PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 115(129) (2024), 145–155 DOI: https://doi.org/10.2298/PIM2429145J

CONNECTIONS BETWEEN NORMALIZED WRIGHT FUNCTIONS WITH FAMILIES OF ANALYTIC FUNCTIONS

Santosh Joshi, Sayali Joshi, and Haridas Pawar

Abstract. We establish sufficient conditions for normalized Wright functions to be in certain subclasses of analytic univalent functions in the unit disc $\mathcal{U}$. Furthermore, we examine some geometric properties of integral transforms involving normalized Wright functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k, \tag{1.1}$$

which are analytic in the unit disc $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$, and are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Also, let us denote by $\mathcal{S}$ the subclass of $\mathcal{A}$, which consists of functions that are univalent in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be *starlike of order* α $(0 \leqslant \alpha < 1)$, if and only if $\operatorname{Re}\big(\frac{zf'(z)}{f(z)}\big) > \alpha$, $z \in \mathcal{U}$, and is said to be *convex of order* α $(0 \leqslant \alpha < 1)$, if and only if $\operatorname{Re}\big(1 + \frac{zf''(z)}{f'(z)}\big) > \alpha$, $z \in \mathcal{U}$, and we denote these classes by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. Note that $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the classes of *starlike and convex functions*, respectively (for details, see [**4**] and [**19**]).

The following class $\mathcal{D}(\alpha, \beta, \gamma)$ was defined and studied by Kulkarni [**9**]:

Definition 1.1. [**9**] A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{D}(\alpha, \beta, \gamma)$, if and only if it satisfies the inequality

$$\Big|\frac{f'(z) - 1}{2\gamma(f'(z) - \alpha) - (f'(z) - 1)}\Big| < \beta, \quad z \in \mathcal{U},$$

where $0 < \beta \leqslant 1$, $0 \leqslant \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \leqslant \gamma \leqslant 1$.

2020 *Mathematics Subject Classification:* Primary 30C45; Secondary 30C50.
Key words and phrases: Wright functions, Bassel functions, analytic functions.
Communicated by Miloš Arsenović.

For suitable choices of α, β and γ, we obtain the following subclasses studied by various researchers in earlier works:

(i) For $\gamma = 1$, we obtain the class $\mathcal{D}(\alpha,\beta,1) = R(\alpha,\beta)$ of functions $f \in A$ satisfying the condition $\left|\frac{f'(z)-1}{f'(z)+1-2\alpha}\right| < \beta$, $z \in \mathcal{U}$, studied by Junenja and Mogra [**7**].

(ii) For $\alpha = 0$ and $\gamma = 1$, we obtain the class $\mathcal{D}(0,\beta,1) = D(\beta)$ of functions $f \in A$ satisfying the condition $\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta$, $z \in \mathcal{U}$, studied by Caplinger and Cauchy [**3**] and Padmanabhan [**13**].

To establish our results we need the following lemma (for the detailed proof, see [**9**]).

Lemma 1.1. [**9**] *A sufficient condition for a function f defined by* (1.1) *belongs to $\mathcal{D}(\alpha,\beta,\gamma)$ is* $\sum_{n=2}^{\infty}[1+\beta(1-2\gamma)]na_n \leqslant 2\beta\gamma(1-\alpha)$.

The special functions of mathematical physics are found to be extremely helpful for finding solutions of initial and/or boundary value problems governed by partial differential equations and fractional differential equations. Special functions have extensive applications in other areas of mathematics. Several special functions, called recently special functions of fractional calculus, play a very significant and fascinating role as solutions of fractional order differential equations, such as the Mittag-Leffler function, Wright function with its auxiliary functions, and Fox's H-function.

