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EXISTENCE AND UNIQUENESS INTEGRAL EQUATIONS IN C*-ALGEBRA-VALUED S_b-METRIC SPACES BY SOME COUPLED FIXED POINT THEOREMS

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Abstract. We study some coupled fixed point theorems in C*-algebra-valued S_b-metric spaces. As applications, existence and uniqueness results for one type of integral equation

$$x(t) = \int_E (K_1(t,s) + K_2(t,s))(f(s,x(s)) + g(s,x(s)))ds + h(t), \quad t \in E$$

where E is the Lebesque measurable set and $m(E) < +\infty$, and under some other conditions are given.

1. Introduction

Metric spaces have very wide applications in mathematics and applied sciences. Therefore, many authors have tried to introduce the generalizations of metric spaces in many ways. In 1989, Gahler [**2**, **3**], introduced the notion of 2-metric spaces and Dhage [**1**] introduced the notion of D-metric spaces. They proved some results related to 2-metric and D-metric spaces. After this Mustafa and Sims [**5**] proved that most of the results of Dhage's D-metric spaces are not valid. So, they introduced the new concept of generalized metric space called G-metric space. Now, recently Sedghi et al [**9**] have introduced the notion of S-metric spaces as the generalization of G-metric and D*-metric spaces. They proved some fixed point results in S-metric spaces. Some results have been obtained in [**9**, **10**] by Sedghi et al. The authors in [**13**] motivated the study of S_b-metric spaces as generalization of the b-metric space and presented some fixed point results under various natures of contractions in complete S_b-metric spaces. For more results in S_b-metric spaces see [**7**, **8**, **11**, **12**, **14**]. In [**4**], Ma and Jiang introduced the concept of C*-algebra-valued b-metric spaces. In [**6**] the authors introduced C*-algebra-valued S_b-metric space and studied some fixed point results for maps defined in this space.

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In the present paper, we prove some coupled fixed point results in C*-algebra-valued S_b-metric space and then we apply some results to study of one type of existence and uniquness Integral equation.

2. Basic definitions

For the reader's convenience, we recall the following definitions and notations which will be needed in the sequel. We start by some facts about C^*-algebra. Suppose that $\mathcal{A}$ is an unital C*-algebra with the unit I. Set $\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$. We say $a \in \mathcal{A}$ is a positive element and denote it by $a \geqslant 0_{\mathcal{A}}$ if $a = a^*$ and $\sigma(a) \subseteq [0, +\infty)$, where $0_{\mathcal{A}}$ is the zero element in $\mathcal{A}$ and $\sigma(a)$ is the spectrum of a.

There is a natural partial ordering on $\mathcal{A}_h$ given by $a \leqslant b$ if and only if $b-a \geqslant 0_{\mathcal{A}}$. From now on, we will denote $\mathcal{A}_+$ and $\mathcal{A}'$ for the set $\{a \in \mathcal{A} : a \geqslant 0_{\mathcal{A}}\}$ and the set $\{a \in \mathcal{A} : ab = ba, \ \text{for all}\, b \in \mathcal{A}\}$, respectively.

Now we give some known lemmas which are used to prove our main results.

Lemma 2.1. *Suppose that $\mathcal{A}$ is a unital C*-algebra with unit $1_{\mathcal{A}}$.*

(1) *For any $x \in \mathcal{A}_+$, we have $x \leqslant 1_{\mathcal{A}}$ if and only if $\|x\| \leqslant 1$.*
(2) *If $a \in \mathcal{A}_+$ with $\|a\| < \frac{1}{2}$, then $1_{\mathcal{A}} - a$ is invertible and $\|a(1_{\mathcal{A}} - a)^{-1}\| < 1$.*
(3) *Suppose that $a, b \in \mathcal{A}$ with $a, b \geqslant 0_{\mathcal{A}}$ and $ab = ba$, then $ab \geqslant 0_{\mathcal{A}}$.*
(4) *Let $a \in \mathcal{A}'$, if $b, c \in \mathcal{A}$ with $b \geqslant c \geqslant 0_{\mathcal{A}}$, and $1_{\mathcal{A}} - a \in \mathcal{A}'_+$ is an invertible operator, then $(1_{\mathcal{A}} - a)^{-1}b \geqslant (1_{\mathcal{A}} - a)^{-1}c$.*
(5) *If $0_{\mathcal{A}} \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$.*

Lemma 2.2. *Suppose that $\mathcal{A}$ is a unital C*-algebra with unit $1_{\mathcal{A}}$.*

(1) *If $\{b_n\}_{n=1}^{+\infty} \subseteq \mathcal{A}$ and $\lim_{n\to+\infty} b_n = 0_{\mathcal{A}}$, then for any $a \in \mathcal{A}$, $\lim_{n\to+\infty} a^* b_n a = 0_{\mathcal{A}}$.*
(2) *If $a, b \in \mathcal{A}_h$ and $c \in \mathcal{A}'_+$, then $a \leqslant b$ deduces $ca \leqslant cb$, where $\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.*
(3) *If $a, b \in \mathcal{A}_+$, then $a + b \in \mathcal{A}_+$.*

The authors in [**6**] introduced the following notion:

Definition 2.1. Let X be a nonempty set and $b \in \mathcal{A}'$ such that $b \geqslant 1_{\mathcal{A}}$. Let the mapping $S_b \colon X \times X \times X \to \mathcal{A}$ satisfies:

(1) $S_b(x, y, z) \geqslant 0_{\mathcal{A}}$ for all $x, y, z \in X$;
(2) $S_b(x, y, z) = 0$ if and only if $x = y = z$;
(3) $S_b(x, y, z) \leqslant b[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$ for all $x, y, z, a \in X$,

then S_b is said to be C*-algebra-valued S_b-metric on X and $(X, \mathcal{A}, S_b)$ is said to be a C*-algebra-valued S_b-metric space.

Definition 2.2. A C*-algebra-valued S_b-metric S_b is said to be symmetric if

$$S_b(x, x, y) = S_b(y, y, x) \quad \text{for all} \quad x, y \in X.$$

For the sake of transparency, we list the basic properties of C*-algebra-valued S_b-metric spaces:

Definition 2.3. Let $(X, \mathcal{A}, S_b)$ be a C^*-algebra-valued S_b-metric space and $\{x_n\}$ be a sequence in X:

(1) If $\|S_b(x_n, x_n, x)\| \to 0$, $(n \to +\infty)$ then it is said that $\{x_n\}$ converges to x, and we denote it by $\lim_{n\to+\infty} x_n = x$.
(2) If for any $p \in \mathbb{N}$, $\|S_b(x_{n+p}, x_{n+p}, x_n)\| \to 0$, $(n \to +\infty)$, then $\{x_n\}$ is called a Cauchy sequence in X.
(3) If every Cauchy sequence is convergent in X, then $(X, \mathcal{A}, S_b)$ is called a complete C*-algebra-valued S_b-metric space.

