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A CLASS OF DEFINITE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. Recently Masjed-Jamei and Koepf obtained generalizations of various classical summation theorems for the ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_6F_5$ generalized hypergeometric series. We evaluate a new class of integrals involving generalized hypergeometric function by employing the results given by Masjed-Jamei and Koepf and MacRobert integral, and we give several special cases.

1. Introduction

In mathematics, the Gaussian or ordinary hypergeometric function ${}_2F_1$ is a special function represented by the hypergeometric series, that includes many other special functions as specific or limiting cases.

The Gauss hypergeometric function is defined by [**1**, **3**, **11**, **12**, **14**]

$$ {}_2F_1\left[\begin{matrix}\lambda, & \mu \\ & \rho\end{matrix};\zeta\right] = \sum_{k=0}^{\infty}\frac{(\lambda)_k(\mu)_k}{(\rho)_k}\frac{\zeta^k}{k!} \qquad (|\zeta|<1,\ \rho\neq 0,-1,-2,\dots) $$

and confluent hypergeometric function is defined by [**1**, **3**, **11**, **12**, **14**]

$$ {}_1F_1\left[\begin{matrix}\lambda \\ \rho\end{matrix};\zeta\right] = \sum_{k=0}^{\infty}\frac{(\lambda)_k}{(\rho)_k}\frac{\zeta^k}{k!} $$

which converges everywhere. Both the above functions are the special cases of the generalized hypergeometric function with r numerator and s denominator parameters defined by [**1**, **3**, **14**–**17**]

$$ {}_rF_s\left(\begin{matrix}\lambda_1, & \cdots, & \lambda_r \\ \mu_1, & \cdots, & \mu_s\end{matrix};\zeta\right) = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{r}(\lambda_i)_k}{\prod_{i=1}^{s}(\mu_i)_k}\frac{\zeta^k}{k!} \tag{1.1} $$

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where $(\lambda)_k$ denotes the well known Pochhammer symbol [**5**] for any complex number λ defined as

$$(1.2)\qquad (\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0, \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}, \lambda \in \mathbb{C}), \end{cases}$$

where $\Gamma(\zeta)$ is the well known gamma function defined by $\Gamma(\zeta) = \int_0^\infty e^{-x} x^{\zeta-1} dx$ provided $\mathrm{Re}(\zeta) > 0$. Further by the ratio test [**1**, **2**] it can be easily verified that series (1.1) is convergent for all $r \leqslant s$. Also it converges in $|\zeta| < 1$ for $r = s+1$, converges everywhere for $r < s+1$ and converges nowhere ($\zeta \neq 0$) for $r > s+1$.

Further, if $r = s+1$, it converges absolutely for $|z| = 1$ provided

$$\delta = \mathrm{Re}\left(\sum_{j=1}^{s} \mu_j - \sum_{j=1}^{r} \lambda_j\right) > 0$$

holds and is conditionally convergent for $|\zeta| = 1$ and $\zeta \neq 1$ if $-1 < \delta \leqslant 0$ and diverges for $|\zeta| = 1$ and $\zeta \neq 1$ if $\delta \leqslant -1$. For more details, we refer [**14**].

It is interesting to mention here that, whenever a generalized hypergeometric function reduces to the gamma function, the results are very important from the applications point of view. Moreover here we shall mention the following summation theorems for the series ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_7F_6$ so that the paper may be self contained.

1) Gauss Theorem: For $\mathrm{Re}(\rho - \lambda - \mu) > 0$

$$(1.3)\qquad {}_2F_1\left[\begin{matrix}\lambda, \quad \mu \\ \rho\end{matrix}; 1\right] = \frac{\Gamma(\rho)\Gamma(\rho-\lambda-\mu)}{\Gamma(\rho-\lambda)\Gamma(\rho-\mu)}$$

2) Kummer's Theorem:

$$(1.4)\qquad {}_2F_1\left[\begin{matrix}\lambda, \quad \mu \\ 1+\lambda-\mu\end{matrix}; -1\right] = \frac{\Gamma(1+\lambda-\mu)\Gamma(1+\frac{1}{2}\lambda)}{\Gamma(1-\mu+\frac{1}{2}\lambda)\Gamma(1+\lambda)}$$

3) Second Gauss Theorem:

$$(1.5)\qquad {}_2F_1\left[\begin{matrix}\lambda, \quad \mu \\ \frac{1}{2}(\lambda+\mu+1)\end{matrix}; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(\lambda+\mu+1))}{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(\frac{1}{2}(\mu+1))}$$

4) Bailey's Theorem:

$$(1.6)\qquad {}_2F_1\left[\begin{matrix}\lambda, \quad 1-\lambda \\ \mu\end{matrix}; \tfrac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}\mu)\Gamma(\frac{1}{2}(\mu+1))}{\Gamma(\frac{1}{2}(\lambda+\mu))\Gamma(\frac{1}{2}(\mu-\lambda+1))}$$

5) Dixon's Theorem: For $\mathrm{Re}(\lambda - 2\mu - 2\rho) > -2$

$$(1.7)\qquad {}_3F_2\left[\begin{matrix}\lambda, \quad \mu, \quad \rho \\ 1+\lambda-\mu, \quad 1+\lambda-\rho\end{matrix}; 1\right] = \frac{\Gamma(1+\frac{1}{2}\lambda)\Gamma(1+\lambda-\mu)\Gamma(1+\lambda-\rho)\Gamma(1-\mu-\rho+\frac{1}{2}a)}{\Gamma(1+\lambda)\Gamma(1-\mu+\frac{1}{2}\lambda)\Gamma(1-\rho+\frac{1}{2}\lambda)\Gamma(1+\lambda-\mu-\rho)}$$

6) Watson's Theorem: For $\text{Re}(2\rho-\lambda-\mu)>1$

$$(1.8)\quad {}_3F_2\left[\begin{matrix}\lambda, & \mu, & \rho\\ \frac{1}{2}(\lambda+\mu+1), & 2\rho\end{matrix};1\right] = \frac{\sqrt{\pi}\,\Gamma(\rho+\frac{1}{2})\Gamma(\frac{1}{2}(\lambda+\mu+1))\Gamma(\rho-\frac{1}{2}(\lambda+\mu-1))}{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(\frac{1}{2}(\mu+1))\Gamma(\rho-\frac{1}{2}(\lambda-1))\Gamma(\rho-\frac{1}{2}(\mu-1))}$$

7) Whipple's Theorem: For $\text{Re}(\mu)>0$

$$(1.9)\quad {}_3F_2\left[\begin{matrix}\lambda, & 1-\lambda, & \mu\\ \rho, & 2\mu-\rho+1\end{matrix};1\right] = \frac{\pi\,2^{1-2\mu}\,\Gamma(\rho)\Gamma(2\mu-\rho+1)}{\Gamma(\frac{1}{2}(\lambda+\rho))\Gamma(\mu+\frac{1}{2}(\lambda-\rho+1))\Gamma(\frac{1}{2}(1-\lambda+\rho))\Gamma(\mu+1-\frac{1}{2}(\lambda+\rho))}$$

8) Saalschutz's Theorem:

$$(1.10)\quad {}_3F_2\left[\begin{matrix}\lambda, & \mu, & -k\\ \rho, & 1+\lambda+\mu-\rho-k\end{matrix};1\right] = \frac{(\rho-\lambda)_k(\rho-\mu)_k}{(\rho)_k(\rho-\lambda-\mu)_k}$$

9) Second Whipple's Theorem:

$$(1.11)\quad {}_4F_3\left[\begin{matrix}\lambda, & 1+\frac{1}{2}\lambda, & \mu, & \rho\\ \frac{1}{2}\lambda, & \lambda-\mu+1, & \lambda-\rho+1\end{matrix};-1\right] = \frac{\Gamma(\lambda-\mu+1)\Gamma(\lambda-\rho+1)}{\Gamma(\lambda+1)\Gamma(\lambda-\mu-\rho+1)}$$

