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ON THE SHARPENING OF AN INEQUALITY DUE TO RIVLIN

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Abstract. Some results on the sharpening and generalizations of an inequality due to Rivlin [**15**] are obtained. Our results give bounds sharper than the bounds given by all the earlier results in this direction, and also improve upon the recent works of Kumar and Milovanović [**12**].

1. Introduction and statements of results

For any polynomial $P(z)$ of degree n, let $M(P,R)=\max_{|z|=R}|P(z)|$. If $P(z)$ is a polynomial of degree n, then the well-known Bernstein inequalities [**3**] on polynomials are given by

$$M(P',1)\leqslant nM(P,1), \tag{1.1}$$

$$M(P,R)\leqslant R^nM(P,1), \tag{1.2}$$

whenever $R\geqslant 1$. The equality holds in both the above inequalities for $P(z)=\alpha z^n$ where α is any complex number.

Inequality (1.1) is a direct consequence of Bernstein's Theorem on the derivative of a trigonometric polynomial [**16**], and inequality (1.2) follows from the maximum modulus theorem (see [**14**, problem 269]).

As mentioned above, equality in (1.2) holds if $P(z)$ has all its zeros at the origin, and therefore it is quite natural to seek improvements under appropriate assumptions on the zeros of $P(z)$. Thus in this direction Ankeny and Rivlin [**1**] proved that, if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then

$$M(P,R)\leqslant \frac{R^n+1}{2}M(P,1),\quad R>1 \tag{1.3}$$

with equality for $P(z)=\alpha+\beta z^n$, whenever $|\alpha|=|\beta|$. A refinement of (1.3) may be seen in Kumar [**10**]. For more information on the growth of polynomials in a disc we refer to the book Milovanović et al. [**13**]. It is equally interesting to extend the inequality (1.2) for the case $R<1$. Varga [**17**] settled this problem and obtained

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the reverse of inequality (1.2) by proving that if $P(z)$ is a polynomial of degree n, then $M(P,r) \geqslant r^n M(P,1)$, $r \leqslant 1$ with equality holding for polynomials having all their zeros at the origin.

The reverse inequality of (1.3) for $0 \leqslant R < 1$ was given by Rivlin [**15**] and he proved that

Theorem A. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \geqslant 1$, then*

$$M(P,r) \geqslant \Big(\frac{1+r}{2}\Big)^n M(P,1), \tag{1.4}$$

whenever $0 \leqslant r < 1$. Equality in (1.4) holds whenever $P(z) = (z+1)^n$.

Aziz [**2**] generalized Rivlin's inequality (1.4) for polynomials having no zeros in $|z| < K$, $K \geqslant 1$ by proving that

Theorem B. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geqslant 1$, then*

$$M(P,r) \geqslant \Big(\frac{K+r}{K+1}\Big)^n M(P,1), \quad 0 \leqslant r < 1. \tag{1.5}$$

Inequality (1.5) is best possible with equality holding for the polynomials $P(z) = (z+a)^n$ satisfying $|a| = K$.

Govil [**7**] generalized inequality (1.4) by studying the relative growth of polynomials $P(z)$ having no zeros in the open unit disc, with respect to two circles $|z| = r$ and $|z| = R$ whenever $0 \leqslant r < R \leqslant 1$.

Theorem C. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $0 \leqslant r < R \leqslant 1$,*

$$M(P,r) \geqslant \left(\frac{1+r}{1+R}\right)^n M(P,1).$$

Inequalities (1.3) and (1.4) are best possible with equality holding for polynomials having all their zeros on $|z| = 1$. But the flip side of the bounds given in (1.3) and (1.4) does not address the issue of how far the zeros lie outside the unit circle. Now naturally a question arises: is there any way to refine (1.3) and (1.4) for the class of polynomials satisfying the hypotheses, by capturing some information on the moduli of zeros? Can we obtain a bound involving the extreme coefficients of $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ which are informative to some extent on the distance of zeros from the origin? In view of the example for the equality case in (1.3) and (1.4) which holds with the property $|a_0| = |a_n|$, it should be possible to improve upon the bound for polynomials $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ satisfying $|a_0| \neq |a_n|$ and the hypotheses of inequalities (1.3) and (1.4). The bound involving the extreme coefficients were introduced to the inequalities for the derivatives of polynomials by Dubinin [**5**] and it is very well explained in the book [**6**] (see also [**11**]). Recently Kumar and Milovanović [**12**] (see also [**9**]) settled the problem of sharpening (1.4) by introducing the moduli of extreme coefficients of the polynomial into it by proving that, if

$P(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$M(P,r) \geqslant \Big(\frac{r+1}{2}\Big)^n \Big[1 + \frac{(1-r)^n}{(r+1)^n}\Big(\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\Big)\Big] M(P,1), \tag{1.6}$$

for any $r < 1$.

At the same time, we see many refinements of the inequality (1.3) in the literature, the latest one due to Dalal and Govil [**4**] in which the authors presented the results in terms of LerchPhi function.