The following examples show that a C*-algebra-valued S_b-metric space is not necessarily a C*-algebra-valued S-metric space.

Example 2.1. Let $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{R})$ be all 2×2-matrices with the usual operations of addition, scalar multiplication and matrix multiplication. It is clear that $\|A\| = \big(\sum_{i,j=1}^{2} |a_{ij}|^2\big)^{1/2}$ defines a norm on $\mathcal{A}$ where $A = (a_{ij}) \in \mathcal{A}$. $*\colon \mathcal{A} \to \mathcal{A}$ defines an involution on $\mathcal{A}$ where $\mathcal{A}^* = \mathcal{A}$. Then $\mathcal{A}$ is a C^*-algebra. For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{A}$, a partial order on $\mathcal{A}$ can be given as follows:

$$A \leqslant B \quad \text{if and only if} \quad (a_{ij} - b_{ij}) \leqslant 0 \quad \text{for all} \quad i, j = 1, 2$$

Let (X, d) be a b-metric space with $b \geqslant 1$ and $S_b\colon X \times X \times X \to M_2(\mathbb{R})$ be defined by

$$S_b(x, y, z) = \begin{bmatrix} d(x,z) + d(y,z) & 0 \\ 0 & d(x,z) + d(y,z) \end{bmatrix}$$

then it is a C^*-algebra-valued S_b-metric space for all $x, y, z \in X$. So $(X, \mathcal{A}, S_b)$ is a C^*-algebra-valued S_b-metric space.

Example 2.2. Let $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$ and (X, d) be a metric space. Let the function $S_b\colon X \times X \times X \to A$ be defined as:

$$S_b(x, y, z) = \begin{bmatrix} (d(x,y) + d(y,z) + d(x,z))^p & 0 \\ 0 & (d(x,y) + d(y,z) + d(x,z))^p \end{bmatrix}$$

where $p > 1$ and $x, y, z \in X$. For $A = (a_{ij})$ and $B = (b_{ij})$ in A, a partial order on A can be given by $A \leqslant B$ if and only if $(a_{ij} - b_{ij}) \leqslant 0$ for all $i, j = 1, 2$ It can be shown that (X, A, S_b) is an C^*-algebra-valued S_b-metric with $b = 2^{3(p-1)}$, but (X, A, S_b) is not necessarily a C^*-algebra-valued S-metric.

Definition 2.4. Let $(X, \mathcal{A}, S_b)$ be a C^*-algebra-valued S_b-metric space and $\{x_n\}$ be a sequence in X:

(1) If $\|S_b(x_n, x_n, x)\| \to 0$, $(n \to +\infty)$ then it is said that $\{x_n\}$ converges to x, and we denote it by $\lim_{n\to+\infty} x_n = x$.
(2) If for any $p \in \mathbb{N}$, $\|S_b(x_{n+p}, x_{n+p}, x_n)\| \to 0$, $(n \to +\infty)$, then $\{x_n\}$ is called a Cauchy sequence in X.
(3) If every Cauchy sequence is convergent in X, then $(X, \mathcal{A}, S_b)$ is called a complete C*-algebra-valued S_b-metric space.

Some concepts of this space are listed in the next definition:

Definition 2.5. Let $(X, \mathcal{A}, S_b)$ and $(X_1, \mathcal{A}_1, S_{b_1})$ be C^*-algebra-valued S_b-metric spaces, and let $f\colon (X, \mathcal{A}, S_b) \to (X_1, \mathcal{A}_1, S_{b_1})$ be a function, then f is said to be continuous at a point $x \in X$ if and only if for every sequence $\{x_n\}$ in X,

$S_b(x_n, x_n, x) \to 0_{\mathcal{A}}$, $(n \to +\infty)$ implies $S_{b_1}(f(x_n), f(x_n), f(x)) \to 0_{\mathcal{A}}$, $(n \to +\infty)$. A function f is continuous at X if and only if it is continuous at all $x \in X$.

LEMMA 2.3. *Let $(X, \mathcal{A}, S_b)$ be a symmetric C^*-algebra-valued S_b-metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and y, respectively, then $x = y$.*

Consider the coupled fixed point definition.

DEFINITION 2.6. Let $(X, \mathcal{A}, S_b)$ be a C^*-algebra-valued S_b-metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F\colon X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

3. Main results

By using the above results, we are now ready to prove some of our main theorems.

THEOREM 3.1. *Let $(X, \mathcal{A}, S_b)$ be a complete C^*-algebra-valued S_b-metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the condition*

$$S_b(F(x,y), F(x,y), F(u,v)) \leqslant a^* S_b(x,x,u)a + a^* S_b(y,y,v)a, \tag{3.1}$$

for every $x, y, u, v \in X$ where $a \in \mathcal{A}$ with $\|a\| < 1/\sqrt{2}$. Then F has a unique coupled fixed point. Moreover, F has a unique fixed point in X.

PROOF. Let x_0, y_0 be two arbitrary points in X. Set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Continuing this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. From (3.1), we get

$$\begin{aligned} S_b(x_n, x_n, x_{n+1}) &= S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leqslant a^* S_b(x_{n-1}, x_{n-1}, x_n)a + a^* S_b(y_{n-1}, y_{n-1}, y_n)a \\ &\leqslant a^*(S_b(x_{n-1}, x_{n-1}, x_n) + S_b(y_{n-1}, y_{n-1}, y_n))a. \end{aligned}$$

Similarly,

$$\begin{aligned} S_b(y_n, y_n, y_{n+1}) &= S_b(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leqslant a^* S_b(y_{n-1}, y_{n-1}, y_n)a + a^* S_b(x_{n-1}, x_{n-1}, x_n)a \\ &\leqslant a^*(S_b(y_{n-1}, y_{n-1}, y_n) + S_b(x_{n-1}, x_{n-1}, x_n))a. \end{aligned}$$