The *Wright function* $W_{\lambda,\mu}(z)$ is defined by

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\lambda n+\mu)} \frac{z^n}{n!}, \quad \lambda > -1, \;\; \mu, z \in \mathbb{C}. \tag{1.2}$$

This series is absolutely convergent in $\mathbb{C}$, when $\lambda > -1$ and absolutely convergent in open unit disc for $\lambda = -1$. Also, for $\lambda > -1$, the Wright function is an entire function. The Wright function $W_{\lambda,\mu}(z)$ was defined by Wright in [**22**]. This function appeared for the first time in the case $\lambda > 0$ in connection with Wright's investigations in the asymptotic theory of partitions [**22**]. Later on, many other applications were found in Mikusinski operational calculus and in the theory of integral transforms of the Hankel type. Furthermore, extending the methods of Lie groups in partial differential equations to the partial differential equations of fractional order, it was shown that some of the group-invariant solutions of these equations can be given in terms of the Wright functions and the generalized Wright functions.

If λ is a positive rational number, then the Wright function $W_{\lambda,\mu}(z)$ can be represented in terms of the more familiar generalized hypergeometric functions (see [**6**, Section 2.1]). In particular, when $\lambda = 1$ and $\mu = p+1$, the functions $W_{1,p+1}(-z^2/4)$ are expressed in terms of the Bessel functions $J_p(z)$, given as follows:

$$J_p(z) = \Big(\frac{z}{2}\Big)^p \, W_{1,p+1}\Big(\frac{-z^2}{4}\Big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(z/2)^{2n+p}}{\Gamma(n+p+1)}.$$

Furthermore, the function $W_{\lambda,p+1}(-z) \equiv J_p^{\lambda}(z)$ $(\lambda > 0,\ p > -1)$ is known as the generalized Bessel function (sometimes it is also called the Bessel–Wright function). Also, the Wright function generalizes various simple functions such as the Array function, the Whittaker function, (Wright-type) entire auxiliary functions, etc. For the details, refer to [**6**].

Many researchers have investigated classes of analytic functions involving special function $F \subset \mathcal{A}$, to find different conditions such that the members of F have certain geometric properties such as starlikeness and convexity in $\mathcal{U}$. There is a widespread literature dealing with various properties, generalizations and applications of different types of hypergeometric functions, especially for the generalized Gaussian, Kummer hypergeometric and Bessel functions (see [**1**, **10**, **14**–**17**]).

Note that, the Wright function $W_{\lambda,\mu}(z)$ defined by (1.2) does not belong to the class $\mathcal{A}$. Thus it is natural to consider the following two kinds of normalization of the Wright function:

$$W_1(\lambda,\mu;z) := \Gamma(\mu)zW_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \frac{z^{n+1}}{n!}, \quad z \in \mathcal{U},\ \lambda > -1,\ \mu > 0,$$

$$\begin{aligned} W_2(\lambda,\mu;z) &:= \Gamma(\lambda+\mu)\Big[W_{\lambda,\mu}(z) - \frac{1}{\Gamma(\mu)}\Big] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\lambda+\mu)} \frac{z^{n+1}}{(n+1)!}, \quad z \in \mathcal{U},\ \lambda > -1,\ (\lambda+\mu) > 0. \end{aligned}$$

From this, we can easily write:

$$W_1(\lambda,\mu;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^n}{(n-1)!}, \quad z \in \mathcal{U},\ \lambda > -1,\ \mu > 0, \tag{1.3}$$

$$W_2(\lambda,\mu;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^n}{n!}, \quad z\in\mathcal{U},\ \lambda>-1,\ (\lambda+\mu) > 0. \tag{1.4}$$

Furthermore, we observe that the normalized Wright functions $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$ are satisfying the following relations:

$$\begin{aligned} \lambda z(W_1(\lambda,\mu;z))' &= (\mu-1)W_1(\lambda,\mu-1;z) + (\lambda-\mu+1)W_1(\lambda,\mu;z), \\ \lambda z(W_2(\lambda,\mu;z))' &= (\lambda+\mu-1)W_2(\lambda,\mu-1;z) + (1-\mu)W_2(\lambda,\mu;z), \\ z(W_2(\lambda,\mu;z))' &= W_1(\lambda,\lambda+\mu;z), \end{aligned}$$

$$-W_1(1,p+1;-z) = \bar{J}_p(z) := \Gamma(p+1)z^{1-(p/2)}J_p\big(2\sqrt{z}\big) \tag{1.5}$$

Here, $\bar{J}_p(z)$ is the normalized Bessel function.