10) Dougall's Theorem: For $\text{Re}(\lambda-\rho-\alpha-\beta)>-1$

$$(1.12)\quad {}_5F_4\left[\begin{matrix}\lambda, & 1+\frac{1}{2}\lambda, & \rho, & \alpha, & \beta\\ \frac{1}{2}\lambda, & \lambda-\rho+1, & \lambda-\alpha+1, & \lambda-\beta+1\end{matrix};1\right] = \frac{\Gamma(\lambda-\rho+1)\Gamma(\lambda-\alpha+1)\Gamma(\lambda-\beta+1)\Gamma(\lambda-\rho-\alpha-\beta+1)}{\Gamma(\lambda+1)\Gamma(\lambda-\alpha-\beta+1)\Gamma(\lambda-\rho-\beta+1)\Gamma(\lambda-\rho-\alpha+1)}$$

11) Second Dougall's Theorem:

$$(1.13)\quad {}_7F_6\left[\begin{matrix}\lambda, & 1+\frac{1}{2}\lambda, & \mu, & \rho, & \alpha, & 1+2\lambda-\mu-\rho-\alpha+k, & -k\\ \frac{1}{2}\lambda, & \lambda-\mu+1, & \lambda-\rho+1, & \lambda-\alpha+1, & \mu+\rho+\alpha-\lambda-k, & \lambda+1+k\end{matrix};1\right] = \frac{(\lambda+1)_k(\lambda-\mu-\rho+1)_k(\lambda-\mu-\beta+1)_k(\lambda-\rho-\beta+1)_k}{(\lambda+1-\mu)_k(\lambda+1-\rho)_k(\lambda+1-\alpha)_k(\lambda+1-\mu-\rho-\alpha)_k}$$

For finite sums of hypergeometric series, we will use the following symbol

$$\overset{(\ell)}{{}_rF_s}\left[\begin{matrix}\lambda_1, & \cdots, & \lambda_r\\ \mu_1, & \cdots, & \mu_s\end{matrix}\right];\zeta = \sum_{k=0}^{\ell}\frac{\prod_{i=1}^{r}(\lambda_i)_k}{\prod_{i=1}^{s}(\mu_i)_k}\,\frac{\zeta^k}{k!},$$

where for instance

$$\overset{(-1)}{{}_rF_s}(\zeta)=0,\quad \overset{(0)}{{}_rF_s}(\zeta)=1,\quad \overset{(1)}{{}_rF_s}(\zeta)=1+\frac{\lambda_1\cdots\lambda_r}{\mu_1\cdots\mu_s}\zeta.$$

By using the following relation [**13**]

$$(1.14)\quad {}_rF_s\left[\begin{matrix}\lambda_1,\cdots,\lambda_{r-1}, & 1\\ \mu_1,\cdots,\mu_{s-1}, & \ell\end{matrix};\zeta\right]$$

$$= \frac{\Gamma(\mu_1)\cdots\Gamma(\mu_{s-1})}{\Gamma(\lambda_1)\cdots\Gamma(\lambda_{r-1})}\,\frac{\Gamma(\lambda_1-\ell+1)\cdots\Gamma(\lambda_{r-1}-\ell+1)}{\Gamma(\mu_1-\ell+1)\cdots\Gamma(\mu_{s-1}-\ell+1)}\,\frac{(\ell-1)!}{\zeta^{\ell-1}}$$
$$\times\Bigg\{{}_{r-1}F_{s-1}\left[\begin{matrix}\lambda_1-\ell+1,\cdots,\lambda_{r-1}-\ell+1\\ \mu_1-\ell+1,\cdots,\mu_{s-1}-\ell+1\end{matrix};\zeta\right]$$
$$-\,{}_{r-1}\overset{(\ell-2)}{F}_{s-1}\left[\begin{matrix}\lambda_1-\ell+1,\cdots,\lambda_{r-1}-\ell+1\\ \mu_1-\ell+1,\cdots,\mu_{s-1}-\ell+1\end{matrix};\zeta\right]\Bigg\},$$

very recently Masjed-Jamei and Koepf [**10**] have established generalizations of the classical summation theorems (1.3) to (1.13) in the following form:

$$(1.15)\quad {}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ & \rho, & \ell\end{matrix};1\right]=\frac{\Gamma(\ell)\Gamma(\rho)\Gamma(\lambda-\ell+1)\Gamma(\mu-\ell+1)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\rho-\ell+1)}$$
$$\times\left\{\frac{\Gamma(\rho-\ell+1)\Gamma(\rho-\lambda-\mu+\ell-1)}{\Gamma(\rho-\lambda)\Gamma(\rho-\mu)}-{}_2\overset{(\ell-2)}{F}_1\left[\begin{matrix}\lambda-\ell+1,\ -\ell+1\\ \rho-\ell+1\end{matrix};1\right]\right\}=\Phi_1$$

$$(1.16)\quad {}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ \lambda-\mu+\ell, & \ell\end{matrix};-1\right]=(-1)^{\ell-1}\frac{\Gamma(\ell)\Gamma(\lambda-\mu+\ell)\Gamma(\lambda-\ell+1)\Gamma(\mu-\ell+1)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda-\mu+1)}$$
$$\times\left\{\frac{\Gamma(\lambda-\mu+1)\Gamma(1+\frac{1}{2}(\lambda-\ell+1))}{\Gamma(2+\lambda-\ell)\Gamma(\ell-\mu+\frac{1}{2}(\lambda-\ell+1))}-{}_2\overset{(\ell-2)}{F}_1\left[\begin{matrix}\lambda-\ell+1,\ \mu-\ell+1\\ \lambda-\mu+1\end{matrix};-1\right]\right\}=\Phi_2$$

$$(1.17)\quad {}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ \frac{1}{2}(\lambda+\mu+1), & \ell\end{matrix};\frac{1}{2}\right]=2^{\ell-1}\frac{\Gamma(\ell)\Gamma(\lambda+\mu+1)\Gamma(\lambda-\ell+1)\Gamma(\mu-\ell+1)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(-\ell+1+\frac{1}{2}(\lambda+\mu+1))}$$
$$\times\left\{\frac{\sqrt{\pi}\,\Gamma(-\ell+1+\frac{1}{2}(\lambda+\mu+1))}{\Gamma(1+\frac{1}{2}(\lambda-\ell))\Gamma(1+\frac{1}{2}(\mu-\ell))}-{}_2\overset{(\ell-2)}{F}_1\left[\begin{matrix}\lambda-\ell+1,\ \mu-\ell+1\\ -\ell+1+\frac{1}{2}(\lambda+\mu+1)\end{matrix};\frac{1}{2}\right]\right\}=\Phi_3$$

$$(1.18)\quad {}_3F_2\left[\begin{matrix}\lambda, & 2\ell-\lambda-1, & 1\\ & \mu, & \ell\end{matrix};\frac{1}{2}\right]=2^{\ell-1}\frac{\Gamma(\ell)\Gamma(\mu)\Gamma(\lambda-\ell+1)\Gamma(\ell-\lambda)}{\Gamma(\lambda)\Gamma(2\ell-\lambda-1)\Gamma(\mu-\ell+1)}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}(\mu-\ell+1))\Gamma(\frac{1}{2}(\mu-\ell+2))}{\Gamma(-\ell+1+\frac{1}{2}(\lambda+\mu))\Gamma(\frac{1}{2}(\mu-\lambda+1))}-{}_2\overset{(\ell-2)}{F}_1\left[\begin{matrix}\lambda-\ell+1,\ \ell-\lambda\\ \mu-\ell+1\end{matrix};\frac{1}{2}\right]\right\}=\Phi_4$$

