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CURVE FITTING FOR SEISMIC WAVES OF EARTHQUAKE WITH HERMITE POLYNOMIALS

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Abstract. We investigate and study on mathematical structures involving mathematical models and others associated with seismic waves in an earthquake. Our first aim is to give some novel formulas and certain finite sums including the Bernoulli numbers and the Hermite polynomials with the aid of generating functions, the Riemann integral, and the Volkenborn integral. The second aim is to examine the seismic wave propagation in different geological units with the help of special polynomials containing the Hermite polynomials and their graph fitting of functions. To evaluate the shape of the seismic waves propagating within the ground (rock and/or soil), we use comparing method with the graph of the Hermite polynomials and functions and the polynomial Rocking Bearings. Furthermore, we also define generating function for the polynomial type Rocking Bearings. We give open problems on this generating function and earthquake facts. By applying partial derivative operator to the generating function for the m-parametric Hermite type polynomials, we give a novel recurrence relation and derivative formulas for these polynomials. We also give a new general formula for monomials in terms of these polynomials. Moreover, for the purpose of visualizing curve fitting approach to the seismic waves, we draw many plots of the Hermite functions with Mathematica (Version 12.0.0) with their codes. Finally, with the aid of these graphs, we give useful evaluation on the shapes of the seismic waves propagated in the ground (rock and/or soil).

1. Introduction

It is well known that understanding an earthquake and its propagation of seismic waves require long field studies, which include expensive and long applications. So, it is possible to reduce research costs by spending less time with mathematical structures involving mathematical models and others. With the aid of these mathematical structures, a proper site for engineering projects can be chosen within a shorter time.

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There are many mathematical structures for modelling the seismic waves propagated in the ground (rock and/or soil). For instance, in the work [**8**] of Stamatovska, there are some mathematical models which coincide with the fact that seismic waves propagate at frequencies in rock and relatively low frequencies in soil. While a seismic wave is reduced in the rock, it can travel very long distances with low energy consumption on the thick ground. During seismic energy transfer from rock to soil plain, energy increases by 100–1600 times, for details see Figure 1. The soil plains generally correspond to irrigated agricultural areas due to very poor geotechnical characteristics with saturated clayey to silty soil. Consequently, mathematical structures and models will help us not only to minimize natural disasters including earthquakes noticeable throughout the world, but also to select sites properly.

The other mathematical models are also handled by [**7**] considering Gaussian Process Regression. In addition, Buratti [**2**] gave an interesting approach to the seismic waves especially by examining seismic waves with the help of the roots of Hermite polynomials. For other mathematical structures and seismic wave models, the interested reader may consult to the references cite therein [**1**]– [**22**].

Our motivation is to give curve fitting approach to the seismic waves propagated in the ground (rock and/or soil) with many novel graphs of the Hermite functions. Moreover, in order to visualize curve fitting of the seismic waves, we also aim to draw many novel plots of the Hermite functions with MATHEMATICA (Version 12.0.0) by implementing their codes.

Throughout this paper the following notations and definitions are used: $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, real numbers, and complex numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_p$ denotes the p-adic integers.

Generating functions for the m-parametric Hermite type polynomials $\mathcal{K}(d, z, \vec{v}, m)$ are given by the following equation:

$$\mathcal{G}(z, w, \vec{v}, m) = \exp\left(zw + \sum_{k=1}^{m} v_k w^k\right) = \sum_{d=0}^{\infty} \mathcal{K}(d, z, \vec{v}, m)\frac{w^d}{d!}, \tag{1.1}$$

where $m \in \{1, 2, 3, \dots\}$, m-tuples $\vec{v} = (v_1, v_2, \dots, v_m)$, $z = a + ib$, $i^2 = -1$, $a, b, v_1, v_2, \dots, v_m \in \mathbb{R}$, see for detail [**11**]. By using (1.1), we give many new formulas in the next sections.