Silverman [**20**] studied starlikeness and convexity properties for hypergeometric functions, and has also examined a linear operator acting on hypergeometric functions. Geometric properties of the normalized Wright functions were studied extensively by various authors like Baricz et al. [**2**], El-Shahed and Salem [**5**], Kilbas et al. [**8**], Mustafa [**11**], Mustafa and Altintas [**12**] and Prajapat [**18**]. In fact, the more generalized Fox-Wright hypergeometric functions were studied by Srivastava [**21**].

Here we determine sufficient conditions for the normalized Wright functions $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$ to be in the class $\mathcal{D}(\alpha,\beta,\gamma)$. Furthermore, we have obtained sufficient conditions for the integral transforms involving the normalized Wright functions $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$ to be in the class $\mathcal{D}(\alpha,\beta,\gamma)$.

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $\mu > 0$, $\lambda > -1$, $0 < \beta \leqslant 1$, $0 \leqslant \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \leqslant \gamma \leqslant 1$.

2.1. Sufficient conditions for the Wright functions to be in the class $\mathcal{D}(\alpha,\beta,\gamma)$.

THEOREM 2.1. *Let $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2\mu\beta\gamma(1-\alpha) + (\mu+1)[1+\beta(1-2\gamma)] - [1+\beta(1-2\gamma)](\mu+2)e^{\frac{1}{\mu+1}} \geqslant 0. \tag{2.1}$$

Then, the normalized Wright function $W_1(\lambda,\mu;z)$ belongs to the class $\mathcal{D}(\alpha,\beta,\gamma)$.

PROOF. Since

$$W_1(\lambda,\mu;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^n}{(n-1)!},$$

according to Lemma 1.1, we need to show that

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)] \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-1)!} \leqslant 2\beta\gamma(1-\alpha). \tag{2.2}$$

Let

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)] \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-1)!} =: T_1,$$

hence

$$\begin{aligned} T_1 &= [1+\beta(1-2\gamma)] \sum_{n=2}^{\infty} n \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-1)!} \\ &= [1+\beta(1-2\gamma)] \Bigg[\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-2)!} \\ &\qquad + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-1)!} \Bigg] \end{aligned}$$

For every $n \in \mathbb{N}_2 := \mathbb{N}\backslash\{1\} = \{2,3,\cdots\}$, the inequality $\Gamma(\lambda(n-1)+\mu) \geqslant \Gamma(n-1+\mu)$ holds true. Since $\Gamma(n-1+\mu) = \Gamma(\mu)(\mu)_{n-1}$, we have

$$\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \leqslant \frac{1}{(\mu)_{n-1}}, \quad n \in \mathbb{N}_2. \tag{2.3}$$

Here, $(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(n)} = \mu(\mu+1)(\mu+2)\cdots(\mu+n-1)$, $(\mu)_0 = 1$ is the Pochhammer symbol, defined in terms of the Euler gamma function. Using (2.3), we get

$$T_1 \leqslant [1+\beta(1-2\gamma)]\Big[\sum_{n=2}^{\infty}\frac{1}{(\mu)_{n-1}}\frac{1}{(n-2)!}+\sum_{n=2}^{\infty}\frac{1}{(\mu)_{n-1}}\frac{1}{(n-1)!}\Big]$$

Also, the inequality $(\mu)_{n-1} = \mu(\mu+1)(\mu+2)\cdots(\mu+n-2) \geqslant \mu(\mu+1)^{n-2}$, $n \in \mathbb{N}_2$ is clear, which is equivalent to

$$\frac{1}{(\mu)_{n-1}} \leqslant \frac{1}{\mu(\mu+1)^{n-2}}, \quad n \in \mathbb{N}_2. \tag{2.4}$$