$$(1.19)\quad {}_4F_3\left[\begin{matrix}\lambda, & \mu, & \rho, & 1\\ \lambda-\mu+\ell, & \lambda-\rho+\ell, & \ell\end{matrix};1\right]$$
$$=\frac{\Gamma(\ell)\Gamma(\lambda-\mu+\ell)\Gamma(\lambda-\rho+\ell)\Gamma(\lambda+1-\ell)\Gamma(\mu+1-\ell)\Gamma(\rho+1-\ell)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\rho)\Gamma(\lambda-\mu+1)\Gamma(\lambda-\rho+1)}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}(\lambda+3-\ell))\Gamma(\lambda-\mu+1)\Gamma(\lambda-\rho+1)\Gamma(-\mu-\rho+\frac{1}{2}(\lambda+3\ell-1))}{\Gamma(\lambda+2-\ell)\Gamma(-\mu+\frac{1}{2}(\lambda+\ell+1))\Gamma(-\rho+\frac{1}{2}(\lambda+\ell+1))\Gamma(\lambda-\mu-\rho+\ell)}\right.$$
$$\left.-\,{}_3\overset{(\ell-2)}{F}_2\left[\begin{matrix}\lambda-\ell+1,\ \mu-\ell+1,\ \rho-\ell+1\\ \lambda-\mu+1,\ \ \lambda-\rho+1\end{matrix};1\right]\right\}=\Phi_5$$

$$(1.20)\quad {}_4F_3\left[\begin{matrix}\lambda, & \mu, & \rho, & 1\\ \frac{1}{2}(\lambda+\mu+1), & 2\rho+1-\ell, & \ell\end{matrix};1\right]$$

$$= \frac{\Gamma(\ell)\Gamma(\frac{1}{2}(\lambda+\mu+1))\Gamma(2\rho+1-\ell)\Gamma(\lambda+1-\ell)\Gamma(\mu+1-\ell)\Gamma(\rho+1-l)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\rho)\Gamma(-\ell+\frac{1}{2}(\lambda+\mu+3))\Gamma(2\rho-2\ell+2)}$$
$$\times \left\{ \frac{\sqrt{\pi}\,\Gamma(\rho-\ell+\frac{3}{2})\Gamma(-\ell+\frac{1}{2}(\lambda+\mu+3))\Gamma(\rho-\frac{1}{2}(\lambda+\mu-1))}{\Gamma(1+\frac{1}{2}(\lambda-\ell))\Gamma(1+\frac{1}{2}(\mu-\ell))\Gamma(\rho+1-\frac{1}{2}(\lambda+\ell))\Gamma(\rho+1-\frac{1}{2}(\mu+\ell))} \right.$$
$$\left. - \;{}_3F_2^{(\ell-2)} \left[\begin{matrix} \lambda-\ell+1,\ \mu-\ell+1,\ \rho-\ell+1 \\ -\ell+1+\frac{1}{2}(\lambda+\mu+1),\ 2\rho-2\ell+2 \end{matrix}; 1 \right] \right\} = \Phi_6$$

(1.21)
$$ {}_4F_3 \left[\begin{matrix} \lambda, & 2\ell-1-\lambda, & \mu, & 1 \\ \rho, & 2\mu-\rho+1, & \ell \end{matrix}; 1 \right]$$
$$= \frac{\Gamma(\ell)\Gamma(\rho)\Gamma(2\mu-\rho+1)\Gamma(\ell-\lambda)\Gamma(\lambda+1-\ell)\Gamma(\mu+1-\ell)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(2\ell-1-\lambda)\Gamma(\rho+1-\ell)\Gamma(2\mu-\rho-\ell+2)}$$
$$\times \left\{ \frac{\pi\, 2^{2\ell-2\mu-1}\,\Gamma(\rho-\ell+1)}{\Gamma(-\ell+1+\frac{1}{2}(\lambda+\rho))\Gamma(-\ell+1+\mu+\frac{1}{2}(\lambda-\rho+1))\Gamma(\frac{1}{2}(1-\lambda+\rho))} \right.$$
$$\left. \times \frac{\Gamma(2\mu-\rho-\ell+2)}{\Gamma(\mu+1-\frac{1}{2}(\lambda+\rho))} - {}_3F_2^{(\ell-2)} \left[\begin{matrix} \lambda-\ell+1,\ \mu-\ell+1,\ \ell-\lambda \\ \rho-\ell+1,\ 2\mu-\rho-\ell+2 \end{matrix}; 1 \right] \right\} = \Phi_7$$

(1.22)
$$ {}_4F_3 \left[\begin{matrix} \lambda,\ \mu,\ -k+\ell-1,\ 1 \\ \rho,\ 1+\lambda+\mu-\rho-k,\ \ell \end{matrix}; 1 \right] = \frac{(\ell-1)!\,(1-\rho)_{\ell-1}}{(1-\lambda)_{\ell-1}(1-\mu)_{\ell-1}}$$
$$\times \frac{(\rho-\lambda-\mu+k)_{\ell-1}}{(k+2-\ell)_{\ell-1}} \times \left\{ \frac{(\rho-\lambda)_k(\rho-\mu)_k}{(\rho+1-\ell)_k(\rho-\lambda-\mu+\ell-1)_k} \right.$$
$$\left. - {}_3F_2^{(\ell-2)} \left[\begin{matrix} \lambda-\ell+1,\ \mu-\ell+1,\ -k \\ \rho-\ell+1,\ 2+\lambda+\mu-\rho-\ell-k \end{matrix}; 1 \right] \right\} = \Phi_8$$

(1.23)
$$ {}_5F_4 \left[\begin{matrix} \lambda, & \frac{1}{2}(\lambda+\ell+1), & \mu, & \rho, & 1 \\ \frac{1}{2}(\lambda+\ell-1), & \lambda-\mu+\ell, & \lambda-\rho+\ell, & \ell \end{matrix}; -1 \right]$$
$$= (-1)^{\ell-1}\Gamma(\ell)\frac{\Gamma(\frac{1}{2}(\lambda+\ell-1))\Gamma(\lambda-\mu+\ell)\Gamma(\lambda-\rho+\ell)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\rho)}$$
$$\times \frac{\Gamma(\frac{1}{2}(\lambda-\ell+3))\Gamma(\lambda-\ell+1)}{\Gamma(\frac{1}{2}(\lambda+\ell+1))\Gamma(\frac{1}{2}(\lambda-\ell+1))}$$
$$\times \frac{\Gamma(\mu+1-\ell)\Gamma(\rho+1-\ell)}{\Gamma(\lambda-\mu+1)\Gamma(\lambda-\rho+1)} \times \left\{ \frac{\Gamma(1+\lambda-\mu)\Gamma(1+\lambda-\rho)}{\Gamma(2-\ell+\lambda)\Gamma(\ell+\lambda-\mu-\rho)} \right.$$
$$\left. - {}_4F_3^{(\ell-2)} \left[\begin{matrix} \lambda-\ell+1,\ \mu-\ell+1,\ \frac{1}{2}(\lambda-\ell+3),\ \rho-\ell+1 \\ \frac{1}{2}(\lambda-\ell+1),\ \lambda-\mu+1,\ \lambda-\rho+1 \end{matrix}; -1 \right] \right\} = \Phi_9$$

