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PRIMITIVE DIAMETER 2-CRITICAL GRAPHS

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Abstract. We study diameter 2-critical graphs (for short, D2C graphs), i.e. graphs of diameter 2 whose diameter increases after removing any edge. Our results include structural considerations, new examples and a particular relationship with minimal 2-self-centered graphs stating that these graph classes are almost identical. We pay an attention to primitive D2C graphs (PD2C graphs) which, by definition, have no two vertices with the same set of neighbours. It is known that a graph of diameter 2 and order n, which has no dominating vertex, has at least $2n-5$ edges, and the graphs that attain this bound are also known. It occurs that exactly three of them are PD2C. The next natural step is to consider PD2C graphs with $2n-4$ edges. In this context, we determine an infinite family of PD2C graphs which, for every $n \geqslant 6$, contains exactly one graph with $2n-4$ edges. We also prove that there are exactly seven Hamiltonian PD2C graphs with the required number of edges. We show that for $n \leqslant 13$, there exists a unique PD2C graph with $2n-4$ edges that does not belong to the obtained family nor is Hamiltonian. It is conjectured that this is a unique example of such a graph.

1. Introduction

For a finite simple undirected graph $G \cong G(V,E)$, we use n and m to denote its *order* (that is, the number of vertices $|V|$) and *size* (that is, the number of edges $|E|$).

We consider a particular family of graphs with diameter 2 that are minimal in the sense that removal of any edge increases the diameter. Such graphs are called *diameter* 2-*critical* (for short, *D2C* graphs). For example, the cycle on 5 vertices and every complete bipartite graph of diameter 2 are D2C. A study of these graphs dates back to 1960s, and many results can be found in [**3**, **5**, **7**–**10**, **14**, **16**] (characterizations, conjectures, constructions), [**11**–**13**, **18**] (particular cases) and references therein.

In this paper we give some structural properties of D2C graphs, construct an infinite family of such graphs and give a relationship between D2C graphs and the

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so-called minimal 2-self-centered graphs leading to the conclusion that these graph classes are almost identical. Definition of a 2-self-centered graph is given upon the forthcoming Proposition 3.2, and these graphs are studied in [**4**, **15**, **20**] and other references not listed here. According to our result, a study on D2C graphs is equivalent to a study on minimal 2-self-centered graphs.

A *duplication* of a vertex v in a graph is the procedure of adding a new vertex and joining it with all neighbours of v. Clearly, a graph obtained by duplication of any vertex of a D2C graph is also a D2C graph. Therefore, an essential part of the family of D2C graphs are those that do not contain duplicated vertices (since every other D2C graph is obtained by duplicating vertices in some of these graphs). We call these graphs *primitive diameter* 2*-critical graphs* (for short, *PD2C graphs*). An alternative name that can be found in the literature is *twin-free diameter* 2*-critical graphs*.

We know from the Erdős–Rényi theorem [**8**] that the size of a graph of diameter 2 with no dominating vertex (i.e. a vertex joined to all remaining vertices) is not less than $2n-5$. Moreover, Henning and Southey [**14**] have determined all graphs that attain the equality in this lower bound. These graphs are listed in the next section. Exactly three of them are PD2C, and as the next step we consider PD2C graphs with $2n-4$ edges. It occurs that determination of such graphs is a rather difficult problem, and our contribution consists of the following results. We determined an infinite family of such graphs, such that for every order $n \geqslant 6$ there is exactly one member of this family. We also determined all Hamiltonian PD2C graphs with $2n-4$ edges, and found one graph with the required properties that does not belong to the mentioned family and is not Hamiltonian. Our computational experiments suggest that this might be a unique PD2C graph with $2n-4$ edges that do not belong to the union of the obtained families, so we formulate this in the form of a conjecture.

Our results are closely related to the following two conjectures. The first one is a longstanding problem considered by many mathematicians, known as the Murty–Simon conjecture (cf. [**5**,**9**]). It states that every D2C graph of order n has at most $\lfloor \frac{n^2}{4} \rfloor$ edges, with equality if and only if G is a complete bipartite graph whose colour classes differ in size by at most one. There are many results about this conjecture; in particular, it holds for graphs having sufficiently large order and for all graphs with a dominating edge, where an edge is *dominating* if joins a pair of vertices without common non-neighbour [**7**].