By using (2.4), we have

$$\begin{aligned} T_1 &\leqslant [1+\beta(1-2\gamma)]\Big[\sum_{n=2}^{\infty}\frac{1}{\mu(\mu+1)^{n-2}}\frac{1}{(n-2)!}+\sum_{n=2}^{\infty}\frac{1}{\mu(\mu+1)^{n-2}}\frac{1}{(n-1)!}\Big] \\ &\leqslant [1+\beta(1-2\gamma)]\Big\{\frac{1}{\mu}e^{\frac{1}{\mu+1}}+\frac{\mu+1}{\mu}\big[e^{\frac{1}{\mu+1}}-1\big]\Big\}. \end{aligned}$$

In view of (2.1) we can write:

$$[1+\beta(1-2\gamma)]\Big\{\frac{(\mu+2)}{\mu}e^{\frac{1}{\mu+1}}-\frac{(\mu+1)}{\mu}\Big\} \leqslant 2\beta\gamma(1-\alpha),$$

which implies that (2.2) holds true. This completes the proof of Theorem 2.1. □

THEOREM 2.2. *Let $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$\begin{aligned} &2(\lambda+\mu)(\lambda+\mu+1)\beta\gamma(1-\alpha)-[1+\beta(1-2\gamma)](\lambda+\mu+1)^2(\lambda+\mu+2)e^{\frac{1}{\lambda+\mu+1}} \\ &\quad+[1+\beta(1-2\gamma)]((\lambda+\mu+1)^3+(\lambda+\mu+1)^2+(\lambda+\mu+1)) \geqslant 0. \end{aligned} \tag{2.5}$$

Then, the normalized Wright function $W_2(\lambda,\mu;z)$ belongs to the class $\mathcal{D}(\alpha,\beta,\gamma)$.

PROOF. Since

$$W_2(\lambda,\mu;z) = z+\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{z^n}{n!},$$

according to Lemma 1.1, we need to show that

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!} \leqslant 2\beta\gamma(1-\alpha). \tag{2.6}$$

Let

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!} =: T_2,$$

hence

$$T_2 = [1+\beta(1-2\gamma)] \sum_{n=2}^{\infty} n \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n!}$$
$$= [1+\beta(1-2\gamma)] \Big[\sum_{n=2}^{\infty} \Big(1-\frac{1}{n}\Big) \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{(n-1)!} + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n!} \Big]$$

For every $n \in \mathbb{N}_2 := \mathbb{N}\backslash\{1\} = \{2,3,\cdots\}$, the inequality $\Gamma(\lambda(n-1)+\lambda+\mu) \geqslant \Gamma(n-1+\lambda+\mu)$ holds true. Since $\Gamma(n-1+\lambda+\mu) = \Gamma(\lambda+\mu)(\lambda+\mu)_{n-1}$, we have

$$\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \leqslant \frac{1}{(\lambda+\mu)_{n-1}}, \quad n \in \mathbb{N}_2. \tag{2.7}$$

Here, $(\lambda+\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(n)} = (\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+n-1)$, $(\lambda+\mu)_0 = 1$ is the Pochhammer symbol, defined in terms of the Euler gamma function.

Using (2.7), we get

$$T_2 \leqslant [1+\beta(1-2\gamma)] \Big[\sum_{n=2}^{\infty} \frac{1}{(\lambda+\mu)_{n-1}} \frac{1}{(n-1)!} + \sum_{n=2}^{\infty} \frac{1}{(\lambda+\mu)_{n-1}} \frac{1}{n!} \Big]$$

Also, the inequality

$$(\lambda+\mu)_{n-1} = (\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+n-2)$$
$$\geqslant (\lambda+\mu)(\lambda+\mu+1)^{n-2}, \quad n \in \mathbb{N}_2$$

is clear, which is equivalent to

$$\frac{1}{(\lambda+\mu)_{n-1}} \leqslant \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}, \quad n \in \mathbb{N}_2. \tag{2.8}$$