We prove that the index n of the term $(1-r)^n$ in the right-hand side of (1.6) is redundant and therefore the term $(1-r)^n$ can be replaced by $1-r$. As we know $1-r > (1-r)^n$ whenever $0 < r < 1$ and $n > 1$, so our result here improves upon the recent results of Kumar and Milovanović [**12**], and in turn sharpens Rivlin's inequality (1.4) considerably. Let us begin with the following fundamental result.

Theorem 1.1. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n having no zeros in $|z| < 1$, then*

$$M(P,r) \geqslant \Big(\frac{r+1}{2}\Big)^n \left[1 + \frac{(1-r)}{(r+1)^n}\left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right)\right] M(P,1), \tag{1.7}$$

whenever $0 \leqslant r \leqslant 1$. The result is best possible and equality holds in (1.7) *for $P(z) = (a+bz)^n$ with $|a| = |b| = 1$ and also for $P(z) = z + a$ with $|a| \geqslant 1$.*

Since $P(z)$ has no zeros in $|z| < 1$, it is a straightforward fact that $|a_0| - |a_n| \geqslant 0$ and hence $\frac{(1-r)}{(r+1)^n}\Big(\frac{|a_0|-|a_n|}{|a_0|+|a_n|}\Big) \geqslant 0$. Therefore (1.7) sharpens inequality (1.4) significantly, whenever $|a_0| \neq |a_n|$. Further

$$\frac{(1-r)}{(r+1)^n}\left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right) \geqslant \frac{(1-r)^n}{(r+1)^n}\left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right),$$

whenever $0 \leqslant r \leqslant 1$, and therefore the inequality (1.7) sharpens the inequality (1.6) whenever $|a_0| \neq |a_n|$, $0 < r < 1$ and $n > 1$. Now it is evident that Theorem 1.1 significantly improves both existing inequalities (1.4) and (1.6), which will be further illustrated in the following example.

Example 1.1. Let $P(z) = (z+2)^2$. For $r = 1/2$, $\max_{|z|=r} |P(z)| = 6.25$, and the bound from inequality (1.6) is given by $\max_{|z|=r} |P(z)| \geqslant 5.4$, whereas from inequality (1.7) of our result it is $\max_{|z|=r} |P(z)| \geqslant 5.7375$, an improvement of about 6.25% over the bound obtained by inequality (1.6). Also, the bound obtained from Rivlin's inequality (1.4) is $\max_{|z|=r} |P(z)| \geqslant 5.06$ and in this case, the improvement of our bound over the bound obtained by (1.4) is about 13.39%.

As a consequence of Theorem 1.1, we can derive a sufficient condition for a polynomial to have at least one zero in an open disc, as given below.

Corollary 1.1. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n and $\max_{|z|=\rho} |P(z)| > 2^{n-1}(|a_0| + |a_n|\rho^n)$ for some $\rho > 0$, then the polynomial $P(z)$ has at least one zero in $|z| < \rho$.*

Let us prove another interesting inequality for the polynomials having all their zeros in the punctured open unit disc as follows.

Theorem 1.2. *If $P(z) = a_0 + a_1z + \cdots + a_nz^n$ is a polynomial of degree n having all its zeros in $0 < |z| \leqslant 1$, then*

$$M(P,R) \geqslant \Big(\frac{R+1}{2}\Big)^n \left[1 + \frac{R^{n-1}(R-1)}{(R+1)^n}\left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right] M(P,1), \tag{1.8}$$

whenever $R \geqslant 1$.

By ignoring the term containing the extreme coefficients of the polynomial $P(z)$ in (1.8), thereby allowing $P(z)$ to have zeros at the origin, and proceeding similarly as in the proof of Theorem 1.2, we obtain the following result, which can be viewed as a consequence of Rivlin's inequality, and proved independently by Jain [**8**].

Corollary 1.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then $M(P,R) \geqslant \big(\frac{R+1}{2}\big)^n M(P,1)$, whenever $R \geqslant 1$.*

In the next result we prove a generalization of Theorem 1.1 for the class of polynomials having no zeros in $|z| < K, K \geqslant 1$ which sharpens Theorem A.

Theorem 1.3. *If $P(z) = a_0 + a_1z + \cdots + a_nz^n$ is a polynomial of degree n having no zeros in $|z| < K, K \geqslant 1$, then*

$$M(P,r) \geqslant \Big(\frac{r+K}{1+K}\Big)^n \left[1 + \frac{1}{K^{n-1}}\frac{(1-r)}{(r+K)^n}\left(\frac{|a_0| - |a_n|K^n}{|a_0| + |a_n|}\right)\right] M(P,1), \tag{1.9}$$

whenever $0 \leqslant r \leqslant 1$. The result is sharp and equality holds for $P(z) = (z+K)^n$ and also for $P(z) = z + a$ with $|a| \geqslant K$.

Remark 1.1. When $K = 1$, Theorem 1.3 reduces to Theorem 1.1.

Remark 1.2. Since $P(z) = \sum_{j=0}^n a_jz^j$ has all zeros in $|z| \geqslant K$, we have

$$\frac{|a_0| - |a_n|K^n}{|a_0| + |a_n|} \geqslant 0.$$

Hence for all the polynomials satisfying the hypothesis of Theorem 1.3 except those having $|a_0| = |a_n|K^n$, our inequality (1.9) sharpens the inequality (1.5).