For the second conjecture, following [**2**], we denote by $\mathcal{C}_5^+$ the family of graphs obtained by replacing three vertices x_1, x_2, x_3 of the cycle C_5 by three independent sets X_1, X_2, X_3 of vertices, under the following conditions: (a) x_1, x_2 and x_3 are consecutive on the cycle and (b) $|X_2| \in \{\lfloor \frac{n-2}{3} \rfloor, \lceil \frac{n-2}{3} \rceil\}$, where n is the order of the obtained graph. Dailly et al. [**7**] conjectured that a non-bipartite D2C graph G of order n is either H_5 of [**7**, Figure 2] or has at most $\lfloor \frac{(n-1)^2}{4} \rfloor + 1$ edges, with the equality if and only if G belongs to $\mathcal{C}_5^+$ or it is one of the 13 graphs listed in the same reference. Obviously, this conjecture strengthens the previous one by excluding the complete bipartite graphs and H_5. It is interesting that Radosavljević [**17**] has

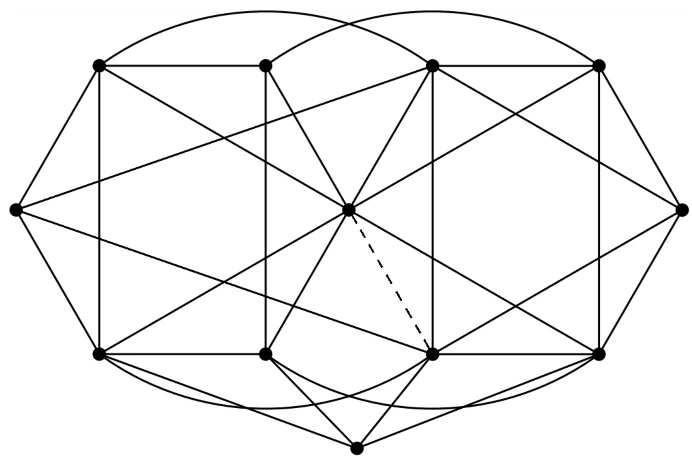

FIGURE 1. A new exception to the conjecture of [**7**]. In this and the forthcoming figures, the purpose of a dashed line is to emphasize a particular edge.

found another graph that should be excluded. We illustrate it in Fig. 1; its order and size are 12 and 32, which leads to $32 > \lfloor \frac{(n-1)^2}{4} \rfloor + 1 = 31$. This graph contains a dominating edge, highlighted in the figure, so in the context of the Murty–Simon conjecture it belongs to a resolved class.

Additional terminology, notation and necessary results are given in Section 2. For basic notions and notation on graphs not given in this paper, we refer the reader to any of [**6**, **19**]. D2C graphs are considered in Section 3. PD2C graphs with $2n-4$ edges are considered in Section 4.

2. Preliminaries

We write $\mathrm{dist}(u, v)$ to denote the distance between the vertices $u, v \in V(G)$. The *eccentricity* $\mathrm{ecc}(u)$ of u is the maximum distance between u and all other vertices. The *diameter* $\mathrm{diam}(G)$ of G is the maximum eccentricity of its vertices.

The degree and the set of neighbours of a vertex u are denoted by $\mathrm{d}(u)$ and $N(u)$, respectively. The *closed neighbourhood* $\{u\} \cup N(u)$ is denoted by $N[u]$. The graph obtained by removal of a vertex v or an edge uv is denoted by $G - u$ and $G - uv$. A cycle and a star on n vertices are denoted by C_n and S_n, respectively. The *double star* S_{n_1,n_2} is obtained by inserting an edge between the centers of stars S_{n_1} and S_{n_2}. A *generalized star* is a tree with exactly one vertex of degree greater than 2. For the graphs G and H, the *join* $G \nabla H$ is obtained by inserting an edge between every vertex of G and every vertex of H. In particular, if G consists of a single vertex, the join reduces to the *cone* over H; if $V(G) = \{v\}$, we also use the term *v-cone*. Accordingly, a star can be seen as a cone over a graph without edges.

Erdős and Rényi [**8**] proved the following classical result on the minimum size of a graph of diameter 2 with no dominating vertex.

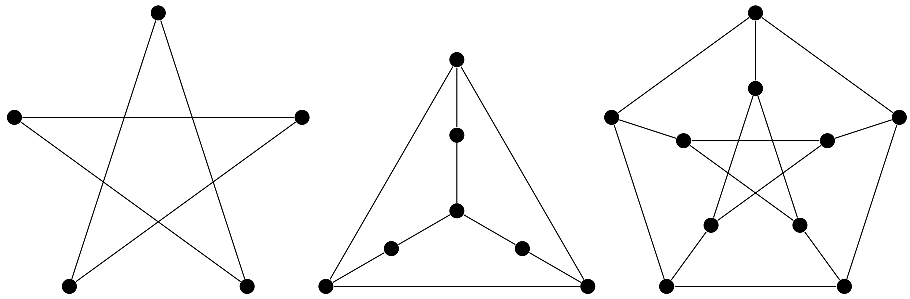

FIGURE 2. PD2C graphs from the $\mathcal{G}$ family with $2n-5$ edges.