By using (2.8), we have

$$T_2 \leqslant [1+\beta(1-2\gamma)] \Big[\sum_{n=2}^{\infty} \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \frac{1}{(n-1)!} + \sum_{n=2}^{\infty} \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \frac{1}{n!} \Big]$$
$$\leqslant [1+\beta(1-2\gamma)] \Big\{ \frac{\lambda+\mu+1}{\lambda+\mu} \big(e^{\frac{1}{\lambda+\mu+1}} - 1\big) + \frac{(\lambda+\mu+1)^2}{(\lambda+\mu)} \Big(e^{\frac{1}{\lambda+\mu+1}} - \frac{1}{(\lambda+\mu+1)} - 1\Big) \Big\}.$$

In view of (2.5) we can write:

$$[1+\beta(1-2\gamma)]\Big\{\frac{\lambda+\mu+1}{\lambda+\mu}\big(e^{\frac{1}{\lambda+\mu+1}}-1\big) + \frac{(\lambda+\mu+1)^2}{(\lambda+\mu)}\Big(e^{\frac{1}{\lambda+\mu+1}}-\frac{1}{(\lambda+\mu+1)}-1\Big)\Big\} \leqslant 2\beta\gamma(1-\alpha),$$

which implies that (2.6) holds true. □

By setting $\gamma = 1$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 2.1. *The normalized Wright function $W_1(\lambda, \mu; z)$ belongs to the class $R(\alpha, \beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2\mu\beta(1-\alpha) + (\mu+1)[1-\beta] - [1-\beta](\mu+2)e^{\frac{1}{\mu+1}} \geqslant 0.$$

Corollary 2.2. *The normalized Wright function $W_2(\lambda, \mu; z)$ belongs to the class $R(\alpha, \beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\lambda+\mu)(\lambda+\mu+1)\beta(1-\alpha) - [1-\beta](\lambda+\mu+1)^2(\lambda+\mu+2)e^{\frac{1}{\lambda+\mu+1}} + [1-\beta]((\lambda+\mu+1)^3 + (\lambda+\mu+1)^2 + (\lambda+\mu+1) \geqslant 0.$$

Also, by taking $\alpha = 0$ and $\gamma = 1$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 2.3. *The normalized Wright function $W_1(\lambda, \mu; z)$ belongs to the class $D(\beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2\mu\beta + (\mu+1)[1-\beta] - [1-\beta](\mu+2)e^{\frac{1}{\mu+1}} \geqslant 0.$$

Corollary 2.4. *The normalized Wright function $W_2(\lambda, \mu; z)$ belongs to the class $D(\beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\lambda+\mu)(\lambda+\mu+1)\beta - [1-\beta](\lambda+\mu+1)^2(\lambda+\mu+2)e^{\frac{1}{\lambda+\mu+1}} + [1-\beta]((\lambda+\mu+1)^3 + (\lambda+\mu+1)^2 + (\lambda+\mu+1)) \geqslant 0.$$

3. Sufficient conditions for the normalized Bessel functions $\bar{J}_p(z)$

Now we will obtain the sufficient conditions for the normalized Bessel function to belong to the class $\mathcal{D}(\alpha, \beta, \gamma)$. Setting $\lambda = 1$, $\mu = p + 1$ and z is replaced by $-z$ in Theorem 2.1, from equation (1.5) we obtain the following results.

Theorem 3.1. *The normalized Bessel function $\bar{J}_p(z)$ belongs to the class $\mathcal{D}(\alpha, \beta, \gamma)$ if*

$$2(p+1)\beta\gamma(1-\alpha) + (p+2)[1+\beta(1-2\gamma)] - [1+\beta(1-2\gamma)](p+3)e^{\frac{1}{p+2}} \geqslant 0.$$

By setting $\gamma = 1$ in Theorem 3.1 we obtain the following corollary.