This paper is structured briefly as follows. In Section 2, we give brief information about the seismic waves that propagate, which is through the rock with great speed, high frequency, low wavelength, and appreciably higher attenuation rate. We give figures for the seismic waves in soil and rock. Finally, we give some properties of the seismic waves propagating in the soil; it is just the opposite in the rock. In Section 3, we give some properties for the m-parametric Hermite type polynomials with the aid of their generating functions and their graphs. For visualizing curve fitting approach of the seismic waves, we present some plots of the Hermite functions in MATHEMATICA package (Version 12.0.0) with their corresponding codes. We also give monomials in terms of the m-parametric Hermite type polynomials. In Section 4, by applying a partial derivative operator to the generating function for the m-parametric Hermite type polynomials, we give a recurrence relation and a derivative formula for these polynomials. In Section 5, we give generating functions and some relations for the polynomial type Rocking Bearing to the waves of rock and soil. We give the relation between these polynomials and the Hermite type polynomials. In Section 6, we give p-adic integral representation and the Riemann integral representation of the Hermite polynomials and the polynomial type Rocking Bearing. A conclusion is given in Section 7.

2. Seismic wave propagation

Here is a brief introduction to the seismic waves that propagate, which is through the rock with great speed, high frequency, low wavelength, and appreciably higher attenuation rate. It is known that the propagation of seismic waves in soil is the opposite of that in rock. The shorter the wavelengths and higher the corresponding frequency the greater energy consumption during propagation. These cases can be given by the energy equation $E = hv$, where h and v denote Planck's constant and frequency, respectively. With aid of h, the wavelength, denoted by λ, and the speed of light, denoted by c, this equation can also be given in the form $E = \lambda^{-1}hc$ (for details see [**5**]).

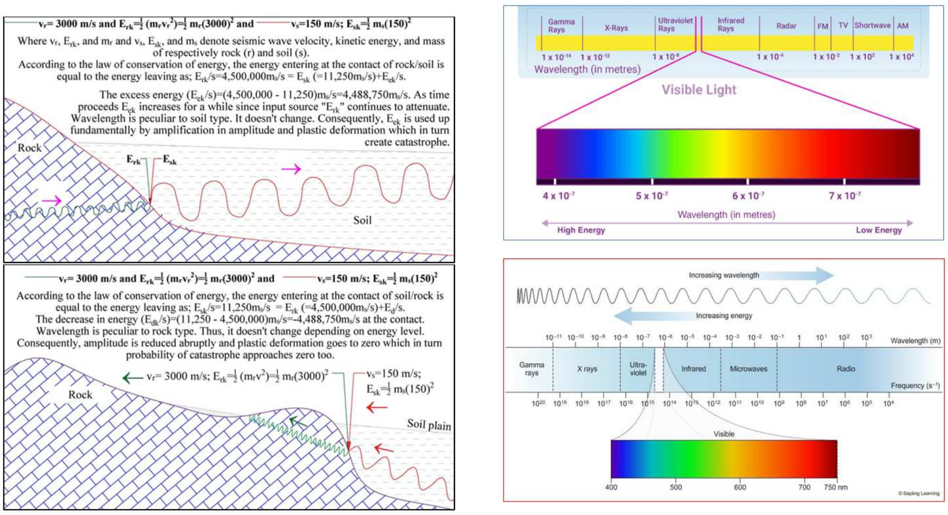

FIGURE 1. (left) Seismic waves in soil and rock; (right) The principles of wavelength, frequency, and energy consumption rate in a respective wave [**17**]

In the light of the equation $E = \lambda^{-1}hc$ and briefly explained in Figure 1 (left), seismic waves propagate with high frequency in rock and comparatively very low frequency in soil. Thus, seismic waves attenuate rapidly in rock, whereas they can travel very long distance in thick (at least 20 m) soil via low energy consumption. It is known that the velocity rate given by Yilmazer et al. [**22**] is transmitted very fast on the ground and causes disaster by magnifying the seismic wave on the ground considerably. On the other hand, the converse situation is valid in rocks.

As all seismic records and relevant literature present that the velocity and energy consumption ratios are directly proportional to the frequency of a wave whereas wave length is inversely proportional to the frequency (see Figure 1 (right)).

Curve fitting consists of steps of function construction that best fits a set of data points subject to constraints that can be made under favorable conditions [**15**]. With this approach, we will try to give the curve fitting model of the graphs of the Hermite functions, which we will explain in the next section, with the graph of the seismic wave propagation given in Figure 1.