By applying Theorem 1.3 to the reciprocal polynomial $z^nP(1/z)$ of $P(z)$, we get an inequality for the class of polynomials having all zeros in $|z| \leqslant K$, $K \leqslant 1$. If $P(z) = a_0 + a_1z + \cdots + a_nz^n$ has all zeros in $|z| \leqslant K, K \leqslant 1$, then $Q(z) = z^nP(1/z)$ has no zeros in $|z| < \frac{1}{K}$, $\frac{1}{K} \geqslant 1$.

Applying Theorem 1.3 to $Q(z)$ with $r = \frac{1}{R}$, $R \geqslant 1$, we get

$$\max_{|z|=R}|Q(z)| \geqslant \left(\frac{\frac{1}{R}+\frac{1}{K}}{1+\frac{1}{K}}\right)^n \left[1 + K^{n-1}\frac{(1-\frac{1}{R})}{(\frac{1}{R}+\frac{1}{K})^n}\left(\frac{|a_n| - |a_0|\big(\frac{1}{K^n}\big)}{|a_n| + |a_0|}\right)\right] \max_{|z|=1}|Q(z)|.$$

Since $\max_{|z|=R}|Q(z)| = \frac{1}{R^n}\max_{|z|=R}|P(z)|$ and $\max_{|z|=1}|Q(z)| = \max_{|z|=1}|P(z)|$, we have

$$\max_{|z|=R}|P(z)| \geqslant \Big(\frac{K+R}{K+1}\Big)^n \left[1 + (KR)^{n-1}\frac{(R-1)}{(K+R)^n}\left(\frac{|a_n|K^n - |a_0|}{|a_n| + |a_0|}\right)\right] \max_{|z|=1}|P(z)|.$$

Thus, we obtain the following result.

Corollary 1.3. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n having all its zeros in $|z| \leqslant K$, $K \leqslant 1$, then for $R \geqslant 1$,*

$$M(P,R) \geqslant \left(\frac{K+R}{K+1}\right)^n \left[1 + (KR)^{n-1}\frac{(R-1)}{(K+R)^n}\left(\frac{|a_n|K^n - |a_0|}{|a_n| + |a_0|}\right)\right] M(P,1).$$

The result is sharp and equality holds for $P(z) = (z+K)^n$ and also for $P(z) = z+a$ with $|a| \leqslant K$.

In view of the sharpened inequality presented in Theorem 1.3, Theorem C also can be improved considerably and is given below.

Theorem 1.4. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $0 \leqslant r < R \leqslant 1$*

$$M(P,r) \geqslant \left(\frac{r+1}{R+1}\right)^n \left[1 + R^{2(n-1)}\frac{(R-r)}{(r+1)^n}\left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|R^n}\right)\right] M(P,R).$$

The result is best possible and equality holds if $P(z) = (z+a)^n$, where $|a| = 1$ and also for $P(z) = z+a$ with $|a| \geqslant 1$.

Remark 1.3. For $R = 1$, Theorem 1.4 gives a sharpened version of Rivlin's inequality stated in Theorem 1.2.

If the polynomial $P(z)$ has all its zeros in $|z| \leqslant 1$, then its reciprocal polynomial $Q(z) = z^n P(1/z)$ has all zeros in $|z| \geqslant 1$. If $1 \leqslant R < r$, then $\frac{1}{r} < \frac{1}{R} \leqslant 1$. Hence applying Theorem 1.4 to $Q(z)$, we get

$$\max_{|z|=1/r}|Q(z)| \geqslant \left(\frac{\frac{1}{r}+1}{\frac{1}{R}+1}\right)^n \left[1+\frac{1}{R^{2(n-1)}}\frac{\left(\frac{1}{R}-\frac{1}{r}\right)}{\left(\frac{1}{r}+1\right)^n}\left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|\left(\frac{1}{R^n}\right)}\right)\right] \max_{|z|=1/R}|Q(z)|,$$

which is equivalent to

$$\frac{1}{r^n}\max_{|z|=r}|P(z)| \geqslant \frac{R^n}{r^n}\left(\frac{1+r}{1+R}\right)^n \left[1 + \left(\frac{r}{R}\right)^{n-1}\frac{(r-R)}{(1+r)^n}\left(\frac{|a_n| - |a_0|}{|a_n|R^n + |a_0|}\right)\right]\frac{1}{R^n}\max_{|z|=R}|P(z)|,$$

which by simplification yields

$$\max_{|z|=r}|P(z)| \geqslant \left(\frac{1+r}{1+R}\right)^n \left[1 + \left(\frac{r}{R}\right)^{n-1}\frac{(r-R)}{(1+r)^n}\left(\frac{|a_n| - |a_0|}{|a_n|R^n + |a_0|}\right)\right]\max_{|z|=R}|P(z)|.$$

Thus we arrive at the following result.

Corollary 1.4. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $1 \leqslant R < r$, we have*

$$M(P,r) \geqslant \left(\frac{1+r}{1+R}\right)^n \left[1 + \left(\frac{r}{R}\right)^{n-1}\frac{(r-R)}{(1+r)^n}\left(\frac{|a_n| - |a_0|}{|a_n|R^n + |a_0|}\right)\right] M(P,R).$$

The result is best possible and equality holds for $P(z) = (z+a)^n$, where $|a| = 1$ and $P(z) = z+a$, where $|a| \leqslant 1$.