THEOREM 2.1 (Erdős-Rényi theorem). *If G is a graph of diameter 2, order n and size m with no dominating vertex, then $m \geqslant 2n-5$.*

The *degree-2 duplication* of a vertex is the duplication of a vertex of degree 2. Let $\mathcal{G}$ be the family of graphs that (i) contains three graphs of Fig. 2, and (ii) is closed under degree-2 vertex duplication. Henning and Southey [**14**] showed that the equality in the previous theorem holds exactly for graphs of $\mathcal{G}$.

3. Diameter 2-critical graphs

In this section we give some theoretical results on D2C graphs, a relationship between D2C graphs and the so-called minimal self-centered graphs, and some constructions of D2C graphs. In the introductory section we have said that D2C graphs have received a great deal of attention in the period of almost 60 years. Therefore, there is a possibility that the first two results are already met in the literature. For the sake of completeness, we give short proofs. The corollary is used in the next section.

PROPOSITION 3.1. *The join $G\nabla H$ is D2C if and only if G and H are empty graphs and at least one of them has more than one vertex.*

PROOF. If G, H are empty graphs and G has at least 2 vertices, then we obviously have $\mathrm{diam}(G\nabla H) = 2$. Moreover, by removing an edge $uv \in E(G\nabla H)$, we get $\mathrm{dist}_{G\nabla H-uv}(u, v) > 2$, which means that $G\nabla H$ is D2C.

Suppose now that $G\nabla H$ is D2C. Clearly, $\mathrm{diam}(G\nabla H) = 2$ yields that at least one of G, H has more than one vertex. By way of contradiction, assume that G contains an edge, say uv. We have

$$\mathrm{dist}_{G\nabla H-uv}(a,b)\begin{cases} =1 & \text{if exactly one of } a, b \text{ belongs to } V(G),\\ \leqslant 2 & \text{otherwise,}\end{cases}$$

which means that $G\nabla H$ is not D2C, a contradiction. □

COROLLARY 3.1. *The following statements hold true.*

(i) A cone is D2C if and only if it is a star with at least 3 vertices.
(ii) A tree is D2C if and only if it is a star with at least 3 vertices.

PROOF. Item (i) is a direct consequence of Proposition 3.1. For (ii), it is well-known that the diameter of a tree is 2 if and only if it is a star with at least 3 vertices. In addition, such a star is D2C by Proposition 3.1. □

Observe that the Erdős-Rényi theorem excludes graphs with a dominating vertex, i.e. cones. The previous corollary gives all cones that are D2C.

We say that a graph G is *self-centered* if all its vertices have the same eccentricity. Equivalently, G is self-centered if its diameter is equal to its radius $\mathrm{r}(G)$, where the *radius* is defined by $\mathrm{r}(G) = \min\{\mathrm{ecc}(i) : i \in V(G)\}$. A graph is *minimal d-self-centered* if it is self-centered with diameter d and every its edge-deleted subgraph is not self-centered. In other words, by removing any edge the graph loses the property of being self-centered.

PROPOSITION 3.2. *Every minimal* 2*-self-centered graph is D2C. Conversely, every D2C graph distinct from a star is minimal* 2*-self-centered.*

PROOF. Assume that G is minimal 2-self-centered. Then we have $\mathrm{diam}(G) = 2$ by definition. Moreover, the minimality of G leads to the conclusion that removal of any edge results in a graph whose diameter is greater than 2. In other words G is D2C.