Let $\delta_n = S_b(x_n, x_n, x_{n+1}) + S_b(y_n, y_n, y_{n+1})$, and now from the above ralations, we have

$$\begin{aligned} \delta_n &= S_b(x_n, x_n, x_{n+1}) + S_b(y_n, y_n, y_{n+1}) \\ &\leqslant a^*(S_b(x_{n-1}, x_{n-1}, x_n) + S_b(y_{n-1}, y_{n-1}, y_n))a \\ &\quad + a^* S_b(y_{n-1}, y_{n-1}, y_n)a + a^* S_b(x_{n-1}, x_{n-1}, x_n)a \\ &\leqslant (\sqrt{2}a)^*(S_b(x_{n-1}, x_{n-1}, x_n) + S_b(y_{n-1}, y_{n-1}, y_n))(\sqrt{2}a) \\ &\leqslant (\sqrt{2}a)^* \delta_{n-1}(\sqrt{2}a). \end{aligned}$$

Due to the following property: (if $b, c \in \mathcal{A}_h$, then $b \leqslant c$ implies $a^*ba \leqslant a^*ca$), we can obtain for any $n \in \mathbb{N}$,

$$0_{\mathcal{A}} \leqslant \delta_n \leqslant (\sqrt{2}a)^*\delta_{n-1}(\sqrt{2}a) \leqslant \cdots \leqslant ((\sqrt{2}a)^*)^n\delta_0(\sqrt{2}a)^n$$

If $\delta_0 = 0_{\mathcal{A}}$, then from 2 of Definition 2.1, we know that (x_0, y_0) is a coupled fixed point of the mapping F. Now, letting $0_{\mathcal{A}} \leqslant \delta_0$, we can obtain for $n \in \mathbb{N}$ and any $p \in \mathbb{N}$,

$$\begin{aligned}S_b(x_{n+p}, x_{n+p}, x_n) &\leqslant b[S_b(x_{n+p}, x_{n+p}, x_{n+p-1}) + S_b(x_{n+p}, x_{n+p}, x_{n+p-1}) \\ &\quad + S_b(x_n, x_n, x_{n+p-1})] \\ &\leqslant 2bS_b(x_{n+p}, x_{n+p}, x_{n+p-1}) + bS_b(x_n, x_n, x_{n+p-1}) \\ &\leqslant 2bS_b(x_{n+p}, x_{n+p}, x_{n+p-1}) + 2b^2S_b(x_{n+p-1}, x_{n+p-1}, x_{n+p-2}) \\ &\quad + b^2S_b(x_{n+p-2}, x_{n+p-2}, x_n) \\ &\leqslant 2bS_b(x_{n+p}, x_{n+p}, x_{n+p-1}) + 2b^2S_b(x_{n+p-1}, x_{n+p-1}, x_{n+p-2}) \\ &\quad + 2b^3S_b(x_{n+p-2}, x_{n+p-2}, x_{n+p-3}) + \cdots + 2b^pS_b(x_{n+1}, x_{n+1}, x_n)\end{aligned}$$

Similarly,

$$\begin{aligned}S_b(y_{n+p}, y_{n+p}, y_n) &\leqslant 2bS_b(y_{n+p}, y_{n+p}, y_{n+p-1}) + 2b^2S_b(y_{n+p-1}, y_{n+p-1}, y_{n+p-2}) \\ &\quad + 2b^3S_b(y_{n+p-2}, y_{n+p-2}, y_{n+p-3}) + \cdots + 2b^pS_b(y_{n+1}, y_{n+1}, y_n).\end{aligned}$$

Therefore,

$$\begin{aligned}S_b(x_{n+p}, x_{n+p}, x_n) + S_b(y_{n+p}, y_{n+p}, y_n) &\leqslant 2b\delta_{n+p-1} + 2b^2\delta_{n+p-2} + \cdots + 2b^p\delta_n \\ &\leqslant 2\sum_{k=n}^{n+p-1} b^{n+p-k}((\sqrt{2}a)^*)^k\delta_0(\sqrt{2}a)^k,\end{aligned}$$

and then

$$\begin{aligned}\|S_b(x_{n+p}, x_{n+p}, x_n) + S_b(y_{n+p}, y_{n+p}, y_n)\| &\leqslant 2\sum_{k=n}^{n+p-1} \|b\|^{n+p-k}\|\sqrt{2}a\|^{2k}\delta_0 \\ &\leqslant 2\sum_{k=n}^{+\infty} \|b\|^{n+p-k}\|\sqrt{2}a\|^{2k}\delta_0 \\ &= 2\frac{\|b\|^p}{1 - \|b\|^{-1}\|\sqrt{2}a\|^2}\|\sqrt{2}a\|^{2n}\delta_0\end{aligned}$$

Since $\|a\| < \frac{1}{\sqrt{2}}$, we have

$$\|S_b(x_{n+p}, x_{n+p}, x_n) + S_b(y_{n+p}, y_{n+p}, y_n)\| \leqslant 2\frac{\|b\|^p}{1 - \|b\|^{-1}\|\sqrt{2}a\|^2}\|\sqrt{2}a\|^{2n}\delta_0 \to 0,$$

which together with

$$\begin{aligned}S_b(x_{n+p}, x_{n+p}, x_n) &\leqslant S_b(x_{n+p}, x_{n+p}, x_n) + S_b(y_{n+p}, y_{n+p}, y_n), \\ S_b(y_{n+p}, y_{n+p}, y_n) &\leqslant S_b(x_{n+p}, x_{n+p}, x_n) + S_b(y_{n+p}, y_{n+p}, y_n)\end{aligned}$$

yields that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X, so there exist $x, y \in X$ such that $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$. Now we prove that $F(x, y) = x$ and $F(y, x) = y$. For that we have

$$\begin{aligned}
&S_b(F(x,y), F(x,y), x)\\
&\quad\leqslant b[S_b(F(x,y), F(x,y), x_{n+1}) + S_b(F(x,y), F(x,y), x_{n+1}) + S_b(x, x, x_{n+1})]\\
&\quad\leqslant b[2S_b(F(x,y), F(x,y), x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x)]\\
&\quad\leqslant b[2S_b(F(x,y), F(x,y), F(x_n, y_n)) + S_b(x_{n+1}, x_{n+1}, x)]\\
&\quad\leqslant b[2a^*S_b(x, x, x_n)a + 2a^*S_b(y, y, y_n)a + S_b(x_{n+1}, x_{n+1}, x)]\\
&\quad\leqslant b[2a^*S_b(x_n, x_n, x)a + 2a^*S_b(y_n, y_n, y)a + S_b(x_{n+1}, x_{n+1}, x)].
\end{aligned}$$

Taking the limit as $n \to +\infty$ in the above relation, we get $S_b(F(x, y), F(x, y), x) = 0_{\mathcal{A}}$ and hence $F(x, y) = x$. Similarly, $F(y, x) = y$. Therefore, (x, y) is a coupled fixed point of F.