(1.24)
$$ {}_6F_5 \left[\begin{matrix} \lambda, & \frac{1}{2}(\lambda+\ell+1), & \rho, & \alpha, & \beta, & 1 \\ \frac{1}{2}(\lambda+\ell-1), & \lambda-\rho+\ell, & \lambda-\alpha+\ell, & \lambda-\beta+\ell, & \ell \end{matrix}; 1 \right]$$
$$= \frac{\Gamma(\ell)\Gamma(\frac{1}{2}(\lambda+\ell-1))\Gamma(\lambda-\rho+\ell)\Gamma(\lambda-\alpha+\ell)\Gamma(\lambda-\beta+\ell)}{\Gamma(\lambda-\rho+1)\Gamma(\lambda-\alpha+1)\Gamma(\lambda-\beta+1)}$$

$$\times \frac{\Gamma(\lambda-\ell+1)\Gamma(\frac{1}{2}(\lambda-\ell+3))\Gamma(\rho+1-\ell)\Gamma(\alpha+1-\ell)\Gamma(\beta+1-\ell)}{\Gamma(\lambda)\Gamma(\rho)\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{1}{2}(\lambda+\ell+1))\Gamma(\frac{1}{2}(\lambda-\ell+1))}$$
$$\times \left\{ \frac{\Gamma(\lambda-\rho+1)\Gamma(\lambda-\alpha+1)\Gamma(\lambda-\beta+1)\Gamma(\lambda-\rho-\alpha-\beta+2\ell-1)}{\Gamma(2-\ell+\lambda)\Gamma(\lambda-\rho-\beta+\ell)\Gamma(\lambda-\alpha-\beta+\ell)\Gamma(\lambda-\rho-\alpha+\ell)} \right.$$
$$\left. - \sum^{(\ell-2)} {}_5F_4\left[\begin{matrix}\lambda-\ell+1,\ \rho-\ell+1,\ \frac{1}{2}(\lambda-\ell+3),\ \alpha-\ell+1,\ \beta-\ell+1\\ \frac{1}{2}(\lambda-\ell+1),\quad \lambda-\rho+1,\quad \lambda-\alpha+1,\quad \lambda-\beta+1\end{matrix};1\right]\right\} = \Phi_{10}$$

$$(1.25)\quad {}_8F_7\left[\begin{matrix}\lambda,\ \frac{1}{2}(\lambda+\ell+1),\ \mu,\ \rho,\ \alpha,\ A_1,\ A_2,\ 1\\ \frac{1}{2}(\lambda+\ell-1),\ \lambda-\mu+\ell,\ \lambda-\rho+\ell,\ \lambda-\alpha+\ell,\ B_1,\ B_2,\ \ell\end{matrix};1\right]$$
$$= (-1)^{\ell-1}(\ell-1)!\frac{(\frac{1}{2}(3-\lambda-\ell))_{\ell-1}(1-\lambda+\mu-\ell)_{\ell-1}}{(\frac{1}{2}(1-\lambda-\ell))_{\ell-1}(1-\lambda)_{\ell-1}}$$
$$\times \frac{(1-\lambda+\rho-\ell)_{\ell-1}(1-\lambda+\alpha-\ell)_{\ell-1}}{(1-\mu)_{\ell-1}(1-\rho)_{\ell-1}(1-\alpha)_{\ell-1}}$$
$$\times \frac{(\ell+k+\lambda-\mu-\rho-\alpha)_{\ell-1}(-\lambda-k)_{\ell-1}}{(\mu+\rho+\alpha-2\lambda+2-2\ell-k)_{\ell-1}(k+2-\ell)_{\ell-1}}$$
$$\times \left\{ \frac{(\lambda-\ell+2)_k(\lambda-\mu-\rho+\ell)_k(\lambda-\mu-\alpha+\ell)_k(\lambda-\rho-\alpha+\ell)_k}{(\lambda-\mu+1)_k(\lambda-\rho+1)_k(\lambda-\alpha+1)_k(\lambda-\mu-\rho-\alpha+2\ell-1)_k} \right.$$
$$\left. - \sum^{(\ell-2)} {}_7F_6\left[\begin{matrix}\lambda-\ell+1,\ \frac{1}{2}(\lambda-\ell+3),\ \mu-\ell+1,\ \rho-\ell+1,\ \alpha-\ell+1,\ A_3,\ -k\\ \frac{1}{2}(\lambda-\ell+1),\ \lambda-\mu+1,\ \lambda-\rho+1,\ \lambda-\alpha+1,\ B_3,\ B_4\end{matrix};1\right]\right\} = \Phi_{11}$$

where $A_1 = 2\lambda-\mu-\rho-\alpha+2\ell-1+k$, $A_2 = \ell-k-1$, $B_1 = \mu+\rho+\alpha-\lambda+1-\ell-k$, $B_2 = \lambda+k+1$, $A_3 = 2\lambda-\mu-\rho-\alpha+\ell+k$, $B_3 = \mu+\rho+\alpha-\lambda+2-2\ell-k$, and $B_4 = \lambda-\ell+k+2$.

It is interesting to mention here that for $\ell = 1$, results (1.15) to (1.25) reduce to results (1.3) to (1.13) respectively. For other generalizations and extensions of results (1.3) to (1.9), we refer [**4**, **6**–**8**, **15**]. Our aim is to establish eleven new class of integrals involving generalized hypergeometric function by employing summation theorems (1.15) to (1.25) in the following integral due to MacRobert [**9**],

$$(1.26)\qquad \int_0^{\frac{\pi}{2}} e^{i(\psi+\nu)\theta}(\sin\theta)^{\psi-1}(\cos\theta)^{\nu-1}d\theta = e^{\frac{i\pi\psi}{2}}\frac{\Gamma(\psi)\Gamma(\nu)}{\Gamma(\psi+\nu)}$$

provided $\mathrm{Re}(\psi) > 0$ and $\mathrm{Re}(\nu) > 0$.

2. Main esults

The eleven new classes of integrals involving generalized hypergeometric functions paper are given in the following theorems.

THEOREM 2.1. *For* $\ell \in \mathbb{N}$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\rho-\mu) > 0$ *and* $\mathrm{Re}(\rho-\lambda-\mu+\ell) > 1$, *the following result holds true.*

$$(2.1)\quad \int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\rho-\mu-1}{}_2F_1\left[\begin{matrix}\lambda,\ 1\\ \ell\end{matrix};e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\rho-\mu)}{\Gamma(\rho)}\Phi_1,$$

where Φ_1 *is the same as given in* (1.15).

PROOF. In order to prove result (2.1) asserted in Theorem 2.1, we proceed as follows. Denoting the left-hand side of (2.1) by I, we have

$$I = \int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\rho-\mu-1}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & \ell\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$

Now, expressing ${}_2F_1$ as a series, change the order of integration and summation (which are easily seen to be justified due to uniform convergence of the series in the interval $(0, \frac{\pi}{2})$), we have

$$I = \sum_{n=0}^{\infty}\frac{(\lambda)_k(1)_k}{(\ell)_k k!}e^{-i\frac{\pi}{2}k}\int_0^{\frac{\pi}{2}} e^{i(\rho+k)\theta}(\sin\theta)^{\mu+k-1}(\cos\theta)^{\rho-\mu-1}d\theta$$

Now evaluating the integral with the help of the MacRobert's integral (1.26), we have

$$I = \sum_{k=0}^{\infty}\frac{(\lambda)_k(1)_k}{(\ell)_k k!}\; e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu+k)\Gamma(\rho-\mu)}{\Gamma(\rho+k)}.$$

Using the identity $(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$, we have

$$I = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\rho-\mu)}{\Gamma(\rho)}\sum_{k=0}^{\infty}\frac{(\lambda)_k(\mu)_k(1)_k}{(\rho)_k(\ell)_k}\frac{1}{k!}.$$

Finally summing up the series, we have

$$I = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\rho-\mu)}{\Gamma(\rho)}{}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ & \rho, & \ell\end{matrix}; 1\right].$$

Now, we observe that the ${}_3F_2$ appearing can be evaluated with the help of (1.15) and we easily arrive at the right-hand side of (2.1). □

COROLLARY 2.1. *In Theorem* 2.1*, if we take* $\ell = 1, 2, 3$*, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\rho-\mu-1}(1-e^{i(\theta-\frac{\pi}{2})}\sin\theta)^{-\lambda}d\theta = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\rho-\lambda-\mu)}{\Gamma(\rho-\lambda)}$$

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\rho-\mu-1}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & 2\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\rho-\mu)}{\Gamma(\rho)}\frac{(\rho-1)}{(\lambda-1)(\mu-1)}\left[\frac{\Gamma(\rho-1)\Gamma(\rho-\lambda-\mu+1)}{\Gamma(\rho-\lambda)\Gamma(\rho-\mu)}-1\right],$$

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\rho-\mu-1}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & 3\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= 2e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\rho-\mu)}{\Gamma(\rho)}\frac{(\rho-2)_2}{(\lambda-2)_2(\mu-2)_2}\left[\frac{\Gamma(\rho-2)\Gamma(\rho-\lambda-\mu+2)}{\Gamma(\rho-\lambda)\Gamma(\rho-\mu)}\right.$$
$$\left.-\frac{\lambda\mu+\rho-2\lambda-2\mu+2}{(\rho-2)}\right]$$

The following Theorems 2.2 to 2.11 and the corresponding Corollaries 2.2 to 2.11 can be obtained similarly by employing the results (1.16) to (1.25). Hence, they are given here without proof.