Remark 1.4. It is easy to see that all the above results clearly sharpen the corresponding generalizations obtained recently by Kumar and Milovanović [**12**].

2. Lemmas

We need the following lemmas to prove our results.

Lemma 2.1. *For any* $0 \leqslant r \leqslant 1$, $R_1 \geqslant 1$, $R_2 \geqslant 1$,

$$\frac{R_1-1}{R_1+1}+\frac{R_2-1}{R_2+1}+(1-r)\left(\frac{R_1-1}{R_1+1}\right)\left(\frac{R_2-1}{R_2+1}\right) \geqslant \frac{R_1R_2-1}{R_1R_2+1}. \tag{2.1}$$

Proof. To prove (2.1) we need to establish that

$$\begin{aligned} f(R_1,R_2,r) = {} & \frac{R_1-1}{R_1+1}+\frac{R_2-1}{R_2+1} \\ & + (1-r)\left(\frac{R_1-1}{R_1+1}\right)\left(\frac{R_2-1}{R_2+1}\right) - \frac{R_1R_2-1}{R_1R_2+1} \geqslant 0. \end{aligned}$$

But

$$\begin{aligned} f(R_1,R_2,r) = {} & \frac{(R_1R_2+1)(3R_1R_2-R_1-R_2-1-rR_1R_2+rR_1+rR_2-r)}{(R_1+1)(R_2+1)(R_1R_2+1)} \\ & - \frac{(R_1+1)(R_2+1)(R_1R_2-1)}{(R_1+1)(R_2+1)(R_1R_2+1)}. \end{aligned}$$

Therefore $f(R_1,R_2,r) \geqslant 0$ if

$$\begin{aligned} & (R_1R_2+1)(3R_1R_2-R_1-R_2-1-rR_1R_2+rR_1+rR_2-r) \\ & \qquad -(R_1+1)(R_2+1)(R_1R_2-1) \geqslant 0. \end{aligned}$$

But then

$$\begin{aligned} & (R_1R_2+1)(3R_1R_2-R_1-R_2-1-rR_1R_2+rR_1+rR_2-r) \\ & \quad - (R_1+1)(R_2+1)(R_1R_2-1) \\ & \quad = r(-R_1^2R_2^2+R_1^2R_2+R_1R_2^2-2R_1R_2+R_1+R_2-1) \\ & \quad + 2(R_1^2R_2^2-R_1^2R_2-R_1R_2^2+R_1R_2) \\ & \quad = r(R_1R_2+1)(-R_1R_2+R_1+R_2-1)-2R_1R_2(-R_1R_2+R_1+R_2-1) \\ & \quad = (R_1R_2-R_1-R_2+1)[2R_1R_2-r(R_1R_2+1)] \\ & \quad = (R_1-1)(R_2-1)[R_1R_2(2-r)-r] \\ & \quad \geqslant 0 \end{aligned}$$

since $R_1R_2(2-r)-r \geqslant 2-r-r = 2(1-r) \geqslant 0$.

Therefore we established that $f(R_1,R_2,r) \geqslant 0$ for any $0 \leqslant r \leqslant 1$ and $R_1 \geqslant 1$, $R_2 \geqslant 1$, and thus the proof is complete. ☐

Remark 2.1. One can observe that the sharpest version of (2.1) is obtained by taking $r = 1$ in (2.1) and it is given by

$$\frac{R_1-1}{R_1+1}+\frac{R_2-1}{R_2+1} \geqslant \frac{R_1R_2-1}{R_1R_2+1},$$

whenever $R_1 \geqslant 1$, $R_2 \geqslant 1$, which improves upon Lemma 1 in the paper due to Kumar [**9**].

LEMMA 2.2. *For any* $0 \leqslant r \leqslant 1$, $R_k \geqslant 1$, *for all* k, $1 \leqslant k \leqslant n$,

$$\prod_{k=1}^{n} \left(\frac{R_k + r}{R_k + 1}\right) \geqslant \left(\frac{r+1}{2}\right)^n \left[1 + \frac{(R_1 \dots R_n - 1)(1-r)}{(R_1 \dots R_n + 1)(r+1)^n}\right]. \tag{2.2}$$

PROOF. We use induction on n. For $n = 1$, inequality (2.2) becomes

$$\frac{R_1 + r}{R_1 + 1} \geqslant \left(\frac{r+1}{2}\right) \left[1 + \frac{(R_1 - 1)(1-r)}{(R_1 + 1)(r+1)}\right],$$

which is true because a simple check shows that

$$\frac{R_1 + r}{R_1 + 1} = \left(\frac{r+1}{2}\right) \left[1 + \frac{(R_1 - 1)(1-r)}{(R_1 + 1)(r+1)}\right]. \tag{2.3}$$