Assume now that G is D2C. If G is a star, then $\mathrm{r}(G) = 1$ which means that G is not minimal 2-self-centered. If G is not a star then by Corollary 3.1, G is not a cone, which means that $\mathrm{r}(G) \neq 1$. Since $\mathrm{r}(G) \leqslant \mathrm{diam}(G) = 2$, we necessarily have $\mathrm{r}(G) = \mathrm{diam}(G)$, and so G is self-centered with diameter 2. By assuming that G is not minimal 2-self-centered, we get that there exists an edge e such that $G - e$ is self-centered, necessarily with diameter 2. But $\mathrm{diam}(G) = \mathrm{diam}(G - e)$ contradicts our initial assumption that G is D2C. Hence, G is minimal 2-self-centered, and we are done. □

We proceed with a particular construction. For a graph G with vertex set $V(G) = \{u_1, u_2, \dots, u_n\}$ and $s \in \mathbb{N}$, we consider the graph $G^{\textcircled{s}}$ with vertex set $V(G^{\textcircled{s}}) = \{(u_p, v) : 1 \leqslant p \leqslant n, 1 \leqslant v \leqslant s\}$, such that two vertices (a, b) and (c, d) are adjacent if and only if a is adjacent to c in G. The graph $G^{\textcircled{s}}$ can be seen as the strong product (defined in [**6**, p. 66]) of G and the complete graph with s vertices from which the edges between the vertices (a, b) and (a, c) are removed for all $b \neq c$. Evidently, $G \cong G^{\textcircled{1}}$. An example in which G is a 4-vertex cycle is illustrated in Fig. 3.

PROPOSITION 3.3. *If G is a triangle-free D2C graph, so is $G^{\textcircled{s}}$, for every $s \in \mathbb{N}$.*

PROOF. We first prove that $G^{\textcircled{s}}$ is triangle-free. Assume, by way of contradiction, that it contains a triangle with vertices (a, b), (c, d) and (e, f). By definition, this means that the vertices a, c, e are adjacent in G, which is a contradiction since G is triangle-free.

We now prove that diameter of $G^{\textcircled{s}}$ is 2. If $\mathrm{diam}(G^{\textcircled{s}}) > 2$, then there are non-adjacent vertices (a, b) and (c, d) such that they have no common neighbours. In other words, every vertex $(e, f) \notin \{(a, b), (c, d)\}$ is adjacent to at most one of

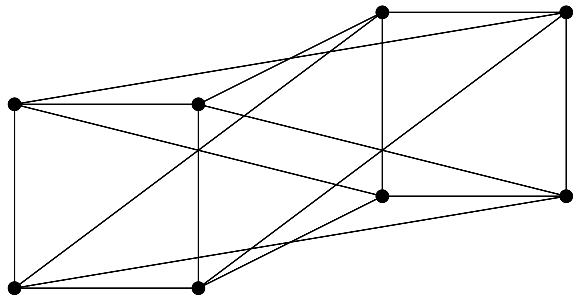

FIGURE 3. The graph $C_4^{②}$.

these vertices. This means that the vertex e is adjacent to at most one of a, c in G, but this leads to $\mathrm{dist}_G(a, c) > 2$, a contradiction.

We finally prove that $G^{ⓢ}$ is D2C. Assume contrary to the statement that by removing an edge located between a pair of vertices, say again (a, b) and (c, d), we obtain a graph with diameter 2. This, in particular, means that

$$\mathrm{dist}_{G^{ⓢ}-\{(a,b),(c,d)\}}((a, b), (c, d)) = 2.$$

In other words, these two vertices have a common neighbour in $G^{ⓢ} - (a, b)(c, d)$ and consequently in $G^{ⓢ}$, but the latter is impossible since, together with this neighbour, they form a triangle in $G^{ⓢ}$ (we have already shown that $G^{ⓢ}$ is triangle-free). $\square$

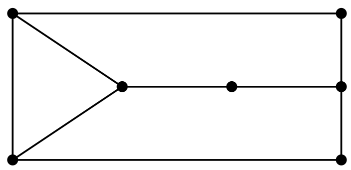

FIGURE 4. The graph for Remark 3.1.

REMARK 3.1. It is worth mentioning that the assumption that G is triangle-free is essential in the previous proposition. For example, it is not difficult to verify that if G is the D2C graph illustrated in Fig. 4 and $s \geqslant 2$, then removing the edge between the vertices (a, b) and (c, b) where a, c are the vertices of the triangle in G, results in the graph with diameter 2 (meaning that $G^{ⓢ}$ is not D2C).

Proposition 3.3 gives constructions of infinite families of D2C graphs. Namely, it is sufficient to select a single triangle-free D2C graph G, and then different choices for $s \in \mathbb{N}$ give different D2C graphs $G^{ⓢ}$. All of them are triangle-free, as well. This construction is essentially equivalent to the following one.

Corollary 3.2. *Let G be a triangle-free D2C graph. For $k \geqslant 0$, the graph obtained by the following iterative procedure*

$$\begin{cases} G_0 \cong G, \\ G_{k+1} \cong G_k^{ⓢ}, \ \text{for } s \geqslant 2, \end{cases}$$

is triangle free and D2C.