THEOREM 2.2. *For $\ell \in \mathbb{N}$, $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(\lambda - 2\mu + \ell) > 0$, the following result holds true.*

$$(2.2)\quad \int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+\ell)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+\ell-1} \times {}_2F_1\left[\begin{matrix}\lambda, & 1\\ & \ell\end{matrix}; -e^{i(\theta-\frac{\pi}{2})}\sin\theta\right] d\theta = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\lambda-2\mu+\ell)\Gamma(\mu)}{\Gamma(\lambda-\mu+\ell)}\Phi_2$$

where Φ_2 is the same as given in (1.16).

COROLLARY 2.2. *In Theorem 2.2, if we take $\ell = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+1)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu}(1+e^{i(\theta-\frac{\pi}{2})}\sin\theta)^{-\lambda}d\theta = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(1+\lambda-2\mu)\Gamma(1+\frac{1}{2}\lambda)}{\Gamma(1+\lambda)\Gamma(1+\frac{1}{2}(\lambda-\mu))},$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+2)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+1}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & 2\end{matrix}; -e^{i(\theta-\frac{\pi}{2})}\sin\theta\right] d\theta = e^{i\frac{\pi}{2}\mu}\frac{(\lambda-\mu+1)}{(\lambda-1)(\mu-1)}\,\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+2)}{\Gamma(\lambda-\mu+2)}\left\{1-\frac{\Gamma(1+\lambda-\mu)\Gamma(\frac{1}{2}\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2}\lambda-\mu+\frac{3}{2})}\right\},$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+3)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+2}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & 3\end{matrix}; -e^{i(\theta-\frac{\pi}{2})}\sin\theta\right] d\theta = 2e^{i\frac{\pi}{2}\mu}\frac{(\lambda-\mu+1)_2}{(\lambda-2)_2(\mu-2)_2}\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+3)}{\Gamma(\lambda-\mu+3)}\left\{\frac{\Gamma(\frac{1}{2}\lambda)\Gamma(1+\lambda-\mu)}{\Gamma(\lambda-1)\Gamma(\frac{1}{2}\lambda-\mu+2)} - \frac{3\lambda+b-\lambda\mu-3}{1+\lambda-\mu}\right\}.$$

THEOREM 2.3. *For $\ell \in \mathbb{N}$, $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(\lambda - \mu + 1) > 0$, the following result holds true.*

$$(2.3)\quad \int_0^{\frac{\pi}{2}} e^{i\frac{1}{2}(\lambda+\mu+1)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\frac{1}{2}(\lambda-\mu-1)}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & \ell\end{matrix}; \tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right] d\theta = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\frac{1}{2}(\lambda-\mu+1))}{\Gamma(\frac{1}{2}(\lambda+\mu+1))}\Phi_3,$$

where Φ_3 is the same as given in (1.17).

COROLLARY 2.3. *In Theorem 2.3, if we take $\ell = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i\frac{1}{2}(\lambda+\mu+1)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\frac{1}{2}(\lambda-\mu-1)}\Big(1-\frac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\Big)^{-\lambda}d\theta = e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\frac{1}{2})\Gamma(\mu)\Gamma(\frac{1}{2}(\lambda-\mu+1))}{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(\frac{1}{2}(\mu+1))},$$

$$\int_0^{\frac{\pi}{2}} e^{i\frac{(\lambda+\mu+1)}{2}\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\frac{1}{2}(\lambda-\mu-1)}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & 2\end{matrix};\tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta = e^{i\frac{\pi}{2}\mu}\frac{(\lambda+\mu-1)}{(\lambda-1)(\mu-1)}\frac{\Gamma(\mu)\Gamma(\frac{1}{2}(\lambda-\mu+1))}{\Gamma(\frac{1}{2}(\lambda+\mu+1))}\bigg\{\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(\lambda+\mu-1))}{\Gamma(\frac{1}{2}\lambda)\Gamma(\frac{1}{2}\mu)}-1\bigg\},$$

$$\int_0^{\frac{\pi}{2}} e^{i\frac{1}{2}(\lambda+\mu+1)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\frac{1}{2}(\lambda-\mu-1)}{}_2F_1\left[\begin{matrix}\lambda, & 1\\ & 3\end{matrix};\tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta = e^{i\frac{\pi}{2}\mu}\frac{2(\lambda+\mu-1)(\lambda+\mu-3)}{(\lambda-2)_2(\mu-2)_2}\frac{\Gamma(\mu)\Gamma(\frac{1}{2}(\lambda-\mu+1))}{\Gamma(\frac{1}{2}(\lambda+\mu+1))} \times\bigg\{\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(\lambda+\mu-3))}{\Gamma(\frac{1}{2}(\lambda-1))\Gamma(\frac{1}{2}(\mu-1))}-\frac{\lambda\mu-\lambda-\mu+1}{\lambda+\mu-3}\bigg\}.$$

THEOREM 2.4. *For $\ell\in\mathbb{N}$, $\mathrm{Re}(\lambda)>0$ and $\mathrm{Re}(\mu-\lambda)>0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i\mu\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\mu-\lambda-1}{}_2F_1\left[\begin{matrix}2\ell-\lambda-1, & 1\\ \ell & \end{matrix};\tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta = e^{i\frac{\pi}{2}\lambda}\frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}\Phi_4, \tag{2.4}$$

where Φ_4 is the same as given in (1.18).

COROLLARY 2.4. *In Theorem 2.4, if we take $\ell=1,2,3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i\mu\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\mu-\lambda-1}\Big(1-\frac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\Big)^{\lambda-1}d\theta = 2^{1-\mu}e^{i\frac{\pi}{2}\lambda}\frac{\Gamma(\frac{1}{2})\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\frac{1}{2}(\lambda+\mu))\Gamma(\frac{1}{2}(\mu-\lambda+1))},$$

$$\int_0^{\frac{\pi}{2}} e^{i\mu\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\mu-\lambda-1}{}_2F_1\left[\begin{matrix}3-\lambda, & 1\\ 2 & \end{matrix};\tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta = 2e^{i\frac{\pi}{2}\lambda}\frac{(1-\mu)}{(1-\lambda)_2}\frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}\bigg\{\frac{\Gamma(\frac{1}{2}(\mu-1))\Gamma(\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}(\lambda+\mu)-1)\Gamma(\frac{1}{2}(\mu-\lambda+1))}-1\bigg\},$$

$$\int_0^{\frac{\pi}{2}} e^{i\mu\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\mu-\lambda-1}{}_2F_1\left[\begin{matrix}5-\lambda, & 1\\ 3 & \end{matrix};\tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$

$$= 8e^{i\frac{\pi}{2}\lambda}\frac{(\mu-2)_2}{(\lambda-4)_4}\frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}(\mu-1))\Gamma(\frac{1}{2}(\mu-2))}{\Gamma(\frac{1}{2}(\lambda+\mu)-2)\Gamma(\frac{1}{2}(\mu-\lambda+1))}-\frac{5\lambda-\lambda^2+2\mu-10}{2(\mu-2)}\right\}.$$

THEOREM 2.5. *For $\ell\in\mathbb{N}$, $\mathrm{Re}(\rho)>0$ and $\mathrm{Re}(\lambda-2\rho+\ell)>0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+\ell)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho+\ell-1} \times {}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ \lambda-\mu+\ell, & \ell\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta = e^{i\frac{\pi}{2}\rho}\frac{\Gamma(\rho)\Gamma(\lambda-2\rho+\ell)}{\Gamma(\lambda-\rho+\ell)}\Phi_5, \tag{2.5}$$

where Φ_5 is the same as given in (1.19).