Corollary 3.1. *The normalized Bessel function $\bar{J}_p(z)$ belongs to the class $R(\alpha, \beta)$ if*

$$2(p+1)\beta(1-\alpha) + (p+2)[1-\beta] - [1-\beta](p+3)e^{\frac{1}{p+2}} \geqslant 0.$$

Also, by taking $\alpha = 0$ and $\gamma = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *The normalized Bessel function $\bar{J}_p(z)$ belongs to the class $D(\beta)$ if*

$$2(p+1)\beta + (p+2)[1-\beta] - [1-\beta](p+3)e^{\frac{1}{p+2}} \geqslant 0.$$

4. Sufficient conditions for the integrals involving normalized Wright functions

In this section, some sufficient conditions for the integrals involving the normalized Wright functions $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$ are given. Let,

$$G_1(\lambda,\mu;z) = \int_0^z \frac{W_1(\lambda,\mu;t)}{t}dt, \quad z \in U, \tag{4.1}$$

$$G_2(\lambda,\mu;z) = \int_0^z \frac{W_2(\lambda,\mu;t)}{t}dt, \quad z \in U, \tag{4.2}$$

where $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$ are functions, defined by (1.3) and (1.4) respectively. From (4.1) and (4.2) it is easy to prove that $G_1(\lambda,\mu;z)$ and $G_2(\lambda,\mu;z)$ are in the class $\mathcal{A}$.

Theorem 4.1. *Let $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$\begin{aligned}2(\mu)(\mu+1)\beta\gamma(1-\alpha) - [1+\beta(1-2\gamma)](\mu+1)^2(\mu+2)e^{\frac{1}{\mu+1}}\\ + [1+\beta(1-2\gamma)]((\mu+1)^3 + (\mu+1)^2 + (\mu+1)) \geqslant 0.\end{aligned}$$

Then, the function $G_1(\lambda,\mu;z)$ belongs to the class $\mathcal{D}(\alpha,\beta,\gamma)$.

Proof. The proof of Theorem 4.1 is similar to the proof of Theorem 2.2. Indeed, from the definition of function $G_1(\lambda,\mu;z)$, we get

$$G_1(\lambda,\mu;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^n}{n!} = W_2(\lambda,\mu-\lambda;z).$$

Hence, the details of the proof of Theorem 4.1 are omitted. □

Theorem 4.2. *Let $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$\begin{aligned}2(\lambda+\mu)\beta\gamma(1-\alpha) - [1+\beta(1-2\gamma)](\lambda+\mu+1)^2 e^{\frac{1}{\lambda+\mu+1}}\\ + [1+\beta(1-2\gamma)]((\lambda+\mu+1)(\lambda+\mu+2)) \geqslant 0.\end{aligned} \tag{4.3}$$

Then, the normalized Wright function $G_2(\lambda,\mu;z)$ belongs to the class $\mathcal{D}(\alpha,\beta,\gamma)$.

Proof. Since

$$G_2(\lambda,\mu;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^n}{n.n!},$$

according to Lemma 1.1, we need to show that

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)] \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n.n!} \leqslant 2\beta\gamma(1-\alpha). \tag{4.4}$$

Let

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n.n!} =: T_4,$$

hence

$$\begin{aligned} T_4 &= [1+\beta(1-2\gamma)]\sum_{n=2}^{\infty} n\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n.n!} \\ &= [1+\beta(1-2\gamma)]\Big[\sum_{n=2}^{\infty}\Big(1-\frac{1}{n}\Big)\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!} \\ &\qquad + \sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n.n!}\Big] \end{aligned}$$

For every $n \in \mathbb{N}_2 := \mathbb{N}\backslash\{1\} = \{2, 3, \cdots\}$, the inequality $\Gamma(\lambda(n-1)+\lambda+\mu) \geqslant \Gamma(n-1+\lambda+\mu)$ holds true. Since $\Gamma(n-1+\lambda+\mu) = \Gamma(\lambda+\mu)(\lambda+\mu)_{n-1}$, we have

$$\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \leqslant \frac{1}{(\lambda+\mu)_{n-1}}, \quad n \in \mathbb{N}_2. \tag{4.5}$$

Here, $(\lambda+\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(n)} = (\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+n-1)$, $(\lambda+\mu)_0 = 1$ is the Pochhammer symbol, defined in terms of the Euler gamma function.