3. Some properties and identities for the m-parametric Hermite type polynomials

In this section, we give new and some known properties of the Hermite type polynomials with their generating functions and their graphs. With the aid of generating function for the m-parametric Hermite type polynomials, we derive a general formula for monomials in terms of these polynomials. In the next section, to give applications of these polynomials with these new results, we compare these graphs with that of Figure 1 with the red and green solid lines.

We now give the symmetry condition for the polynomials $\mathcal{K}(d, z, \vec{v}, m)$. In the m-tuples $\vec{v} = (v_1, v_2, \ldots, v_m)$, we choose only an even index entries $v_2, v_4, \ldots, v_{2k}$ are positive integers and all odd index entries $v_1 = v_3 = \cdots = v_{2k+1} = 0$.

Let $\vec{U} = (0, -v_2, 0, -v_4, \ldots, 0, -v_{2k})$ be $2k$-tuples. Then we modified (1.1) as follows:

$$\exp(zw - v_2 w^2 - v_4 w^4 - \cdots - v_{2k} w^{2k}) = \sum_{d=0}^{\infty} \mathcal{K}(d, z, \vec{U}, 2k) \frac{w^d}{d!}.$$

Replacing z by $-z$ in the above equation, after some elementary calculations, we get the following symmetry condition for the $2k$-parametric Hermite type polynomials:

$$\mathcal{K}(d, -z, \vec{U}, 2k) = (-1)^d \mathcal{K}(d, z, \vec{U}, 2k).$$

Substituting $\vec{U} = (0, v_2, 0, 0, \ldots, 0, 0)$ into the above equation, we have symmetry condition for the classical Hermite type polynomials. The symmetry property is used not only in mathematics, but also in many social sciences, medical sciences, and natural sciences. Symmetry property in mathematics, which is seen in geometry, functions, and other branches of mathematics, has many applications. The most important feature of symmetry is invariance. In other words, looking at the symmetry property mathematically, an object remains unchanged under many operations or transformations. This means that symmetry is the match that protects the structure when a structured y object is given from any kind of structure. As for the $2k$-parametric Hermite type polynomials, when d is an even integer, the $2k$-parametric Hermite type polynomials are even functions and thus their graphs are symmetric with respect to the y-axis. This property implies that the graphs of the $2k$-parametric Hermite type polynomials remains unchanged after reflection

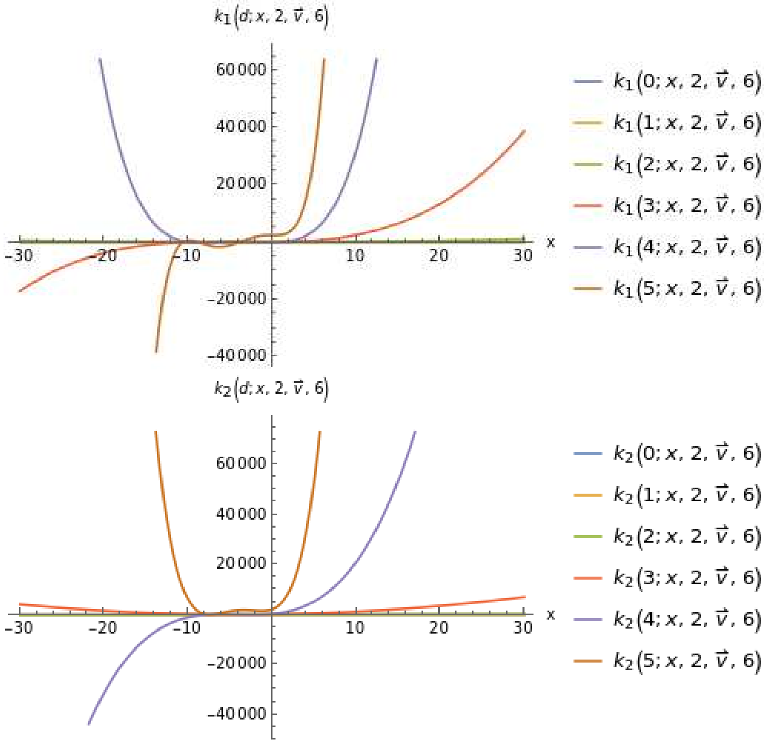

FIGURE 2. Plots of the polynomials $k_1(d; x, 2, \vec{v}, 6)$ and $k_2(d; x, 2, \vec{v}, 6)$ when $x \in [-30, 30]$, $d \in \{0, 1, 2, 3, 4, 5\}$ and $\vec{v} = (4, -1, 1, 6, 2, -3, 0, 0, \dots, 0)$