So assume that the result is true for $n = m$, and thus we have

$$\prod_{k=1}^{m} \left(\frac{R_k + r}{R_k + 1}\right) \geqslant \left(\frac{r+1}{2}\right)^m \left[1 + \frac{(R_1 \dots R_m - 1)(1-r)}{(R_1 \dots R_m + 1)(r+1)^m}\right]. \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned}
&\prod_{k=1}^{m+1} \left(\frac{R_k + r}{R_k + 1}\right) \\
&\quad = \left(\frac{R_{m+1} + r}{R_{m+1} + 1}\right) \prod_{k=1}^{m} \left(\frac{R_k + r}{R_k + 1}\right) \\
&\quad \geqslant \left(\frac{r+1}{2}\right)^{m+1} \left[1 + \frac{(R_{m+1} - 1)(1-r)}{(R_{m+1} + 1)(r+1)}\right] \left[1 + \frac{(R_1 \dots R_m - 1)(1-r)}{(R_1 \dots R_m + 1)(r+1)^m}\right] \\
&\quad = \left(\frac{r+1}{2}\right)^{m+1} \left[1 + \frac{(R_{m+1} - 1)(1-r)}{(R_{m+1} + 1)(r+1)} + \frac{(R_1 \dots R_m - 1)(1-r)}{(R_1 \dots R_m + 1)(r+1)^m}\right] \\
&\qquad + \left(\frac{r+1}{2}\right)^{m+1} \left[\frac{(R_{m+1} - 1)(1-r)}{(R_{m+1} + 1)(r+1)}\right] \left[\frac{(R_1 \dots R_m - 1)(1-r)}{(R_1 \dots R_m + 1)(r+1)^m}\right] \\
&\quad \geqslant \left(\frac{r+1}{2}\right)^{m+1} \left[1 + \frac{(1-r)}{(r+1)^{m+1}} \left\{\frac{(R_{m+1} - 1)}{(R_{m+1} + 1)} + \frac{(R_1 \dots R_m - 1)}{(R_1 \dots R_m + 1)}\right\}\right] \\
&\qquad + \left(\frac{r+1}{2}\right)^{m+1} \left[\frac{(1-r)}{(r+1)^{m+1}} \frac{(R_{m+1} - 1)(R_1 \dots R_m - 1)(1-r)}{(R_{m+1} + 1)(R_1 \dots R_m + 1)}\right].
\end{aligned}$$

Now using Lemma 2.1 on the appropriate terms contained in the last two lines of the above inequality, we get

$$\prod_{k=1}^{m+1} \left(\frac{R_k + r}{R_k + 1}\right) \geqslant \left(\frac{r+1}{2}\right)^{m+1} \left[1 + \frac{(R_1 \dots R_{m+1} - 1)(1-r)}{(R_1 \dots R_{m+1} + 1)(r+1)^{m+1}}\right]. \qquad \square$$

LEMMA 2.3. *For any* $c \geqslant K^m$, $d \geqslant K$, *where* $K \geqslant 1$ *and* m *is any positive integer, then*

$$\frac{1}{K^{m-1}} \frac{c - K^m}{c+1} + \frac{d - K}{d+1} \geqslant \frac{1}{K^m} \frac{cd - K^{m+1}}{cd + 1}. \tag{2.5}$$

PROOF. To establish (2.5), we need to prove

$$\frac{1}{K^{m-1}}\frac{c-K^m}{c+1}+\frac{d-K}{d+1}-\frac{1}{K^m}\frac{cd-K^{m+1}}{cd+1}\geqslant 0.$$

Equivalently, we need to prove

$$K(cd+1)(c-K^m)(d+1)+K^m(d-K)(cd+1)(c+1)-(cd-K^{m+1})(c+1)(d+1)\geqslant 0.$$

Since $(c+1)(d+1)=cd+1+c+d$, it is enough to show that

$$\begin{aligned}K(cd+1)(c-K^m)(d+1)+K^m(d-K)(cd+1)(c+1)&\\ -(cd-K^{m+1})(cd+1)-(cd-K^{m+1})(c+d)&\geqslant 0.\end{aligned}\tag{2.6}$$

It is a simple exercise to verify the following two identities:

$$K(cd+1)(c-K^m)(d+1)=Kd(cd+1)(c-K^m)+K(cd+1)(c-K^m),\tag{2.7}$$

$$K^m(d-K)(cd+1)(c+1)=K^mc(d-K)(cd+1)+K^m(d-K)(cd+1).\tag{2.8}$$

In view of (2.7), (2.8) and (2.6), we need to show that

$$\begin{aligned}&Kd(cd+1)(c-K^m)+K(cd+1)(c-K^m)+K^mc(d-K)(cd+1)\\ &\quad+K^m(d-K)(cd+1)-(cd-K^{m+1})(cd+1)-(cd-K^{m+1})(c+d)\geqslant 0.\end{aligned}$$

Since $cd+1\geqslant c+d$ for $c\geqslant 1, d\geqslant 1$, it suffices to show that

$$\begin{aligned}&Kd(cd+1)(c-K^m)+K(cd+1)(c-K^m)+K^mc(d-K)(cd+1)\\ &\quad+K^m(d-K)(cd+1)-(cd-K^{m+1})(cd+1)-(cd-K^{m+1})(cd+1)\geqslant 0,\end{aligned}$$

equivalently we need to establish that

$$\begin{aligned}Kd(c-K^m)+K(c-K^m)+K^mc(d-K)&\\ +K^m(d-K)-(cd-K^{m+1})-(cd-K^{m+1})&\geqslant 0.\end{aligned}\tag{2.9}$$

