{2k}(x)}.$$

Summing their product for k from 2 to $n+1$, and then taking replacement $k \to k+1$, we find by telescoping another formula.

Theorem 5.10.

$$\sum_{k=1}^{n}\operatorname{artanh}\frac{\mathrm{P}_{k+1}^2(x)+2(-1)^k x^2}{(2x^2+1)\,\mathrm{P}_{k+1}^2(x)}\operatorname{artanh}\frac{2(-1)^{k+1}x}{\mathrm{P}_{2k+2}(x)}$$
$$=\operatorname{artanh}^2\frac{\mathrm{P}_{n+1}(x)}{\mathrm{P}_{n+3}(x)}+\operatorname{artanh}^2\frac{\mathrm{P}_{n}(x)}{\mathrm{P}_{n+2}(x)}-\operatorname{artanh}^2\frac{1}{4x^2+1}.$$

The limiting case as $n\to\infty$ is given by the following corollary.

Corollary 5.10.

$$\sum_{k=1}^{\infty}\operatorname{artanh}\frac{\mathrm{P}_{k+1}^2(x)+2(-1)^k x^2}{(2x^2+1)\,\mathrm{P}_{k+1}^2(x)}\operatorname{artanh}\frac{2(-1)^{k+1}x}{\mathrm{P}_{2k+2}(x)}$$
$$=2\operatorname{artanh}^2\beta^2-\operatorname{artanh}^2\frac{1}{4x^2+1}.$$

For application, we record two identities about Fibonacci and Pell numbers:

$$x=\frac{1}{2}:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{2F_{k+1}^2+(-1)^k}{3F_{k+1}^2}\operatorname{artanh}\frac{(-1)^{k+1}}{F_{2k+2}}=\frac{1}{8}\ln^2 5-\frac{1}{4}\ln^2 3,$$
$$x=1:\quad \sum_{k=1}^{\infty}\operatorname{artanh}\frac{P_{k+1}^2+2(-1)^k}{3P_{k+1}^2}\operatorname{artanh}\frac{2(-1)^{k+1}}{P_{2k+2}}=\frac{1}{8}\ln^2 2-\frac{1}{4}\ln^2\frac{3}{2}.$$

5.4. Now we are going to work out the counterpart formulae for $\mathrm{Q}(x)$ by employing another Cassini–like formula (cf. Koshy [**9**, §14.10])

(5.12)
$$\mathrm{Q}_{k+\lambda}(x)\,\mathrm{Q}_{k-\lambda}(x)-\mathrm{Q}_k^2(x)=4(-1)^{k+\lambda}(1+x^2)\,\mathrm{P}_\lambda^2(x)=(-1)^{k+\lambda}(\mathrm{Q}_\lambda^2(x)-4(-1)^\lambda)$$

as well as two reformulated ones by (1.2) and (1.3):

(5.13)
$$\operatorname{artanh}\frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+\lambda}(x)}+\operatorname{artanh}\frac{\mathrm{Q}_{k-\lambda}(x)}{\mathrm{Q}_k(x)}=\begin{cases}\operatorname{artanh}\frac{2\,\mathrm{Q}_k^2(x)+(-1)^k(\mathrm{Q}_\lambda^2(x)-4)}{\mathrm{Q}_\lambda(x)\,\mathrm{Q}_k^2(x)}, & \lambda\text{ even};\\ \operatorname{artanh}\frac{2\,\mathrm{Q}_k^2(x)-(-1)^k(\mathrm{Q}_\lambda^2(x)+4)}{4(x^2+1)\,\mathrm{P}_\lambda(x)\,\mathrm{P}_{2k}(x)}, & \lambda\text{ odd};\end{cases}$$

(5.14)
$$\operatorname{artanh}\frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+\lambda}(x)}-\operatorname{artanh}\frac{\mathrm{Q}_{k-\lambda}(x)}{\mathrm{Q}_k(x)}=\begin{cases}\operatorname{artanh}\frac{(-1)^k(4-\mathrm{Q}_\lambda^2(x))}{4(x^2+1)\,\mathrm{P}_\lambda(x)\,\mathrm{P}_{2k}(x)}, & \lambda\text{ even};\\ \operatorname{artanh}\frac{(-1)^k(4+\mathrm{Q}_\lambda^2(x))}{\mathrm{Q}_\lambda(x)\,\mathrm{Q}_k^2(x)}, & \lambda\text{ odd};\end{cases}$$

§5.4A. . Letting $\lambda=1$ in (5.13) and (5.14), we have

$$\operatorname{artanh}\frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+1}(x)}+\operatorname{artanh}\frac{\mathrm{Q}_{k-1}(x)}{\mathrm{Q}_k(x)}=\operatorname{artanh}\frac{\mathrm{Q}_k^2(x)-2(-1)^k(x^2+1)}{2(x^2+1)\,\mathrm{P}_{2k}(x)},$$
$$\operatorname{artanh}\frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+1}(x)}-\operatorname{artanh}\frac{\mathrm{Q}_{k-1}(x)}{\mathrm{Q}_k(x)}=\operatorname{artanh}\frac{2(-1)^k(x^2+1)}{x\,\mathrm{Q}_k^2(x)}.$$

Multiplying them and then summing the resultant expression for k from 2 to $n+1$, we find, under the replacement $k\to k+1$, the formula below.