COROLLARY 2.5. *In Theorem 2.5, if we take $\ell=1,2,3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+1)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho}{}_2F_1\left[\begin{matrix}\lambda, & \mu, & 1\\ \lambda-\mu+1\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\rho}\frac{\Gamma(\rho)\Gamma(1+\lambda-2\rho)\Gamma(1+\frac{1}{2}\lambda)\Gamma(1+\lambda-\mu)\Gamma(1+\frac{1}{2}\lambda-\mu-\rho)}{\Gamma(1+\lambda)\Gamma(1+\frac{1}{2}\lambda-\mu)\Gamma(1+\frac{1}{2}\lambda-\mu)\Gamma(1+\lambda-\mu-\rho)},$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+2)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho+1}{}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ \lambda-\mu+2, & 2\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\rho}\frac{(1+\lambda-\mu)(1+\lambda-\rho)}{(\lambda-1)(\mu-1)(\rho-1)}\frac{\Gamma(\rho)\Gamma(2+\lambda-2\rho)}{\Gamma(\lambda-\rho+2)}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(1+\lambda-\mu)\Gamma(1+\lambda-\rho)\Gamma(\frac{1}{2}\lambda-\mu-\rho+\frac{5}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2}\lambda-\rho+\frac{3}{2})\Gamma(\frac{1}{2}\lambda-\rho+\frac{3}{2})\Gamma(2+\lambda-\mu-\rho)}-1\right\}.$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+3)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho+2}{}_3F_2\left[\begin{matrix}\lambda, & \rho, & \rho\\ \lambda-\mu+3, & 3\end{matrix}; \tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= 2e^{i\frac{\pi}{2}\rho}\frac{(\lambda-\mu+1)_2(\lambda-\rho+1)_2}{(\lambda-2)_2(\mu-2)_2(\rho-2)_2}\frac{\Gamma(\rho)\Gamma(\lambda-2\rho+3)}{\Gamma(\lambda-\rho+3)}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}\lambda)\Gamma(1+\lambda-\mu)\Gamma(1+\lambda-\rho)\Gamma(\frac{1}{2}\lambda-\mu-\rho+4)}{\Gamma(\lambda-1)\Gamma(\frac{1}{2}\lambda-\mu+2)\Gamma(\frac{1}{2}\lambda-\rho+2)\Gamma(3+\lambda-\mu-\rho)}\right.$$
$$\left.-\frac{(\lambda-2)(\mu-2)(\rho-2)}{(\lambda-\mu+1)(\lambda-\rho+1)}-1\right\}.$$

THEOREM 2.6. *For $\ell\in\mathbb{N}$ and $\mathrm{Re}(\rho)>0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(2\rho-\ell+1)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\rho-\ell}{}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ \frac{1}{2}(\lambda+\mu+1), & \ell\end{matrix}; \tfrac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta \tag{2.6}$$

$$= e^{i\frac{\pi}{2}\rho}\frac{\Gamma(\rho)\Gamma(\rho-\ell+1)}{\Gamma(2\rho-\ell+1)}\Phi_6,$$

where Φ_6 *is the same as given in* (1.20).

Corollary 2.6. *In Theorem* 2.6, *if we take* $\ell = 1, 2, 3$, *we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{2i\rho\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\rho-1}{}_2F_1\left[\begin{matrix}\lambda, & \mu\\ \frac{1}{2}(\lambda+\mu+1)\end{matrix};\frac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= 2^{1-2\rho}e^{i\frac{\pi}{2}\rho}\frac{\pi\Gamma(\rho)\Gamma(\frac{1}{2}(\lambda+\mu+1))\Gamma(\rho-\frac{1}{2}(\lambda+\mu-1))}{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(\frac{1}{2}(\mu+1))\Gamma(\rho-\frac{1}{2}(\lambda-1))\Gamma(\rho-\frac{1}{2}(\mu-1))},$$

$$\int_0^{\frac{\pi}{2}} e^{i(2\rho-1)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\rho-2}{}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ \frac{1}{2}(\lambda+\mu+1), & 2\end{matrix};\frac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\rho}\frac{(\lambda+\mu-1)}{(\lambda-1)(\mu-1)}\frac{\Gamma(\rho)\Gamma(\rho-1)}{\Gamma(2\rho-1)}$$
$$\times\left\{\frac{\sqrt{\pi}\,\Gamma(\rho-\frac{1}{2})\Gamma(\frac{1}{2}(\lambda+\mu-1))\Gamma(\rho-\frac{1}{2}(\lambda+\mu-1))}{\Gamma(\frac{1}{2}\lambda)\Gamma(\frac{1}{2}\mu)\Gamma(\rho-\frac{1}{2}\lambda)\Gamma(\rho-\frac{1}{2}\mu)}-1\right\},$$

$$\int_0^{\frac{\pi}{2}} e^{i(2\rho-2)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\rho-3}{}_3F_2\left[\begin{matrix}\lambda, & \mu, & 1\\ \frac{1}{2}(\lambda+\mu+1), & 3\end{matrix};\frac{1}{2}e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\rho}\frac{(2\rho-3)(\lambda+\mu-1)(\lambda+\mu-3)}{(\rho-1)(\lambda-2)_2(\mu-2)_2}\frac{\Gamma(\rho)\Gamma(\rho-2)}{\Gamma(2\rho-2)}$$
$$\times\left\{\frac{\sqrt{\pi}\,\Gamma(\rho-\frac{3}{2})\Gamma(\frac{1}{2}(\lambda+\mu-3))\Gamma(\rho-\frac{1}{2}(\lambda+\mu-1))}{\Gamma(\frac{1}{2}(\lambda-1))\Gamma(\frac{1}{2}(\mu-1))\Gamma(\rho-\frac{1}{2}(\lambda+1))\Gamma(\rho-\frac{1}{2}(\mu+1))}\right.$$
$$\left.-\frac{(\lambda-2)(\mu-2)}{\lambda+\mu-3}-1\right\}.$$

Theorem 2.7. *For* $\ell \in \mathbb{N}$, $\mathrm{Re}(\lambda) > 0$ *and* $\mathrm{Re}(\rho-\lambda) > 0$, *the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_3F_2\left[\begin{matrix}2\ell-1-\lambda, & \mu, & 1\\ 2\mu-\rho+1, & \ell\end{matrix};e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta \tag{2.7}$$
$$= e^{i\frac{\pi}{2}\lambda}\frac{\Gamma(\lambda)\Gamma(\rho-\lambda)}{\Gamma(\rho)}\Phi_7,$$

where Φ_7 *is the same as given in* (1.21).