Again, it is a simple exercise to verify the following two identities:

$$K(c-K^m)+K^m(d-K)-(cd-K^{m+1})=(K^m-c)(d-K),\tag{2.10}$$

$$\begin{aligned}Kd(c-K^m)+K^mc(d-K)-(cd-K^{m+1})&\\ =(c-K^m)(K-1)d+K^m(c-1)(d-K).&\end{aligned}\tag{2.11}$$

In view of (2.9), (2.10) and (2.11) we finally need to show that

$$\begin{aligned}&(c-K^m)(K-1)d+(d-K)[K^m(c-1)+K^m-c]\geqslant 0\\ \text{i.e., }&(c-K^m)(K-1)d+(d-K)(K^m-1)c\geqslant 0,\end{aligned}$$

which is true because $c\geqslant K^m$, $d\geqslant K$ and $K\geqslant 1$. □

LEMMA 2.4. *For any* $0\leqslant r\leqslant 1$, $R_j\geqslant K\geqslant 1$, *for* $1\leqslant j\leqslant n$, *we have*

$$\prod_{j=1}^{n}\left(\frac{R_j+r}{R_j+1}\right)\geqslant\left(\frac{K+r}{K+1}\right)^n\left[1+\frac{1}{K^{n-1}}\frac{(1-r)}{(K+r)^n}\frac{(R_1\dots R_n-K^n)}{(R_1\dots R_n+1)}\right].\tag{2.12}$$

PROOF. Let us use induction on n. For $n = 1$, (2.12) becomes

$$\frac{R_1 + r}{R_1 + 1} \geqslant \left(\frac{K + r}{K + 1}\right)\left[1 + \frac{(1 - r)}{(K + r)}\left(\frac{R_1 - K}{R_1 + 1}\right)\right],$$

which is true because a simple check shows that

$$\frac{R_1 + r}{R_1 + 1} = \left(\frac{K + r}{K + 1}\right)\left[1 + \frac{(1 - r)}{(K + r)}\left(\frac{R_1 - K}{R_1 + 1}\right)\right]. \tag{2.13}$$

Let us assume that the result is true for $n = m$, and thus we have

$$\prod_{j=1}^{m}\left(\frac{R_j + r}{R_j + 1}\right) \geqslant \left(\frac{K + r}{K + 1}\right)^{m}\left[1 + \frac{1}{K^{m-1}}\frac{(1 - r)}{(K + r)^m}\frac{(R_1 \dots R_m - K^m)}{(R_1 \dots R_m + 1)}\right]. \tag{2.14}$$

Using (2.13) and (2.14), we obtain

$$\begin{aligned}
\prod_{k=1}^{m+1}\left(\frac{R_k+r}{R_k+1}\right) &= \left(\frac{R_{m+1}+r}{R_{m+1}+1}\right)\prod_{k=1}^{m}\left(\frac{R_k+r}{R_k+1}\right)\\
&\geqslant \left(\frac{K+r}{K+1}\right)^{m+1}\left[1 + \frac{(1-r)}{(K+r)}\frac{(R_{m+1}-K)}{(R_{m+1}+1)}\right]\left[1 + \frac{1}{K^{m-1}}\frac{(1-r)}{(K+r)^m}\frac{(R_1 \dots R_m - K^m)}{(R_1 \dots R_m+1)}\right]\\
&= \left(\frac{K+r}{K+1}\right)^{m+1}\left[1 + \frac{(1-r)}{(K+r)}\frac{(R_{m+1}-K)}{(R_{m+1}+1)} + \frac{1}{K^{m-1}}\frac{(1-r)}{(K+r)^m}\frac{(R_1 \dots R_m - K^m)}{(R_1 \dots R_m+1)}\right]\\
&\quad + \frac{1}{K^{m-1}}\left(\frac{K+r}{K+1}\right)^{m+1}\left[\frac{(1-r)}{(K+r)}\frac{(R_{m+1}-K)}{(R_{m+1}+1)}\right]\left[\frac{(1-r)}{(K+r)^m}\frac{(R_1 \dots R_m - K^m)}{(R_1 \dots R_m+1)}\right]\\
&\quad \left(\frac{K+r}{K+1}\right)^{m+1}\left[1 + \frac{(1-r)}{(K+r)^{m+1}}\left\{\frac{(R_{m+1}-K)}{(R_{m+1}+1)} + \frac{1}{K^{m-1}}\frac{(R_1 \dots R_m - K^m)}{(R_1 \dots R_m+1)}\right\}\right]\\
&\quad + \left(\frac{K+r}{K+1}\right)^{m+1}\left[\frac{(1-r)}{(K+r)^{m+1}}\frac{1}{K^{m-1}}\frac{(R_{m+1}-K)(R_1 \dots R_m - K^m)(1-r)}{(R_{m+1}+1)(R_1 \dots R_m+1)}\right].
\end{aligned}$$