Now if (x', y') is another coupled fixed point of F, then

$$\begin{aligned}
S_b(x, x, x') &= S_b(F(x,y), F(x,y), F(x', y')) \leqslant a^*S_b(x, x, x')a + a^*S_b(y, y, y')a,\\
S_b(y, y, y') &= S_b(F(y,x), F(y,x), F(y', x')) \leqslant a^*S_b(y, y, y')a + a^*S_b(x, x, x')a,
\end{aligned}$$

and hence

$$S_b(x, x, x') + S_b(y, y, y') \leqslant (\sqrt{2}a)^*(S_b(x, x, x') + S_b(y, y, y'))(\sqrt{2}a),$$

which further induces that

$$\|S_b(x, x, x') + S_b(y, y, y')\| \leqslant \|\sqrt{2}a\|^2 \|S_b(x, x, x') + S_b(y, y, y')\|.$$

Since $\|\sqrt{2}a\| < 1$, then $\|S_b(x, x, x') + S_b(y, y, y')\| = 0$. Hence we get $(x', y') = (x, y)$, which means the coupled fixed point is unique.

In order to show that F has a unique fixed point, we only have to show that $x = y$. Notice that

$$S_b(x, x, y) = S_b(F(x,y), F(x,y), F(y,x)) \leqslant a^*S_b(x, x, y)a + a^*S_b(y, y, x)a$$

and then

$$\|S_b(x, x, y)\| \leqslant \|a\|^2\|S_b(x, x, y)\| + \|a\|^2\|S_b(y, y, x)\| \leqslant 2\|a\|^2\|S_b(x, x, y)\|.$$

It follows from the fact that $\|a\| < \frac{1}{\sqrt{2}}$ that $\|S_b(x, x, y)\| = 0$, thus $x = y$. □

THEOREM 3.2. *Let $(X, \mathcal{A}, S_b)$ be a complete C^*-algebra-valued S_b-metric space. Suppose the mapping $F\colon X \times X \to X$ satisfies the following condition*

$$S_b(F(x,y), F(x,y), F(u,v)) \leqslant a_1 S_b(F(x,y), F(x,y), x) + a_2 S_b(F(u,v), F(u,v), u), \tag{3.2}$$

For every $x, y, u, v \in X$ where $a_1, a_2 \in \mathcal{A}'_+$ with $(\|a_1\| + \|a_2\|)\|b\| < 1$. Then F has a unique coupled fixed point. Moreover, F has a unique fixed point in X.

PROOF. Since $a_1, a_2 \in \mathcal{A}'_+$, then we have

$$a_1 S_b(F(x,y), F(x,y), x) + a_2 S_b(F(u,v), F(u,v), u)$$

is a positive element. Choose $x_0, y_0 \in X$. Set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ for $n = 0, 1, \ldots$. Applying (3.2), we have

$$\begin{aligned} S_b(x_n, x_n, x_{n+1}) &= S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leqslant a_1 S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), x_{n-1}) \\ &\quad + a_2 S_b(F(x_n, y_n), F(x_n, y_n), x_n) \\ &\leqslant a_1 S_b(x_n, x_n, x_{n-1}) + a_2 S_b(x_{n+1}, x_{n+1}, x_n) \\ &\leqslant a_1 S_b(x_n, x_n, x_{n-1}) + a_2 S_b(x_n, x_n, x_{n+1}). \end{aligned}$$

So $(1_{\mathcal{A}} - a_2) S_b(x_n, x_n, x_{n+1}) \leqslant a_1 S_b(x_n, x_n, x_{n-1})$. Since $a_1, a_2 \in \mathcal{A}'_+$ with $\|a_1\| + \|a_2\| < \frac{1}{\|b\|} \leqslant 1$, we have $1_{\mathcal{A}} - a_2$ is invertible and $(1_{\mathcal{A}} - a_2)^{-1} a_1 \in \mathcal{A}'_+$. Hence $S_b(x_n, x_n, x_{n+1}) \leqslant (1_{\mathcal{A}} - a_2)^{-1} a_1 S_b(x_n, x_n, x_{n-1})$. Inductively, for all $n \in \mathbb{N}$, we have

$$S_b(x_n, x_n, x_{n+1})v \leqslant k^n \delta_0, \tag{3.3}$$

where $k = (1_{\mathcal{A}} - a_2)^{-1} a_1$ and $\delta_0 = S_b(x_1, x_1, x_0)$. Since $\|a_1\|\|b\| + \|a_2\| \leqslant (\|a_1\| + \|a_2\|)\|b\| < 1$, we have

$$\begin{aligned} \|bk\| = \|(1_{\mathcal{A}} - a_2)^{-1} a_1 b\| &\leqslant \|(1_{\mathcal{A}} - a_2)^{-1}\| \|a_1\| \|b\| \\ &= \sum_{i=0}^{+\infty} \|a_2\|^i \|a_1\| \|b\| = \frac{\|a_1\| \|b\|}{1 - \|a_2\|} < 1. \end{aligned}$$

and $\|k\| \leqslant \|bk\| < 1$ by Lemma 2.1(5).

Let $m, n \in \mathbb{N}$ with $m > n$ by using Definition 2.1, (3.2), (3.3), we have