Using (4.5), we get

$$T_4 \leqslant [1+\beta(1-2\gamma)]\Big[\sum_{n=2}^{\infty}\frac{1}{(\lambda+\mu)_{n-1}}\frac{1}{n!}\Big]$$

Also, the inequality

$$\begin{aligned} (\lambda+\mu)_{n-1} &= (\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+n-2) \\ &\geqslant (\lambda+\mu)(\lambda+\mu+1)^{n-2}, \quad n \in \mathbb{N}_2 \end{aligned}$$

is clear, which is equivalent to

$$\frac{1}{(\lambda+\mu)_{n-1}} \leqslant \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}, \quad n \in \mathbb{N}_2. \tag{4.6}$$

By using (4.6), we have

$$\begin{aligned} T_4 &\leqslant [1+\beta(1-2\gamma)]\Big[\sum_{n=2}^{\infty}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}\frac{1}{n!}\Big] \\ &\leqslant [1+\beta(1-2\gamma)]\Big\{\frac{(\lambda+\mu+1)^2}{(\lambda+\mu)}\Big(e^{\frac{1}{\lambda+\mu+1}} - \frac{1}{(\lambda+\mu+1)} - 1\Big)\Big\}. \end{aligned}$$

In view of (4.3) we can write:

$$[1+\beta(1-2\gamma)]\Big\{\frac{(\lambda+\mu+1)^2}{(\lambda+\mu)}\Big(e^{\frac{1}{\lambda+\mu+1}} - \frac{1}{(\lambda+\mu+1)} - 1\Big)\Big\} \leqslant 2\beta\gamma(1-\alpha),$$

which implies that (4.4) holds true. This completes the proof of Theorem 4.2. □

By setting $\gamma = 1$ in Theorems 4.1 and 4.2, we obtain the following corollaries.

Corollary 4.1. *The normalized Wright function $G_1(\lambda,\mu;z)$ belongs to the class $R(\alpha,\beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\mu)(\mu+1)\beta(1-\alpha) - [1-\beta](\mu+1)^2(\mu+2)e^{\frac{1}{\mu+1}} + [1-\beta]((\mu+1)^3 + (\mu+1)^2 + (\mu+1)) \geqslant 0.$$

Corollary 4.2. *The normalized Wright function $G_2(\lambda,\mu;z)$ belongs to the class $R(\alpha,\beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\lambda+\mu)\beta(1-\alpha) - [1-\beta](\lambda+\mu+1)^2 e^{\frac{1}{\lambda+\mu+1}} + [1-\beta]((\lambda+\mu+1)(\lambda+\mu+2) \geqslant 0.$$

Also, by taking $\alpha = 0$ and $\gamma = 1$ in Theorem 4.1 and Theorem 4.2, we obtain the following corollaries.

Corollary 4.3. *The normalized Wright function $G_1(\lambda,\mu;z)$ belongs to the class $D(\beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\mu)(\mu+1)\beta - [1-\beta](\mu+1)^2(\mu+2)e^{\frac{1}{\mu+1}} + [1-\beta]((\mu+1)^3 + (\mu+1)^2 + (\mu+1)) \geqslant 0.$$

Corollary 4.4. *The normalized Wright function $G_2(\lambda,\mu;z)$ belongs to the class $D(\beta)$ if $\lambda \geqslant 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\lambda+\mu)\beta - [1-\beta](\lambda+\mu+1)^2 e^{\frac{1}{\lambda+\mu+1}} + [1-\beta]((\lambda+\mu+1)(\lambda+\mu+2)) \geqslant 0.$$

Acknowledgement. The authors express deepest gratitude to the esteemed referee(s) for their review of the manuscript and their valuable suggestions and remarks aimed at enhancing the quality of the paper. The work is supported by NBHM research project(NBHM/(R.P)/R & D II/14977) sanctioned to second author.