Proof. This result is a direct consequence of Proposition 3.3 stating that if G_k is triangle-free and D2C, so is $G_{k+1} \cong G_k^{ⓢ}$. □

4. PD2C graphs of size $2n-4$

An obvious consequence of the Erdős-Rényi theorem is that there are only three PD2C graphs of order $n \leqslant 13$ and size $2n-5$, see Fig. 2; these are the only primitive graphs in $\mathcal{G}$. The next natural step is to consider PD2C graphs of the second smallest size, i.e. of size $2n-4$. In the previous work [**17**], the first author of this paper has obtained the list of all PD2C graphs of order $n \leqslant 13$. Starting from this list, we obtained the sub-list of all PD2C graphs of size $m \leqslant 2n-4$. In this section we give some theoretical results concerning these graphs, construct an infinite family, and determine all Hamiltonian PD2C graphs.

We start by looking at PD2C graphs of order $n \leqslant 13$ and size $2n-4$ divided into three sets, see Figs. 5–7.

Let the graph Z_n, for $n \geqslant 6$, be defined as follows. Let x be the remainder when dividing n by 2, and set $y = (n-2-x)/2$. Let T be a generalized star with center c, x paths of length 1 attached at c and y paths of length 2 attached at c. The graph Z_n is obtained from T by adding a new vertex v and joining v to all vertices in $V(T) \smallsetminus \{c\}$. Let $\mathcal{Z} = \{Z_n : n \geqslant 6\}$. Fig. 5 illustrates Z_n, for $6 \leqslant n \leqslant 13$, as the first set of PD2C graphs of order $6 \leqslant n \leqslant 13$ and size $2n-4$.

In [**7**], D2C graphs of order n with maximum degree $n-2$ are characterized. It turns out that they are exactly the graphs of the family $\mathcal{Z}$, with a dominating edge and $2n-4$ edges.

Let $\mathcal{T} \supset \mathcal{Z}$ be the family of graphs having a vertex whose removal results in a tree. In the next subsection we determine all PD2C graphs in $\mathcal{T}$ of size $2n-4$. Fig. 6 illustrates the seven Hamiltonian PD2C graphs of size $2n-4$. In Subsection 4.2, we prove that there is no other Hamiltonian PD2C graphs of given size. Finally, Fig. 7 illustrates a unique PD2C graph of order $n \leqslant 13$ and size $2n-4$ which is not Hamiltonian and which does not belong to $\mathcal{Z}$.

4.1. The family $\mathcal{Z}$. Here we prove that a graph of $\mathcal{T}$ is PD2C of size $2n-4$ if and only if it belongs to $\mathcal{Z} \cup \{C_5\}$. Since every graph of $\mathcal{Z} \cup \{C_5\}$ is PD2C with the required number of edges, one implication follows immediately. For the opposite one, we need the following lemma concerning a general setting.

Lemma 4.1. *Let $G \in \mathcal{T}$ be a D2C graph and let $v \in V(G)$ be a vertex such that $T \cong G - v$ is a tree. Then* $\operatorname{diam}(T) \leqslant 4$.

Proof. Assume that $\operatorname{diam}(T) > 4$. There are two cases, depending on parity of $\operatorname{diam}(T)$. We show first that in both cases G is a v-cone over T.

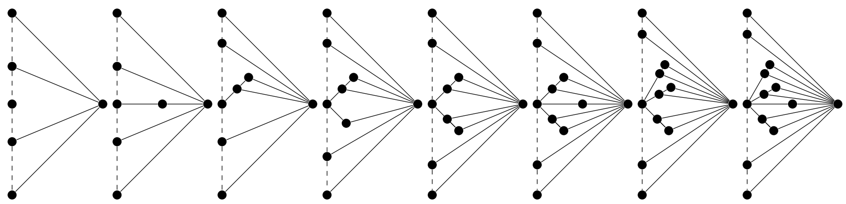

FIGURE 5. Graphs Z_n of order n ($6 \leqslant n \leqslant 13$) and size $2n-4$.