Now using Lemma 2.3 on the appropriate terms in the last two lines of the above inequality, we get

$$\prod_{k=1}^{m+1}\left(\frac{R_k+r}{R_k+1}\right) \geqslant \left(\frac{K+r}{K+1}\right)^{m+1}\left[1 + \frac{1}{K^m}\frac{(1-r)}{(K+r)^{m+1}}\frac{(R_1 \dots R_{m+1} - K^{m+1})}{(R_1 \dots R_{m+1}+1)}\right]. \qquad \square$$

3. Proofs of theorems

PROOF OF THEOREM 1.1. Let $z_j = R_j e^{i\phi_j}$, $j = 1, 2, \dots, n$ be the zeros of $P(z)$. Since $P(z)$ has no zeros in $|z| < 1$, we must have $R_j \geqslant 1$ for $j = 1, 2, \dots, n$. Suppose that $|P(e^{i\beta})| = M(p, 1) = 1$. Observe that for $r = 1$, the result is obvious. So for any $r < 1$, we have $M(P, r) = \frac{M(P,r)}{M(P,1)} \geqslant \frac{P(re^{i\beta})}{P(e^{i\beta})}$. Now

$$\frac{|P(re^{i\beta})|}{|P(e^{i\beta})|} = \prod_{j=1}^{n}\frac{|re^{i\beta} - R_j e^{i\phi_j}|}{|e^{i\beta} - R_j e^{i\phi_j}|}. \tag{3.1}$$

Firstly let us extract the following inequality from Rivlin's paper [**15**];

$$\left|\frac{re^{i\beta}-R_je^{i\phi_j}}{e^{i\beta}-R_je^{i\phi_j}}\right| \geqslant \frac{r+R_j}{1+R_j}, \quad j=1,2,\dots,n. \tag{3.2}$$

For the sake of completion, let us present the proof of (3.2). In fact the inequality (3.2) can be obtained by proving the non-negativity of the function

$$\begin{aligned} f(\phi_j) &= \frac{|re^{i\beta}-R_je^{i\phi_j}|^2}{|e^{i\beta}-R_je^{i\phi_j}|^2} - \frac{(r+R_j)^2}{(1+R_j)^2}, \ \text{for} \ \ 0 \leqslant \phi_j < 2\pi, \ \ j, 1 \leqslant j \leqslant n. \\ &= \frac{|re^{i\beta}-R_je^{i\phi_j}|^2(1+R_j)^2 - |e^{i\beta}-R_je^{i\phi_j}|^2(r+R_j)^2}{|e^{i\phi_j}-R_je^{i\phi_j}|^2(1+R_j)^2} = \frac{f_1(\phi_j)}{f_2(\phi_j)}, \end{aligned}$$

where

$$\begin{aligned} f_1(\phi_j) &= (re^{i\beta}-R_je^{i\phi_j})(re^{-i\beta}-R_je^{-i\phi_j})(1+R_j^2+2R_j) \\ &\quad - (e^{i\beta}-R_je^{i\phi_j})(e^{-i\beta}-R_je^{-i\phi_j})(r^2+R_j^2+2rR_j) \\ &= 2R_j^3 - 2rR_j^3 - 2rR_j + 2r^2R_j - 2rR_j\left[\frac{e^{i(\beta-\phi_j)}+e^{-i(\beta-\phi_j)}}{2}\right] \\ &\quad - 2rR_j^3\left[\frac{e^{i(\beta-\phi_j)}+e^{-i(\beta-\phi_j)}}{2}\right] + 2r^2R_j\left[\frac{e^{i(\beta-\phi_j)}+e^{-i(\beta-\phi_j)}}{2}\right] \\ &\quad + 2R_j^3\left[\frac{e^{i(\beta-\phi_j)}+e^{-i(\beta-\phi_j)}}{2}\right] \\ &= 2R_j^3 - 2rR_j^3 - 2rR_j + 2r^2R_j - 2rR_j\cos(\beta-\phi_j) - 2rR_j^3\cos(\beta-\phi_j) \\ &\quad + 2r^2R_j\cos(\beta-\phi_j) + 2R_j^3\cos(\beta-\phi_j) \\ &= 2R_j^3[1+\cos(\beta-\phi_j)] - 2rR_j^3[1+\cos(\beta-\phi_j)] - 2rR_j[1+\cos(\beta-\phi_j)] \\ &\quad + 2r^2R_j[1+\cos(\beta-\phi_j)] \\ &= (2R_j^3 - 2rR_j^3 - 2rR_j + 2r^2R_j)[1+\cos(\beta-\phi_j)] \\ &= 2R_j(R_j^2 - rR_j^2 - r + r^2)[1+\cos(\beta-\phi_j)] \\ &= 2R_j(R_j^2-r)(1-r)[1+\cos(\beta-\phi_j)] \geqslant 0. \\ &= \frac{2R_j(R_j^2-r)(1-r)[1+\cos(\beta-\phi_j)]}{|e^{i\phi_j}-R_je^{i\phi_j}|^2(1+R_j)^2} \geqslant 0, \ \ \text{for every} \ \ j, 1 \leqslant j \leqslant n, \end{aligned}$$

which establishes inequality (3.2).