Corollary 2.7. *In Theorem* 2.7, *if we take* $\ell = 1, 2, 3$, *we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_2F_1\left[\begin{matrix}1-\lambda, & \mu\\ 2\mu-\rho+1\end{matrix};e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}\lambda}\frac{\pi 2^{1-2\mu}\,\Gamma(\lambda)\Gamma(\rho-\lambda)\Gamma(2\mu-\rho+1)}{\Gamma(\frac{1}{2}(\lambda+\rho))\Gamma(\mu+\frac{1}{2}(\lambda-\rho+1))\Gamma(\frac{1}{2}(1-\lambda+\rho))\Gamma(\mu+1-\frac{1}{2}(\lambda+\rho))},$$

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_3F_2\left[\begin{matrix}3-\lambda, & \mu, & 1\\ 2\mu-\rho+1, & 2\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\lambda}\frac{(\rho-1)(\rho-2\mu)}{(\lambda-2)_2(\mu-1)}\frac{\Gamma(\lambda)\Gamma(\rho-\lambda)}{\Gamma(\rho)}$$
$$\times\left\{\frac{\pi 2^{3-2\mu}\Gamma(\rho-1)\Gamma(2\mu-\rho)}{\Gamma(\frac{1}{2}(\lambda+\rho)-1)\Gamma(\mu+\frac{1}{2}(\lambda-\rho-1))\Gamma(\frac{1}{2}(1-\lambda+\rho))\Gamma(\mu+1-\frac{1}{2}(\lambda+\rho))}-1\right\},$$

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_3F_2\left[\begin{matrix}5-\lambda, & \mu, & 1\\ 2\mu-\rho+1, & 3\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\lambda}\frac{\Gamma(\lambda)\Gamma(\rho-\lambda)}{\Gamma(\rho)}$$
$$\times\left\{\frac{\pi 2^{5-2\mu}\,\Gamma(\rho-2)\Gamma(2\mu-\rho+1)}{\Gamma(\frac{1}{2}(\lambda+\rho)-2)\Gamma(\mu+\frac{1}{2}(\lambda-\rho-3))\Gamma(\frac{1}{2}(1-\lambda+\rho))\Gamma(\mu+1-\frac{1}{2}(\lambda+\rho))}\right.$$
$$\left.-\frac{(\lambda-2)(3-\lambda)(\mu-2)}{(\rho-2)(2\mu-\rho-1)}-1\right\}.$$

THEOREM 2.8. *For $\ell \in \mathbb{N}$, $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Re}(\rho-\lambda) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_3F_2 v\left[\begin{matrix}-k+\ell-1, & \mu, & 1\\ 1+\lambda+\mu-\rho-k, & \ell\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta = e^{i\frac{\pi}{2}\lambda}\frac{\Gamma(\lambda)\Gamma(\rho-\lambda)}{\Gamma(\rho)}\Phi_8, \tag{2.8}$$

where Φ_8 is the same as given in (1.22).

COROLLARY 2.8. *In Theorem 2.8, if we take $\ell = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_2F_1\left[\begin{matrix}-k, & \rho\\ 1+\lambda+\mu-\rho-k\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\lambda}\frac{\Gamma(\lambda)\Gamma(\rho-\lambda)}{\Gamma(\rho)}\frac{(\rho-\lambda)_k(\rho-\mu)_k}{(\rho)_k(\rho-\lambda-\mu)_k},$$

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_3F_2\left[\begin{matrix}-k+1, & \mu, & 1\\ 1+\lambda+\mu-\rho-k, & 2\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\lambda}\frac{(1-\rho)(\rho-\lambda-\mu+k)}{k(1-\lambda)(1-\mu)}\frac{\Gamma(\lambda)\Gamma(\rho-\lambda)}{\Gamma(\rho)}\left\{\frac{(\rho-\lambda)_k(\rho-\mu)_k}{(\rho)_k(\rho-\lambda-\mu+1)_k}-1\right\},$$

$$\int_0^{\frac{\pi}{2}} e^{i\rho\theta}(\sin\theta)^{\lambda-1}(\cos\theta)^{\rho-\lambda-1}{}_3F_2\left[\begin{matrix}-k+2, & \mu, & 1\\ 1+\lambda+\mu-\rho-k, & 3\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$

$$= 2e^{i\frac{\pi}{2}\lambda}\frac{(1-\rho)_2(\rho-\lambda-\mu+k)_2}{(1-\lambda)_2(1-\mu)_2}\frac{\Gamma(\lambda)\Gamma(\rho-\lambda)}{\Gamma(\rho)}$$
$$\times\left\{\frac{(\rho-\lambda)_k(\rho-\mu)_k}{(\rho-2)_k(\rho-\lambda-\mu+2)_k}+\frac{k(\lambda-2)(\mu-2)}{(\rho-2)(\lambda+\mu-\rho-k-1)}-1\right\}.$$

THEOREM 2.9. *For* $\ell \in \mathbb{N}$, $\mathrm{Re}(\mu) > 0$ *and* $\mathrm{Re}(\lambda - 2\mu + \ell) > 0$, *the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+\ell)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+\ell-1} \tag{2.9}$$
$$\times\,{}_4F_3\left[\begin{matrix}\lambda, & \frac{1}{2}\lambda+\ell+1), & \rho, & 1\\ \frac{1}{2}(\lambda+\ell-1), & \lambda-\rho+\ell, & \ell\end{matrix}; -e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+\ell)}{\Gamma(\lambda-\mu+\ell)}\Phi_9,$$

where Φ_9 *is the same as given in* (1.23).

COROLLARY 2.9. *In Theorem* 2.9, *if we take* $\ell = 1, 2, 3$, *we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+1)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu}\,{}_3F_2\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+2), & \rho\\ \frac{1}{2}\lambda, & \lambda-\rho+1\end{matrix}; -e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(1+\lambda-\rho)\Gamma(1+\lambda-2\mu)}{\Gamma(1+\lambda)\Gamma(1+\lambda-\mu-\rho)},$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+2)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+1}$$
$$\times\,{}_4F_3\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+3), & \rho, & 1\\ \frac{1}{2}(\lambda+1), & \lambda-\rho+2, & 2\end{matrix}; -e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{(1+\lambda-\mu)(1+\lambda-\rho)}{(\lambda+1)(\mu-1)(\rho-1)}\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+2)}{\Gamma(2+\lambda-\mu)}\left\{1-\frac{\Gamma(1+\lambda-\mu)\Gamma(1+\lambda-\rho)}{\Gamma(\lambda)\Gamma(2+\lambda-\mu-\rho)}\right\}.$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+3)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-\mu+2}$$
$$\times\,{}_4F_3\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+4), & \rho, & 1\\ \frac{1}{2}(\lambda+3), & \lambda-\rho+3, & 3\end{matrix}; -e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{2(1+\lambda-\mu)_2(1+\lambda-\rho)_2}{(\lambda+2)(\lambda-1)(\mu-2)_2(\rho-2)_2}\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+3)}{\Gamma(3+\lambda-\mu)}$$
$$\times\left\{\frac{\Gamma(1+\lambda-\mu)\Gamma(1+\lambda-\rho)}{\Gamma(\lambda-1)\Gamma(3+\lambda-\mu-\rho)}+\frac{\lambda(\mu-2)(\rho-2)}{(1+\lambda-\mu)(1+\lambda-\rho)}-1\right\}.$$

THEOREM 2.10. *For* $\ell \in \mathbb{N}$, $\mathrm{Re}(\mu) > 0$ *and* $\mathrm{Re}(\lambda - 2\rho + \ell) > 0$, *the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+\ell)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho+\ell-1} \tag{2.10}$$

$$\times {}_5F_4\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+\ell+1), & \alpha, & \beta, & 1\\ \frac{1}{2}(\lambda+\ell-1), & \lambda-\alpha+\ell, & \lambda-\beta+\ell, & \ell\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\rho}\frac{\Gamma(\rho)\Gamma(\lambda-2\rho+\ell)}{\Gamma(\lambda-\rho+\ell)}\Phi_{10},$$

where Φ_{10} *is the same as given in* (1.24).