$$\begin{aligned} S_b(x_n, x_n, x_m) &\leqslant S_b(x_n, x_n, x_{n+1}) + S_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m x_{n+1}) \\ &\leqslant 2b S_b(x_n, x_n, x_{n+1}) + b[2b S_b(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad + b S_b(x_{n+2}, x_{n+2}, x_m)] \\ &\leqslant 2b S_b(x_n, x_n, x_{n+1}) + 2b^2 S_b(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad + 2b^3 S_b(x_{n+2}, x_{n+2}, x_{n+3}) + b^3 S_b(x_{n+3}, x_{n+3}, x_m) \\ &\leqslant 2b S_b(x_n, x_n, x_{n+1}) + 2b^2 S_b(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad + 2b^3 S_b(x_{n+2}, x_{n+2}, x_{n+3}) + \cdots + b^{m-n-1} S_b(x_{m-1}, x_{m-1}, x_m) \\ &\leqslant 2b k^n \delta_0 + 2b^2 k^{n+1} \delta_0 + 2b^3 k^{n+2} \delta_0 + \ldots \\ &\quad + 2b^{m-n-1} k^{m-2} \delta_0 + b^{m-n-1} k^{m-1} \delta_0 \\ &= 2 \sum_{i=1}^{m-n-1} b^i k^{n+i-1} \delta_0 + b^{m-n-1} k^{m-1} \delta_0 \\ &= 2 \sum_{i=1}^{m-n-1} |\delta_0^{\frac{1}{2}} k^{\frac{n+i-1}{2}} b^{\frac{i}{2}}|^2 + |\delta_0^{\frac{1}{2}} k^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}|^2 \\ &\leqslant 2 \sum_{i=1}^{m-n-1} \|\delta_0^{\frac{1}{2}} k^{\frac{n+i-1}{2}} b^{\frac{i}{2}}\|^2 1_{\mathcal{A}} + \|\delta_0^{\frac{1}{2}} k^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}\|^2 1_{\mathcal{A}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\|\delta_0\| \sum_{i=1}^{m-n-1} \|(bk)^{\frac{i}{2}}\|^2 \|k^{\frac{n-1}{2}}\|^2 1_{\mathcal{A}} + \|\delta_0\| \|(bk)^{\frac{m-n-1}{2}}\|^2 \|k^{\frac{n}{2}}\|^2 1_{\mathcal{A}} \\
&= 2\|\delta_0\| \|k\|^{n-1} \sum_{i=1}^{m-n-1} \|bk\|^i 1_{\mathcal{A}} + \|\delta_0\| \|bk\|^{m-n-1} \|k\|^n 1_{\mathcal{A}} \\
&= 2\|\delta_0\| \|k\|^{n-1} \frac{\|bk\| - \|bk\|^{m-n}}{1-\|bk\|} 1_{\mathcal{A}} + \|\delta_0\| \|bk\|^{m-n-1} \|k\|^n 1_{\mathcal{A}} \\
&\leqslant \frac{2\|\delta_0\| \|bk\|}{1-\|bk\|} \|k\|^{n-1} 1_{\mathcal{A}} + \|\delta_0\| \|bk\|^{m-n-1} \|k\|^n 1_{\mathcal{A}} \\
&\to 0_{\mathcal{A}} \quad (m, n \to +\infty)
\end{aligned}
$$

Hence $\{x_n\}$ is a Cauchy sequence. Similarly, we can prove that $\{y_n\}$ is also a Cauchy sequence. Since $(X, \mathcal{A}, S_b)$ is complete, there are $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to +\infty$. In the following, we will show that $F(x, y) = x$ and $F(y, x) = y$. From 3.2, we get

$$
\begin{aligned}
S_b(F(x,y), F(x,y), x) &\leqslant b[S_b(F(x,y), F(x,y), x_{n+1}) + S_b(F(x,y), F(x,y), x_{n+1}) \\
&\quad + S_b(x, x, x_{n+1})] \\
&= 2bS_b(x_{n+1}, x_{n+1}, F(x,y)) + bS_b(x_{n+1}, x_{n+1}, x) \\
&= 2bS_b(F(x_n, y_n), F(x_n, y_n), F(x,y)) + bS_b(x_{n+1}, x_{n+1}, x) \\
&\leqslant 2ba_1 S_b(F(x_n, y_n), F(x_n, y_n), x_n) \\
&\quad + 2ba_2 S_b(F(x,y), F(x,y), x) + bS_b(x_{n+1}, x_{n+1}, x) \\
&= 2ba_1 S_b(x_{n+1}, x_{n+1}, x_n) + 2ba_2 S_b(F(x,y), F(x,y)x) \\
&\quad + bS_b(x_{n+1}, x_{n+1}, x),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
S_b(F(x,y), F(x,y), x) &\leqslant (1_{\mathcal{A}} - 2ba_2)^{-1} 2ba_1 S_b(x_{n+1}, x_{n+1}, x_n) \\
&\quad + (1_{\mathcal{A}} - 2ba_2)^{-1} 2ba_1 S_b(x_{n+1}, x_{n+1}, x).
\end{aligned}
$$

Then $S_b(F(x,y), F(x,y), x) = 0_{\mathcal{A}}$ or equivalently $F(x, y) = x$. Similarly, one can obtain $F(y, x) = y$. Now if (x', y') is another coupled fixed point of F, then according to 3.2, we obtain

$$
\begin{aligned}
0_{\mathcal{A}} &\leqslant S_b(x', x', x) = S_b(F(x', y'), F(x', y'), F(x, y)) \\
&\leqslant a_1 S_b(F(x', y'), F(x', y'), x') + a_2 S_b(F(x,y), F(x,y), x) = 0_{\mathcal{A}}.
\end{aligned}
$$

Then $S_b(x', x', x) = 0_{\mathcal{A}}$, which implies that $x' = x$. s Similarly, we obtain that $y' = y$. That is, (x, y) is the unique coupled fixed point of F. In the following we will show the uniqueness of fixed points of F. From (3.2), we can obtain

$$
\begin{aligned}
S_b(x, x, y) &= S_b(F(x,y), F(x,y), F(y,x)) \\
&\leqslant a_1 S_b(F(x,y), F(x,y), x) + a_2 S_b(F(y,x), F(y,x), y) \\
&= a_1 S_b(x, x, x) + a_2 S_b(y, y, y) = 0_{\mathcal{A}},
\end{aligned}
$$

which yields that $x = y$. □

It is worth noting that when the contractive elements in Theorem 3.2 are equal, we have the following corollary.

COROLLARY 3.1. *Let $(X, \mathcal{A}, S_b)$ be a complete C*-algebra-valued S_b-metric space. Suppose the mapping $F\colon X \times X \to X$ satisfies the following condition*

$$S_b(F(x,y), F(x,y), F(u,v)) \leqslant aS_b(F(x,y), F(x,y), x) + aS_b(F(u,v), F(u,v), u),$$

for every $x, y, u, v \in X$ where $a \in \mathcal{A}'_+$ with $\|a\|\|b\| < \frac{1}{2}$. Then F has a unique fixed point in X.

THEOREM 3.3. *Let $(X, \mathcal{A}, S_b)$ be a complete C*-algebra-valued S_b-metric space. Suppose the mapping $F\colon X \times X \to X$ satisfies the following condition*

$$S_b(F(x,y), F(x,y), F(u,v)) \leqslant a_1 S_b(F(x,y), F(x,y), u) + a_2 S_b(F(u,v), F(u,v), x), \tag{3.4}$$

For every $x, y, u, v \in X$ where $a_1, a_2 \in \mathcal{A}'_+$ with $\|a_1 + a_2\|\|b\| < \frac{1}{2}$. Then F has a unique coupled fixed point. Moreover, F has a unique fixed point in X.