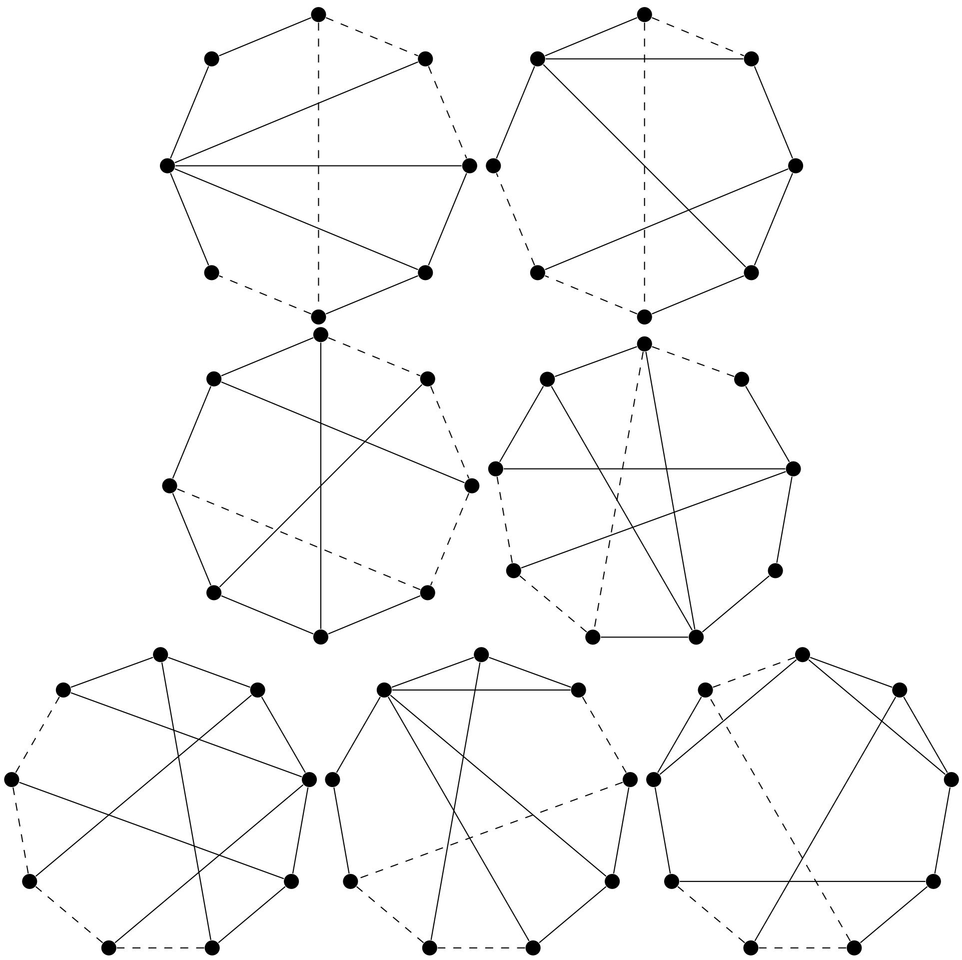

FIGURE 6. Hamiltonian PD2C graphs of order $n \leqslant 13$ and size $2n-4$.

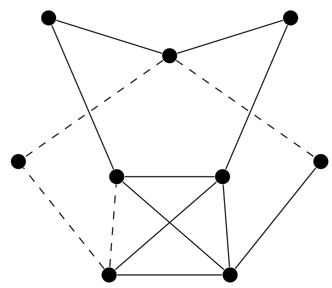

FIGURE 7. A unique PD2C graph of order $n \leqslant 13$ and size $2n-4$, which is not Hamiltonian and not in $\mathcal{Z}$.

$\mathrm{diam}(T) = 2k+1$, $k \geqslant 2$: Then T has exactly two central vertices. Denote these vertices by r_1 and r_2. Let R_1, R_2 be the trees that are the connected components of $T - r_1r_2$, and let $r_i \in V(R_i)$, $i = 1, 2$.

- Let $u_i \in V(R_i)$, $i = 1, 2$, denote any two vertices, $u_1 \neq r_1$, $u_2 \neq r_2$; then $\mathrm{dist}_T(u_1, u_2) \geqslant 3$. From $\mathrm{dist}_G(u_1, u_2) \leqslant 2$ it follows $vu_1,\ vu_2 \in E(G)$.
- There is at least one vertex $u_2 \in R_2$, such that $\mathrm{dist}_T(r_1, u_2) \geqslant 3$. From $\mathrm{dist}_G(r_1, u_2) \leqslant 2$ it follows $vr_1 \in E(G)$. Similarly, $vr_2 \in E(G)$.