References

1. A. Baricz, S. Ponnusamy, *Starlikeness and convexity of generalized Bessel functions*, Integral Transforms Spec. Funct. **21**(9) (2010), 641–653.
2. A. Baricz, E. Toklu, E. Kadioglu, *Radii of starlikeness and convexity of Wright functions*, Math. Commun. **23** (2018), 97–117.
3. T. R. Caplinger, W. M. Causey, *A class of univalent functions*, Proc. Am. Math. Soc. **39** (1973), 357–361.
4. P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.
5. M. El-Shahed, A. Salem, *An extension of Wright function and its properties*, J. Math. **2015** (2015), 950728.
6. R. Gorenflo, Yu. Luchko, F. Mainardi, *Analytic properties and applications of Wright functions*, Fract. Calc. Appl. Anal. **2**(4) (1999), 383–414.
7. O. P. Juneja , M. L. Mogra, *A class of univalent functions*, Bull. Sci. Math., II. Sér. **103**(4) (1979), 435–447.
8. A. A. Kilbas, M. Saigo, J. J. Trujillo, *On the generalized Wright function*, Fract. Calc. Appl. Anal. **5**(4) (2002), 437–460.
9. S. R. Kulkarni, *Some Problems Connected with Univalent Functions*, Ph.D. Thesis, Shivaji University, Kolhapur, 1981.

10. S. S. Miller, P. T. Mocanu, *Univalence of Gaussian and confluent hypergeometric functions*, Proc. Am. Math. Soc. **110**(2) (1990), 333–342.
11. N. Mustafa, *Geometric Properties of Normalized Wright Functions*, Math. Comput. Appl. **21**(2) (2016), 14.
12. N. Mustafa, O. Altintas, *Integral operators of the normalized Wright functions and their some geometric properties*, Gazi Univ. J. Sci. **30**(1) (2017), 333–343.
13. K. S. Padmanabhan, *On certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Pol. Math. **23** (1970), 73–81.
14. S. Ponnusamy, F. Ronning, *Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a half plane*, Integral Transforms Spec. Funct. **8** (1999), 121–138.
15. S. Ponnusamy, V. Singh, P. Vasundhra, *Starlikeness and convexity of an integral transform*, Integral Transforms Spec. Funct. **15**(3) (2004), 267–280.
16. S. Ponnusamy, M. Vuorinen, *Univalence and convexity properties for confluent hypergeometric functions*, Complex Variables, Theory Appl. **36**(1) (1998), 73–97.
17. ______, *Univalence and convexity properties for Gaussian hypergeometric functions*, Rocky Mt. J. Math. **31**(1) (2001), 327–353.
18. J. K. Prajapat, *Certain geometric properties of theWright function*, Integral Transforms Spec. Funct. **26** 2015, 203–212.
19. M. S. Robertson, *On the theory of univalent functions*, Annals of Math. **37** (1936), 374–408.
20. H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl. **172** (1993), 574–581.
21. H. M. Srivastava, *Some fox-wright generalized hypergeometric functions and associated families of convolution operators*, Appl. Anal. Discrete Math. **1**(1) (2007), 56–71.
22. E. M. Wright, *On the coefficients of power series having exponential singularities*, J. Lond. Math. Soc. **8** (1933), 71–79.

Department of Mathematics
Walchand College of Engineering
Sangli
India
santosh.joshi@walchandsangli.ac.in; joshisb@hotmail.com

(Received 12 06 2021)

Department of Mathematics
Sanjay Bhokare Group of Institutes
Miraj
India
joshiss@sbgimiraj.org

Department of Mathematics
Sveri's College of Engineering
Pandharpur
India
haridas_pawar007@yahoo.co.in