PROOF. From $a_1, a_2 \in \mathcal{A}'_+$ and Lemma 2.2(3), we see that

$$a_1 S_b(F(x,y), F(x,y), u) + a_2 S_b(F(u,v), F(u,v), x) \in \mathcal{A}'_+$$

Choose $x_0, y_0 \in X$. Set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ for $n = 0, 1, \dots$. Applying (3.4), we have

$$\begin{aligned}
S_b(x_n, x_n, x_{n+1}) &= S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\leqslant a_1 S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), x_n) \\
&\quad + a_2 S_b(F(x_n, y_n), F(x_n, y_n), x_{n-1}) \\
&= a_2 S_b(x_{n+1}, x_{n+1}, x_{n-1}) \\
&\leqslant a_2 b[S_b(x_{n+1}, x_{n+1}, x_n) + S_b(x_{n+1}, x_{n+1}, x_n) \\
&\quad + S_b(x_{n-1}, x_{n-1}, x_n)] \\
&= 2a_2 b S_b(x_n, x_n, x_{n+1}) + a_2 b S_b(x_n, x_n, x_{n-1}),
\end{aligned}$$

which implies that

$$(1_{\mathcal{A}} - 2a_2 b) S_b(x_n, x_n, x_{n+1}) \leqslant a_2 b S_b(x_n, x_n, x_{n-1}) \tag{3.5}$$

Because of the symmetry in (3.4),

$$\begin{aligned}
S_b(x_{n+1}, x_{n+1}, x_n) &= S_b(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\
&\leqslant a_1 S_b(F(x_n, y_n), F(x_n, y_n), x_{n-1}) \\
&\quad + a_2 S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), x_n) \\
&= a_1 S_b(x_{n+1}, x_{n+1}, x_{n-1}) \\
&\leqslant a_1 b[S_b(x_{n+1}, x_{n+1}, x_n) + S_b(x_{n+1}, x_{n+1}, x_n) \\
&\quad + S_b(x_{n-1}, x_{n-1}, x_n)] \\
&= 2a_1 b S_b(x_{n+1}, x_{n+1}, x_n) + a_1 b S_b(x_n, x_n, x_{n-1}),
\end{aligned}$$

that is

$$(1_{\mathcal{A}} - 2a_1 b) S_b(x_{n+1}, x_{n+1}, x_n) \leqslant a_1 b S_b(x_n, x_n, x_{n-1}) \tag{3.6}$$

Now, from (3.5) and (3.6) we obtain

$$(1_{\mathcal{A}} - (a_1 + a_2)b)S_b(x_n, x_n, x_{n+1}) \leqslant \frac{(a_1 + a_2)b}{2} S_b(x_n, x_n, x_{n-1}). \tag{3.7}$$

Since $a_1, a_2, b \in \mathcal{A}'_+$, we have $(a_1 + a_2)b \in \mathcal{A}'_+$ and $\frac{(a_1+a_2)b}{2} \in \mathcal{A}'_+$. Moreover, from the condition $\|a_1 + a_2\|\|b\| < 1$, we get

$$\Big\|\frac{(a_1 + a_2)b}{2}\Big\| \leqslant \frac{1}{2}\|a_1 + a_2\|\|b\| < \frac{1}{2} \quad \text{and} \quad \|(a_1 + a_2)b\| \leqslant \|a_1 + a_2\|\|b\| < 1$$

which implies that $\big(1_{\mathcal{A}} - \frac{(a_1+a_2)b}{2}\big)^{-1} \in \mathcal{A}'_+$ and $(1_{\mathcal{A}} - (a_1 + a_2)b)^{-1} \in \mathcal{A}'_+$ with

$$\|(1_{\mathcal{A}} - (a_1 + a_2)b)^{-1} \frac{(a_1 + a_2)b}{2}\| < 1 \tag{3.8}$$

by Lemma 2.1(2). By (3.7) we have $S_b(x_{n+1}, x_{n+1}, x_n) \leqslant tS_b(x_n, x_n, x_{n-1})$, where $t = (1_{\mathcal{A}} - (a_1 + a_2)b)^{-1} \frac{(a_1+a_2)b}{2}$ with $\|t\| \leqslant \|tb\| < 1$ by (3.8). Inductively, for all $n \in \mathbb{N}$, we have

$$S_b(x_{n+1}, x_{n+1}, x_n) \leqslant t^n S_b(x_1, x_1, x_0) = t^n \delta_0, \tag{3.9}$$

where $\delta_0 = S_b(x_1, x_1, x_0)$. Let $m, n \in \mathbb{N}$ with $m > n$, by using Definition 2.1 and relations (3.8)–(3.9), we have

$$\begin{aligned}
S_b(x_n, x_n, x_m) &\leqslant b[S_b(x_n, x_n, x_{n+1}) + S_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m, x_{n+1})] \\
&= 2bS_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m, x_{n+1}) \\
&\leqslant 2bS_b(x_n, x_n, x_{n+1}) + b[2bS_b(x_{n+1}, x_{n+1}, x_{n+2}) \\
&\quad + bS_b(x_{n+2}, x_{n+2}, x_m)] \\
&\leqslant 2bS_b(x_n, x_n, x_{n+1}) + 2b^2 S_b(x_{n+1}, x_{n+1}, x_{n+2}) \\
&+ 2b^3 S_b(x_{n+2}, x_{n+2}, x_{n+3}) + b^3 S_b(x_{n+3}, x_{n+3}, x_m) \\
&\leqslant 2bS_b(x_n, x_n, x_{n+1}) + 2b^2 S_b(x_{n+1}, x_{n+1}, x_{n+2}) \\
&\quad + 2b^3 S_b(x_{n+2}, x_{n+2}, x_{n+3}) + \cdots + b^{m-n-1} S_b(x_{m-1}, x_{m-1}, x_m) \\
&\leqslant 2bt^n\delta_0 + 2b^2t^{n+1}\delta_0 + 2b^3t^{n+2}\delta_0 + \cdots + b^{m-n-1}t^{m-1}\delta_0 \\
&= 2\sum_{i=1}^{m-n-1} b^i t^{n+i-1}\delta_0 + b^{m-n-1}t^{m-1}\delta_0 \\
&= 2\sum_{i=1}^{m-n-1} |\delta_0{}^{\frac{1}{2}} t^{\frac{n+i-1}{2}} b^{\frac{i}{2}}|^2 + |\delta_0{}^{\frac{1}{2}} t^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}|^2 \\
&\leqslant 2\sum_{i=1}^{m-n-1} \|\delta_0{}^{\frac{1}{2}} t^{\frac{n+i-1}{2}} b^{\frac{i}{2}}\|^2 1_{\mathcal{A}} + \|\delta_0{}^{\frac{1}{2}} t^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}\|^2 1_{\mathcal{A}} \\
&\leqslant 2\|\delta_0\| \sum_{i=1}^{m-n-1} \|(bt)^{\frac{i}{2}}\|^2 \|t^{\frac{n-1}{2}}\|^2 1_{\mathcal{A}} + \|\delta_0\|\|(bt)^{\frac{m-n-1}{2}}\|^2 \|t^{\frac{n}{2}}\|^2 1_{\mathcal{A}}
\end{aligned}$$