$\mathrm{diam}(T) = 2k$, $k \geqslant 3$: Denote by r the center of T. Let $N(r) = \{u_1, u_2, \ldots, u_k\}$, $k \geqslant 2$, and let R_i be a sub-tree of T with root r, such that $u_ir \in R_i$, $1 \leqslant i \leqslant k$. Since r is the center of T, at least two of these sub-trees have depth $\geqslant 3$. Without the loss of generality, suppose that R_1 and R_2 are such sub-trees. Let $v_i \in R_i$, $i \in \{1, 2\}$, be two vertices such that $\mathrm{dist}_T(r, v_i) \geqslant 3$.

- From $\mathrm{dist}_G(r, u_1) \leqslant 2$ it follows $vr \in E(G)$.
- Let $w_1 \in T - R_1$ be any vertex not in R_1. From $\mathrm{dist}_T(w_1, u_1) \geqslant 3$ and $\mathrm{dist}_G(w_1, u_1) \leqslant 2$ it follows $vw_1 \in E(G)$.
- Symmetrically, let $w_2 \in T - R_2$ be any vertex not in R_2. From $\mathrm{dist}_T(w_2, u_2) \geqslant 3$ and $\mathrm{dist}_G(w_2, u_2) \leqslant 2$ it follows $vw_2 \in E(G)$.

Therefore, in both cases G is a v-cone over T. However, such a cone is not a D2C graph, by Corollary 3.1. Thus, $\mathrm{diam}(T) \leqslant 4$. □

We formulate the announced result.

THEOREM 4.1. *Let $G \in \mathcal{T}$ be a PD2C graph and let v be a vertex of G such that $T \cong G - v$ is a tree. Then $G \in \{C_5\} \cup \mathcal{Z}$.*

PROOF. From Lemma 4.1, we have $\mathrm{diam}(T) \leqslant 4$. The proof is performed considering the following cases.

$\mathrm{diam}(T) = 0$: Then $T \cong K_1$ and $G \cong P_1$, which is a diameter 1 graph.

$\mathrm{diam}(T) = 1$: Then $T \cong K_2$ and $G \cong C_3$ or $G \cong P_3$. None of these is a PD2C graph, since $\mathrm{diam}(C_3) = 1$ and P_3 is not primitive.

$\mathrm{diam}(T) = 2$: Then $T \cong S_n$, a star with $n \geqslant 3$ vertices. Let $V(T) = \{c, u_1, u_2, \ldots, u_{n-1}\}$, where c is the center of T.

- If $n = 3$, i.e. $T \cong P_3$, then by adding a vertex v, the five different connected graphs can be obtained. None of them is a PD2C graph, which is confirmed by hand.
- If $n > 3$, then due to $\operatorname{diam}(G) = 2$, v must be either
  – adjacent to c, or
  – adjacent to $u_1, u_2, \ldots, u_{n-1}$, or
  – adjacent to all vertices in $V(T)$.

  In each of these cases G is not primitive, because e.g. u_1 and u_2 have the same set of neighbours.

$\operatorname{diam}(T) = 3$**:** In this case T consists of two stars whose centers, say a and b, are joined by an edge, i.e. $T \cong S_{i,j}$, $i, j \geqslant 2$. Assume that $\mathrm{d}(a) = i$, $\mathrm{d}(b) = j$. Because $\operatorname{diam}(G) = 2$, v must be adjacent to all vertices in $N(a) \cup N(b) \smallsetminus \{a, b\}$. Since we already have $\operatorname{diam}(G) = 2$, by adding the edge va and/or vb, the graph becomes not critical. Hence, a PD2C graph G can be obtained by adding v to T in exactly one way.

If $i \geqslant 3$, then every two vertices in $N(a) \smallsetminus \{b\}$ have the same set of neighbours, and G is not primitive; therefore, $i = 2$. Similarly, it must be $j = 2$. Hence, if $\operatorname{diam}(T) = 3$, then the only PD2C graph obtainable from T by adding vertex v and some edges from v is C_5.

$\operatorname{diam}(T) = 4$**:** Let c be the center of T and let $N(c) = \{v_1, v_2, \ldots, v_k\}$. Then each vertex v_i is the root of a tree of depth $\leqslant 1$, $1 \leqslant i \leqslant k$. Since $\operatorname{diam}(T) = 4$, the number of trees of depth 1 is at least two.

From $\operatorname{diam}(G) = 2$, it follows (similarly as in the proof of the previous lemma) that v must be adjacent to all vertices in $V(T) \smallsetminus \{c\}$. Since we already have $\operatorname{diam}(G) = 2$, after the addition of vc the graph becomes not critical. Thus, by adding a vertex v to T, T can be extended into G in a unique way.