Now from (3.1) and (3.2), we have

$$\frac{|P(re^{i\beta})|}{|P(e^{i\beta})|} \geqslant \prod_{j=1}^{n}\left(\frac{r+R_j}{1+R_j}\right),$$

which implies

$$|P(re^{i\beta})| \geqslant \left[\prod_{j=1}^{n}\left(\frac{r+R_j}{1+R_j}\right)\right]|P(e^{i\beta})|. \tag{3.3}$$

Applying Lemma 2.2 to the right-hand side of inequality (3.3) with the substitution $R_1 \dots R_n = \frac{|a_0|}{|a_n|}$, we have for any real values of β

$$|P(re^{i\beta})| \geqslant \left(\frac{r+1}{2}\right)^n \left[1 + \frac{(|a_0| - |a_n|)(1-r)}{(|a_0| + |a_n|)(r+1)^n}\right] |P(e^{i\beta})|,$$

which gives the required inequality. □

Proof of Corollary 1.1. Suppose $P(z)$ has no zeros in $|z| < \rho$. Taking $r = 0$ in Theorem 1.1, we get that for any polynomial $S(z) = \sum_{k=0}^{n} c_n z^n$ of degree n having no zeros in $|z| < 1$,

$$|S(0)| = |c_0| \geqslant 2^{-n} \left(\frac{2|c_0|}{|c_0| + |c_n|}\right) \max_{|z|=1} |S(z)|. \tag{3.4}$$

Let $Q(z) = P(\rho z)$ where $\rho > 0$, so $Q(z)$ has no zeros in $|z| < 1$ and hence using the inequality (3.4) for $Q(z)$ we have,

$$\begin{aligned} |Q(0)| \geqslant 2^{-n} \left(\frac{2|a_0|}{|a_0| + |a_n|\rho^n}\right) \max_{|z|=1} |Q(z)| \\ \implies |a_0| \geqslant 2^{-n} \left(\frac{2|a_0|}{|a_0| + |a_n|\rho^n}\right) \max_{|z|=\rho} |P(z)| \\ \implies \max_{|z|=\rho} |P(z)| \leqslant 2^{n-1}(|a_0| + |a_n|\rho^n), \end{aligned}$$

a contradiction to the hypothesis. Therefore we proved that, if $\max_{|z|=\rho} |P(z)| > 2^{n-1}(|a_0| + |a_n|\rho^n)$ for some $\rho > 0$, then there exists a zero in the disc $|z| < \rho$. □

Proof of Theorem 1.2. Since $P(z)$ has all zeros in $0 < |z| \leqslant 1$, so $a_0 \neq 0$ and $Q(z) = z^n \overline{P(1/\overline{z})}$ has no zeros in $|z| < 1$. Applying Theorem 1.1 with $r = \frac{1}{R}$ to the polynomial $Q(z)$, we have

$$\begin{aligned} M(Q, 1/R) \geqslant \left(\frac{\frac{1}{R}+1}{2}\right)^n \left[1 + \frac{(1-\frac{1}{R})}{(\frac{1}{R}+1)^n} \left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right] M(Q, 1) \\ \implies \frac{1}{R^n}, (P, R) \geqslant \left(\frac{\frac{1}{R}+1}{2}\right)^n \left[1 + \frac{(1-\frac{1}{R})}{(\frac{1}{R}+1)^n} \left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right] M(P, 1), \end{aligned}$$

which gives the required inequality. □

Proof of Theorem 1.3. Let $z_j = R_j e^{i\phi_j}$, $j = 1, 2, \dots, n$ be the zeros of $P(z)$. Since $P(z)$ has no zeros in $|z| < K$, $K \geqslant 1$, we must have $R_j \geqslant 1$ for $j = 1, 2, \dots, n$. Then from (3.3), we have for any real value of θ,

$$M(P, r) \geqslant \left[\prod_{j=1}^{n} \left(\frac{r + R_j}{1 + R_j}\right)\right] M(P, 1). \tag{3.5}$$

Applying Lemma 2.4 to the right-hand side of the inequality (3.5) with the substitution $R_1 \dots R_n = \frac{|a_0|}{|a_n|}$, we get the required inequality. □

Proof of Theorem 1.4. Since $P(z)$ has no zeros in $|z|<1$, so the polynomial $P(Rz)$ has no zeros in $|z|<\frac{1}{R}$ where $\frac{1}{R}\geqslant 1$. Note that $P(Rz)$ satisfies the hypotheses of Theorem 1.3, so by applying it on $P(Rz)$, we get

$$\max_{|z|=r/R}|P(Rz)| \geqslant \left(\frac{\frac{r}{R}+\frac{1}{R}}{1+\frac{1}{R}}\right)^n\left[1+R^{n-1}\frac{\left(1-\frac{r}{R}\right)}{\left(\frac{r}{R}+\frac{1}{R}\right)^n}\left(\frac{|a_0|-|a_nR^n|\left(\frac{1}{R^n}\right)}{|a_0|+|a_nR^n|}\right)\right]\max_{|z|=1}|P(Rz)|,$$

which is equivalent to

$$\max_{|z|=r}|P(z)|\geqslant\left(\frac{r+1}{R+1}\right)^n\left[1+R^{2(n-1)}\frac{(R-r)}{(r+1)^n}\left(\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right)\right]\max_{|z|=R}|P(z)|. \quad \square$$

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