$$
\begin{aligned}
&= 2\|\delta_0\|\|t\|^{n-1}\sum_{i=1}^{m-n-1}\|bt\|^i 1_{\mathcal{A}} + \|\delta_0\|\|bt\|^{m-n-1}\|t\|^n 1_{\mathcal{A}}\\
&= 2\|\delta_0\|\|t\|^{n-1}\frac{\|bt\|-\|bt\|^{m-n}}{1-\|bt\|}1_{\mathcal{A}} + \|\delta_0\|\|bt\|^{m-n-1}\|t\|^n 1_{\mathcal{A}}\\
&\leqslant \frac{2\|\delta_0\|\|bt\|}{1-\|bt\|}\|t\|^{n-1}1_{\mathcal{A}} + \|\delta_0\|\|bt\|^{m-n-1}\|t\|^n 1_{\mathcal{A}}\\
&\to 0_{\mathcal{A}} \quad (m,n\to+\infty)
\end{aligned}
$$

Hence $\{x_n\}$ is a Cauchy sequence. Similarly, we can prove that $\{y_n\}$ is also a Cauchy sequence. Since $(X,\mathcal{A},S_b)$ is complete, there are $x,y\in X$ such that $x_n\to x$ and $y_n\to y$ as $n\to+\infty$. In the following, we will show that $F(x,y)=x$ and $F(y,x)=y$. From 3.4, we get

$$
\begin{aligned}
S_b(F(x,y),F(x,y),x) &\leqslant b[S_b(F(x,y),F(x,y),x_{n+1})\\
&\quad + S_b(F(x,y),F(x,y),x_{n+1}) + S_b(x,x,x_{n+1})]\\
&= 2bS_b(x_{n+1},x_{n+1},F(x,y)) + bS_b(x_{n+1},x_{n+1},x)\\
&= 2S_b(F(x_n,y_n),F(x_n,y_n),F(x,y)) + bS_b(x_{n+1},x_{n+1},x)\\
&\leqslant 2ba_1S_b(F(x_n,y_n),F(x_n,y_n),x)\\
&\quad + 2ba_2S_b(F(x,y),F(x,y),x_n) + bS_b(x_{n+1},x_{n+1},x)\\
&= 2ba_1S_b(x_{n+1},x_{n+1},x) + 2ba_2S_b(F(x,y),F(x,y),x_n)\\
&\quad + bS_b(x_{n+1},x_{n+1},x)
\end{aligned}
$$

and then

$$
\begin{aligned}
\|S_b(F(x,y),F(x,y),x)\| &\leqslant \|2ba_1\|\|S_b(x_{n+1},x_{n+1},x)\|\\
&\quad + \|2ba_2\|\|S_b(F(x,y),F(x,y),x_n)\| + \|b\|\|S_b(x_{n+1},x_{n+1},x)\|
\end{aligned}
$$

by the continuity of the S_b-metric and the norm, we get

$$
\|S_b(F(x,y),F(x,y),x)\| \leqslant \|2ba_2\|\|S_b(F(x,y),F(x,y),x)\|
$$

Since $0_{\mathcal{A}}\leqslant 2ba_2\leqslant 2(a_1+a_2)b$, we have $\|2ba_2\|\leqslant\|2(a_1+a_2)b\|<2\|a_1+a_2\|b<1$, thus $\|S_b(F(x,y),F(x,y),x)\|=0$, thus $F(x,y)=x$. Similarly $F(y,x)=y$. Hence (x,y) is a coupled fixed point of F. Now if (x',y') ia another coupled fixed point of F, then

$$
\begin{aligned}
0_{\mathcal{A}} \leqslant S_b(x',x',x) &= S_b(F(x',y'),F(x',y'),F(x,y))\\
&\leqslant a_1S_b(F(x',y'),F(x',y'),x) + a_2S_b(F(x,y),F(x,y),x')\\
&= a_1S_b(x',x',x) + a_2S_b(x,x,x')\\
&= a_1S_b(x',x',x) + a_2S_b(x',x',x) = (a_1+a_2)S_b(x',x',x),
\end{aligned}
$$

So, we get

$$
\begin{aligned}
0 \leqslant \|S_b(x',x',x)\| &\leqslant \|a_1+a_2\|\|S_b(x',x',x)\|\\
&< \frac{1}{2\|b\|}\|S_b(x',x',x)\| \leqslant \|S_b(x',x',x)\|,
\end{aligned}
$$

which implies that $\|S_b(x', x', x)\| = 0$, then we have $x = x'$. Similarly, we can get $y = y'$. Hence, the coupled fixed point is unique. In the following we will prove the uniqueness of fixed points of F. By (3.4), we can obtain,

$$\begin{aligned} S_b(x,x,y) &\leqslant S_b(F(x,y),F(x,y),F(y,x)) \\ &\leqslant a_1 S_b(F(x,y),F(x,y),y) + a_2 S_b(F(y,x),F(y,x),x) \\ &= a_1 S_b(x,x,y) + a_2 S_b(y,y,x) \\ &= a_1 S_b(x,x,y) + a_2 S_b(x,x,y) \\ &= (a_1+a_2) S_b(x,x,y). \end{aligned}$$

Then

$$\|S_b(x,x,y)\| \leqslant \|a_1+a_2\| \|S_b(x,x,y)\| < \frac{1}{2\|b\|} \|S_b(x,x,y)\| \leqslant \|S_b(x,x,y)\|$$

which yields, $\|S_b(x,x,y)\| = 0$, then $x = y$. □

The following corollary can be easily deduced from Theorem 3.3.