THEOREM 5.11.

$$\sum_{k=1}^{n} \operatorname{artanh} \frac{2(-1)^{k+1}(x^2+1)}{x\,\mathrm{Q}_{k+1}^2(x)} \operatorname{artanh} \frac{\mathrm{Q}_{k+1}^2(x)+2(-1)^k(x^2+1)}{2(x^2+1)\,\mathrm{P}_{2k+2}(x)} = \operatorname{artanh}^2 \frac{\mathrm{Q}_{n+1}(x)}{\mathrm{Q}_{n+2}(x)} - \operatorname{artanh}^2 \frac{x}{2x^2+1}.$$

As $n \to \infty$, the limiting case evaluates the following infinite series.

COROLLARY 5.11.

$$\sum_{k=1}^{\infty} \operatorname{artanh} \frac{2(-1)^{k+1}(x^2+1)}{x\,\mathrm{Q}_{k+1}^2(x)} \operatorname{artanh} \frac{\mathrm{Q}_{k+1}^2(x)+2(-1)^k(x^2+1)}{2(x^2+1)\,\mathrm{P}_{2k+2}(x)} = \operatorname{artanh}^2(\sqrt{x^2+1}-x) - \operatorname{artanh}^2 \frac{x}{2x^2+1}.$$

Two special cases are produced below as examples:

$$x=\frac{1}{2}: \quad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{5(-1)^{k+1}}{L_{k+1}^2} \operatorname{artanh} \frac{2L_{k+1}^2+5(-1)^k}{5F_{2k+2}} = \frac{1}{4}\ln^2(2+\sqrt{5}) - \frac{1}{4}\ln^2 2,$$

$$x=1: \quad \sum_{k=1}^{\infty} \operatorname{artanh} \frac{4(-1)^{k+1}}{Q_{k+1}^2} \operatorname{artanh} \frac{Q_{k+1}^2+4(-1)^k}{4P_{2k+2}} = \frac{1}{4}\ln^2(1+\sqrt{2}) - \frac{1}{4}\ln^2 2.$$

§5.4B. . Analogously for $\lambda = 2$, we have from (5.13) and (5.14)

$$\operatorname{artanh} \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+2}(x)} + \operatorname{artanh} \frac{\mathrm{Q}_{k-2}(x)}{\mathrm{Q}_k(x)} = \operatorname{artanh} \frac{\mathrm{Q}_k^2(x)+8(-1)^k(x^4+x^2)}{(2x^2+1)\,\mathrm{Q}_k^2(x)}$$

$$\operatorname{artanh} \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+2}(x)} - \operatorname{artanh} \frac{\mathrm{Q}_{k-2}(x)}{\mathrm{Q}_k(x)} = \operatorname{artanh} \frac{2(-1)^{k+1}x}{\mathrm{P}_{2k}(x)}.$$

Summing their product for k from 2 to $n+1$, we derive, after replacing k by $k+1$, the formula in the following theorem.

THEOREM 5.12.

$$\sum_{k=1}^{n} \operatorname{artanh} \frac{2(-1)^k x}{\mathrm{P}_{2k+2}(x)} \operatorname{artanh} \frac{\mathrm{Q}_{k+1}^2(x)-8(-1)^k(x^4+x^2)}{(2x^2+1)\,\mathrm{Q}_{k+1}^2(x)}$$
$$= \operatorname{artanh}^2 \frac{\mathrm{Q}_{n+1}(x)}{\mathrm{Q}_{n+3}(x)} + \operatorname{artanh}^2 \frac{\mathrm{Q}_n(x)}{\mathrm{Q}_{n+2}(x)} - \operatorname{artanh}^2 \frac{1}{4x^2+3} - \operatorname{artanh}^2 \frac{1}{2x^2+1}.$$

Letting $n \to \infty$ in this theorem, we get the infinite series evaluation.

COROLLARY 5.12.

$$\sum_{k=1}^{\infty} \operatorname{artanh} \frac{2(-1)^k x}{\mathrm{P}_{2k+2}(x)} \operatorname{artanh} \frac{\mathrm{Q}_{k+1}^2(x)-8(-1)^k(x^4+x^2)}{(2x^2+1)\,\mathrm{Q}_{k+1}^2(x)} = 2\operatorname{artanh}^2 \beta^2 - \operatorname{artanh}^2 \frac{1}{4x^2+3} - \operatorname{artanh}^2 \frac{1}{2x^2+1}.$$

This formula implies the two infinite series identities:

$$x=\frac{1}{2}: \quad \sum_{k=1}^{\infty} \operatorname{artanh}\frac{(-1)^k}{F_{2k+2}} \operatorname{artanh}\frac{2L_{k+1}^2-5(-1)^k}{3L_{k+1}^2} = -\frac{1}{8}\ln^2 5-\frac{1}{4}\ln^2\frac{5}{3},$$

$$x=\frac{1}{2}: \quad \sum_{k=1}^{\infty} \operatorname{artanh}\frac{2(-1)^k}{P_{2k+2}} \operatorname{artanh}\frac{Q_{k+1}^2-16(-1)^k}{3Q_{k+1}^2} = -\frac{1}{8}\ln^2 2-\frac{1}{4}\ln^2\frac{4}{3}.$$

Concluding Comments. We have presented a systematic treatment to the infinite series of hyperbolic arctangent function by making use of the telescoping method. However, there exist several important series in the literature that are not covered due to the space limitation. For instance, Ling [**10**, **11**] and Zucker [**14**, **15**] evaluated different classes of series involving hyperbolic functions, respectively, by employing Weierstrass and Jacobi elliptic functions. The interested reader are advised to consult these papers and the references therein.

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