If the tree with the root v_i has two children w' and w'', then $N(w') = N(w'')$ holds and so G is not primitive. Therefore, T is a generalized star of diameter 4.

Let T have y paths $cu'_i, u'_iu''_i$ from c of length 2, $1 \leqslant i \leqslant y$, and x paths cw_i from c of length 1, $1 \leqslant i \leqslant x$. Since $\operatorname{diam}(T) = 4$, we have $y \geqslant 2$. If $x \geqslant 2$, then $N_G(w_1) = N_G(w_2)$, and G is not primitive; hence, $x \leqslant 1$. The graph G therefore belongs to $\mathcal{Z}$ and $G \cong Z_n$, where $n = 2y + x + 2 \geqslant 6$.

Since $\operatorname{diam}(G) = 2$, and the diameter of each graph obtained from G by removing any edge (vw_i, vu'_i, vu''_i, cw_i, cu'_i, $u'_iu''_i$) is greater than 2, G is a D2C graph. In addition, the graph G is primitive since:

- there are no two neighbours of c with the same set of neighbours, and
- there are no two neighbours of v with the same set of neighbours.

Therefore, the only PD2C graphs that can be obtained for $\operatorname{diam}(T) \leqslant 4$ are the graphs Z_n, $n \geqslant 6$. □

4.2. Hamiltonian PD2C graphs of size $2n - 4$. Fig. 6 illustrates all the graphs in question having at most 13 vertices; in fact, all of them have at most 9 vertices. To show that they comprise the set of all graphs described in the last title it is sufficient to prove the following result.

THEOREM 4.2. *If the size of a Hamiltonian graph G of order n is equal to $2n - 4$, then $n \leqslant 11$.*

PROOF. Consider a fixed Hamiltonian cycle of G. If $v \in V$, then let $n(v) = \mathrm{d}(v) - 2$ denote the number of diagonals from v in the Hamiltonian cycle. Let v' and v'' be the neighbours of $v \in V$ in the same cycle. The number of vertices at distance at most 2 from v (including v) satisfies

$$|N_2[v]| \leqslant 3n(v) + n(v') + n(v'') + 5.$$

Since $\mathrm{diam}(G) = 2$, the equality $|N_2[v]| = n$ holds. Summing $3n(v) + n(v') + n(v'') + 5 \geqslant n$ over all $v \in V$, we obtain

$$5 \sum_{v \in V} n(v) \geqslant n(n-5).$$

The total number of diagonals relative to the Hamiltonian cycle is $|E| - n = n - 4$, hence $\sum_{v \in V} n(v) = 2(n-4)$. Therefore, n must satisfy the inequality $10(n-4) \geqslant n(n-5)$, i.e. $n^2 - 15n + 40 \leqslant 0$. Since the larger root of the left hand side is $(15 + \sqrt{65})/2 \approx 11.53$, if the inequality is satisfied, then $n \leqslant 11$. □

Consequently, graphs of Fig. 6 are the only Hamiltonian PD2C graphs of order n and size $2n - 4$. On the basis of this result and Theorem 4.1, we formulate the following conjecture.

CONJECTURE 4.1. *All PD2C graphs of order $n \geqslant 10$ and size $2n - 4$ belong to $\mathcal{Z}$.*

By [**17**], the conjecture holds true for $n \leqslant 13$. By employing a sequence of computational experiments on graphs with more than 13 vertices, we did not find any counterexample. It is worth mentioning that this conjecture is closely related to [**7**, Theorem 25], with the condition of having $2n - 4$ edges replaced by non-bipartiteness and maximum vertex degree $n - 2$.

From [**12**, Theorem 3] it follows that, apart from the cycle C_5, every PD2C graph contains P_5 as an induced subgraph. This fact is illustrated by highlighting P_5 in PD2C graphs shown in Figs. 5–7. It also makes a step toward proving Conjecture 4.1.

A spanning tree of a graph with no vertex of degree 2 is called a *homeomorphically irreducible spanning tree* (HIST). Let A_k^s be a generalized star with center c and k paths of length 2 attached at c. Recently, Ando [**1**] (see also [**18**]) has characterized primitive (twin-free) graphs with diameter 2 that contain a HIST in the following way. Let G be a primitive graph of order $n \geqslant 10$ with diameter 2. Then G has a HIST if and only if G is not isomorphic to A_k^s for any $k \geqslant 1$. From this characterization it follows that all PD2C graphs of size $2n - 4$ have a HIST, which is another step in proving Conjecture 4.1.

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