Corollary 2.10. *In Theorem* 2.10*, if we take* $\ell = 1, 2, 3$*, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+1)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho}$$
$$\times {}_4F_3\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+2), & \alpha, & \beta\\ \frac{1}{2}\lambda, & \lambda-\alpha+1, & \lambda-\beta+1\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\rho}\frac{\Gamma(\rho)\Gamma(\lambda-2\rho+1)\Gamma(1+\lambda-\alpha)\Gamma(1+\lambda-\beta)\Gamma(1+\lambda-\rho-\alpha-\beta)}{\Gamma(1+\lambda)\Gamma(1+\lambda-\alpha-\beta)\Gamma(1+\lambda-\rho-\beta)\Gamma(1+\lambda-\rho-\alpha)}.$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+2)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho+1}$$
$$\times {}_5F_4\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+3), & \alpha, & \beta, & 1\\ \frac{1}{2}(\lambda+1), & \lambda-\alpha+2, & \lambda-\beta+2, & 2\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\rho}\frac{(1+\lambda-\rho)(1+\lambda-\alpha)(1+\lambda-\beta)}{(1+\lambda)(\rho-1)(\alpha-1)(\beta-1)}\frac{\Gamma(\rho)\Gamma(\lambda-2\rho+2)}{\Gamma(\lambda-\rho+2)}$$
$$\times\left\{\frac{\Gamma(1+\lambda-\rho)\Gamma(1+\lambda-\alpha)\Gamma(1+\lambda-\beta)\Gamma(3+\lambda-\rho-\alpha-\beta)}{\Gamma(\lambda)\Gamma(2+\lambda-\alpha-\beta)\Gamma(2+\lambda-\rho-\beta)\Gamma(2+\lambda-\rho-\alpha)}-1\right\}.$$

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\rho+3)\theta}(\sin\theta)^{\rho-1}(\cos\theta)^{\lambda-2\rho+2}$$
$$]\times {}_5F_4\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+4), & \alpha, & \beta, & 1\\ \frac{1}{2}(\lambda+2), & \lambda-\alpha+3, & \lambda-\beta+3, & 3\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= 2e^{i\frac{\pi}{2}\rho}\frac{(1+\lambda-\rho)_2(1+\lambda-\alpha)_2(1+\lambda-\beta)_2}{(\lambda-1)(\lambda+2)(\rho-2)_2(\alpha-2)_2(\beta-2)_2}\frac{\Gamma(\rho)\Gamma(\lambda-2\rho+3)}{\Gamma(\lambda-\rho+3)}$$
$$\times\left\{\frac{\Gamma(1+\lambda-\rho)\Gamma(1+\lambda-\alpha)\Gamma(1+\lambda-\beta)\Gamma(5+\lambda-\rho-\alpha-\beta)}{\Gamma(\lambda-1)\Gamma(3+\lambda-\alpha-\beta)\Gamma(3+\lambda-\rho-\beta)\Gamma(3+\lambda-\rho-\alpha)}\right.$$
$$\left.-\frac{\lambda(\rho-2)(\alpha-2)(\beta-2)}{(1+\lambda-\rho)(1+\lambda-\alpha)(1+\lambda-\beta)}\right\}.$$

Theorem 2.11. *For* $\ell\in\mathbb{N}$, $\mathrm{Re}(\mu)>0$ *and* $\mathrm{Re}(\lambda-2\mu+\ell)>0$*, the following result holds true.*

$$(2.11)\quad \int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+\ell)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+\ell-1}$$
$$\times {}_7F_6\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+\ell+1), & \rho, & \alpha, & A_1, & A_2, 1\\ \frac{1}{2}(\lambda+\ell-1), & \lambda-\rho+\ell, & \lambda-\alpha+\ell, & B_1, & B_2, & \ell\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+\ell)}{\Gamma(\lambda-\mu+\ell)}\Phi_{11},$$

where A_1, A_2, B_1, B_2 *and* Φ_{11} *are the same as given in* (1.25).

Corollary 2.11. *In Theorem* 2.11, *if we take* $\ell = 1, 2, 3$, *we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+1)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu} \times {}_6F_5\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+2), & \rho, & \alpha, & A_4, & -k \\ \frac{1}{2}\lambda, & 1+\lambda-\rho, & 1+\lambda-\alpha, & B_5, & B_2\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+1)}{\Gamma(\lambda-\mu+1)}\frac{(1+\lambda)_k(\lambda-\mu-\rho+1)_k}{(1+\lambda-\mu)_k(1+\lambda-\rho)_k}, \times \frac{(\lambda-\mu-\alpha+1)_k(\lambda-\rho-\alpha+1)_k}{(1+\lambda-\alpha)_k(1+\lambda-\mu-\rho-\alpha)_k}$$

where $A_4 = 2\lambda-\mu-\rho-\alpha+k+1$, $B_5 = \mu+\rho+\alpha-\lambda-k$, *and* $B_2 = \lambda+k+1$.

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+2)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+1} \times {}_7F_6\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+3), & \rho, & \alpha, & A_5, & 1-k, & 1 \\ \frac{1}{2}(\lambda+1), & 1+\lambda-\rho, & 1+\lambda-\alpha, & B_6, & B_2, & 2\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{(\mu-\lambda-1)(\rho-\lambda-1)(\alpha-\lambda-1)(k+2+\lambda-\mu-\rho-\alpha)(\lambda+k)}{k(1+\lambda)(1-\mu)(1-\rho)(1-\alpha)(\mu+\rho+\alpha-2\lambda-2-k)} \times \frac{\Gamma(\mu)\Gamma(\lambda-2\mu+1)}{\Gamma(\lambda-\mu+1)}\Omega$$

where $A_5 = 2\lambda-\mu-\rho-\alpha+k+3$, $B_6 = \mu+\rho+\alpha-\lambda-k-1$, $B_2 = \lambda+k+1$, *and* $\Omega = \left\{1 - \frac{(\lambda)_k(\lambda-\mu-\rho+2)_k(\lambda-\mu-\alpha+2)_k(\lambda-\rho-\alpha+2)_k}{(1+\lambda-\mu)_k(1+\lambda-\rho)_k(1+\lambda-\alpha)_k(3+\lambda-\mu-\rho-\alpha)_k}\right\}$.

$$\int_0^{\frac{\pi}{2}} e^{i(\lambda-\mu+3)\theta}(\sin\theta)^{\mu-1}(\cos\theta)^{\lambda-2\mu+2} \times {}_7F_6\left[\begin{matrix}\lambda, & \frac{1}{2}(\lambda+4), & \rho, & \alpha, & A_6, & 2-k, & 1 \\ \frac{1}{2}(\lambda+2), & 3+\lambda-\rho, & 3+\lambda-\alpha, & B_7, & B_2, & 3\end{matrix}; e^{i(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}\mu}\frac{(\lambda-2)(\mu-\lambda-2)_2(\rho-\lambda-2)_2(d-\lambda-2)_2(-\lambda-k)_2}{(\lambda+2)(1-\lambda)_2(1-\mu)_2(1-\rho)_2(1-\alpha)_2(k-1)_2} \times \frac{(3+k+\lambda-\mu-\rho-\alpha)_2}{(\mu+\rho+\alpha-2\lambda-4-k)_2}\frac{\Gamma(\mu)\Gamma(\lambda-2\mu+3)}{\Gamma(\lambda-\mu+2)}$$
$$\left\{\frac{(\lambda-1)_k(\lambda-\mu-\rho+3)_k(\lambda-\mu-\alpha+3)_k(\lambda-\rho-\alpha+3)_k}{(\lambda-\mu+1)_k(\lambda-\rho+1)_k(\lambda-\alpha+1)_k(\lambda-\mu-\rho-\alpha+5)_k} + \frac{k\lambda(\mu-2)(\rho-2)(\alpha-2)(2\lambda-\mu-\rho+k+3)}{(\lambda-\mu+1)(\lambda-\rho+1)(\lambda-\alpha+1)(\mu+\rho+\alpha-\lambda-k-4)(k+\lambda-1)} - 1\right\}$$

where $A_6 = 2\lambda-\mu-\rho-\alpha+k+5$, $B_7 = \mu+\rho+\alpha-\lambda-k-2$, *and* $B_2 = \lambda+k+1$

Conclusion Remark. We evaluated in terms of gamma function, a new class of integrals due to MacRobert involving generalized hypergeometric functions, by employing very recently obtained summation theorems by Masjed-Jamei and Koepf also gave new, interesting and elementary integrals as special cases of our main findings.

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