Corollary 3.2. *Let $(X, \mathcal{A}, S_b)$ be a complete C^*-algebra-valued S_b-metric space. Suppose the mapping $F\colon X \times X \to X$ satisfies the following condition*

$$S_b(F(x,y),F(x,y),F(u,v)) \leqslant a S_b(F(x,y),F(x,y),u) + a S_b(F(u,v),F(u,v),x),$$

For every $x,y,u,v \in X$ where $a \in \mathcal{A}'_+$ with $\|a\|\|b\| < \frac{1}{4}$. Then F has a unique fixed poin in X.

4. Application

As application of contractive mapping theorem on complete C^*-algebra-valued S_b-metric space, existence and uniqueness results for a type of integral equation and operator equation are given.

Theorem 4.1. *Consider the integral equation*

$$x(t) = \int_E (K_1(t,s) + K_2(t,s))(f(s,x(s)) + g(s,x(s)))ds + h(t), \quad t \in E \tag{4.1}$$

where E is the Lebesque measurable set and $m(E) < +\infty$.

In what follows, we always let $X = L^\infty(E)$ denote the class of essentially bounded measurable functions on E, where E is a Lebesgue measurable set such that $m(E) < +\infty$

Now, we consider the functions K_1, K_2, f, g fulfill the following assumptions:

1. *$K_1\colon E \times E\times \to [0,+\infty)$, $K_2\colon E \times E\times \to (-\infty, 0]$, $f,g\colon E \times \mathbb{R} \to \mathbb{R}$ are integrable, and $h \in L^\infty(E)$.*
2. *there exists $l \in (0, \frac{1}{2})$ such that*

   $$0 \leqslant f(t,x) - f(t,y) \leqslant l(x-y) \quad \text{and} \quad -l(x-y) \leqslant g(t,x) - g(t,y) \leqslant 0$$

   for $t \in E$ and $x, y \in \mathbb{R}$;
3. $\sup_{t\in E} \int_E (K_1(t,s) - K_2(t,s))ds \leqslant 1$.

Then the integral equation (4.1) has a unique solution in $L^\infty(E)$.

Proof. Let $X = L^\infty(E)$ and $B(L^2(E))$ be the set of bounded linear operators on a Hilbert space $L^2(E)$. We endow X with the S_b-metric $S_b\colon X \times X \times X \to B(L^2(E))$ defined by $S_b(f,g,h) = \pi_{(|f-h|+|g-h|)^p}$ for all $f,g,h \in X$, where $\pi_h\colon H \to H$ is multiplication operator, $\pi_h(\phi) = h \cdot \phi$ for $\phi \in H$, and $p > 1$. It is clear that $(X, B(L^2(E)), S_b)$ is a complete C^*-algebra-valued S_b-metric space. Define the self-mapping $F\colon X \times X \to X$ by

$$\begin{aligned} F(x,y)(t) = \int_E & K_1(t,s)(f(s,x(s)) + g(s,y(s)))ds \\ & + K_2(t,s)(f(s,y(s)) + g(s,x(s)))ds + h(t), \end{aligned}$$

for all $t \in E$. Now, we have

$$\begin{aligned} S_b(F(x,y),F(x,y),F(u,v)) &= \pi_{(|F(x,y)-F(u,v)|+|F(x,y)-F(u,v)|)^p} \\ &= \pi_{(2|F(x,y)-F(u,v)|)^p}. \end{aligned}$$

We first evaluate the following expression:

$$\begin{aligned} (2|F(x,y)-F(u,v)|)^p &= 2^p\big(\big|K_1(t,s)(f(s,x(s)) + g(s,y(s)))ds \\ &\quad + K_2(t,s)(f(s,y(s)) + g(s,x(s)))ds \\ &\quad - K_1(t,s)(f(s,u(s)) + g(s,v(s)))ds \\ &\quad - K_2(t,s)(f(s,v(s)) + g(s,u(s)))ds\big|\big)^p \\ &= 2^p\Big(\Big|\int_E K_1(t,s)(f(s,x(s)) - f(s,u(s)) \\ &\qquad\qquad + g(s,y(s)) - g(s,v(s)))ds\Big| \\ &\quad + \Big|\int_E K_2(t,s)(f(s,y(s)) - f(s,v(s)) \\ &\qquad\qquad + g(s,x(s)) - g(s,u(s)))ds\Big|\Big)^p \\ &\leqslant 2^p(\sup_{s\in E}[l|x(s)-u(s)| + l|y(s)-v(s)|] \\ &\quad . \int_E (K_1(t,s) - K_2(t,s))ds)^p \\ &\leqslant 2^p(l\|x-u\|_\infty + l\|y-v\|_\infty)^p \\ &\quad \cdot \sup_{t\in E}\int_E ((K_1(t,s) - K_2(t,s))ds)^p \\ &\leqslant 2^p(l\|x-u\|_\infty + l\|y-v\|_\infty)^p \\ &\leqslant 2^p l^p(\|x-u\|_\infty + \|y-v\|_\infty)^p \\ &\leqslant l(2\|x-u\|_\infty + 2\|y-v\|_\infty)^p \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|S_b(F(x,y),F(x,y),F(u,v))\| &= \|\pi_{(2|F(x,y)-F(u,v)|)^p}\| \\ &= \sup_{\|\phi\|=1}\langle \pi_{(2|F(x,y)-F(u,v)|)^p}\phi, \phi\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|\phi\|=1} \langle 2^p |F(x,y) - F(u,v)|^p \phi, \phi \rangle \\
&= \sup_{\|\phi\|=1} \int_E 2^p |(F(x,y) - F(u,v))(t)|^p \phi(t)\overline{\phi(t)} dt \\
&= \sup_{\|\phi\|=1} \int |\phi(t)|^2 dt \cdot (l\|2(x-u)\|_\infty + l\|2(y-v)\|_\infty)^p \\
&\leqslant (l\|2(x-u)\|_\infty + l\|2(y-v)\|_\infty)^p \\
&\leqslant l(\|2(x-u)\|_\infty + \|2(y-v)\|_\infty)^p \\
&= l\|\pi_{(2|x-u|)^p}\| + l\|\pi_{(2|y-v|)^p}\| \\
&= a^* S_b(x,x,u)a + a^* S_b(y,y,v)a.
\end{aligned}
$$

Set $a = \sqrt{l} 1_{B(L^2(E))}$, then $a \in B(L^2(E))$ and $\|a\| = |\sqrt{l}| < \frac{1}{\sqrt{2}}$. Hence, applying Theorem 3.1, we get the desired result. □

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