about the y-axis. On the other hand, if d is an odd integer, the $2k$-parametric Hermite type polynomials are odd functions and thus their graphs are symmetric with respect to the origin. This means that the graphs of the $2k$-parametric Hermite type polynomials remain unchanged after rotation of 180 degrees about the origin (see, for observation, Figure 2). The symmetry feature is very vital in seismic wave movements. In the next sections, some important facts will be given on this topic.

Using (1.1) and the Euler formula, we have the well-known decomposition formulas for the polynomials $\mathcal{K}(d, z, \vec{v}, m)$. These are given by the following generating functions for the polynomials $k_1(d; x, y, \vec{v}, m)$ and $k_2(d; x, y, \vec{v}, m)$:

$$\exp((x + v_1)w + v_2w^2 + \cdots + v_mw^m)\cos(yt) = \sum_{d=0}^{\infty} k_1(d; x, y, \vec{v}, m)\frac{w^d}{d!}, \tag{3.1}$$

$$\exp((x + v_1)w + v_2w^2 + \cdots + v_mw^m)\sin(yt) = \sum_{d=0}^{\infty} k_2(d; x, y, \vec{v}, m)\frac{w^d}{d!}, \tag{3.2}$$

respectively [**11**].

By using (3.1) and (3.2), we present some plots of the polynomials $k_1(d;x,y,\vec{v},m)$ and $k_2(d;x,y,\vec{v},m)$. Here we note that the plots of these polynomials with different values are given in the works of Kilar and Simsek [**11**, **12**]. Moreover, MATHEMATICA codes for these polynomials are similar to those of Kilar's work [**10**]. Figure 2 is obtained by $m=6$, $v_1=4$, $v_2=-1$, $v_3=1$, $v_4=6$, $v_5=2$, $v_6=-3$, $v_7=v_8=\cdots=v_m=0$, $y=2$ and $d\in\{0,1,2,3,4,5\}$ using (3.1) and (3.2) for $x\in[-30,30]$.

Note that as the values of the entries of the vector $\vec{v}$ change, symmetric plots can be obtained according to the appropriate values of x and y. As a result, these entries have different and applicable implications on the plots of the polynomials, see the above figure.

Setting $z=0$ in (1.1), we get the following generating functions for the generalized Hermite–Kampè de Fèriet polynomials $H_n(\vec{v};m)$:

$$e^{v_1w+v_2w^2+\cdots+v_mw^m}=\sum_{d=0}^{\infty}H_d(\vec{v};m)\frac{w^d}{d!} \tag{3.3}$$

[**11**, **19**]. Substituting $v_1=x$, $v_2=y$, $v_3=v_4=\cdots=v_m=0$ into (3.3), we have the generating function for the two variable Hermite polynomials:

$$e^{xw+yw^2}=\sum_{d=0}^{\infty}H_d((x,y,0,0,\ldots,0);2)\frac{w^d}{d!} \tag{3.4}$$

[**4**, **19**]. The two variable Hermite type polynomials have many representations with their generating functions written in different notations. Here, we set

$$H_d((x,y,0,0,\ldots,0);2)=H_d^{(2)}(x,y).$$

Using the above equation, the explicit formula for $H_d^{(2)}(x,y)$ is given by

$$\sum_{d=0}^{\infty}x^d\frac{w^d}{d!}\sum_{d=0}^{\infty}y^d\frac{w^{2d}}{d!}=\sum_{d=0}^{\infty}H_d^{(2)}(x,y)\frac{w^d}{d!}.$$

Therefore

$$\sum_{d=0}^{\infty}\left(\sum_{k=0}^{[d/2]}\frac{x^{d-2k}y^k}{(d-2k)!k!}\right)w^d=\sum_{d=0}^{\infty}H_d^{(2)}(x,y)\frac{w^d}{d!},$$

where $[a]$ denotes the largest integer $\leqslant a$. Comparing the coefficient of w^d on both sides of the above equation, we get the following formula for the polynomials $H_d^{(2)}(x,y)$:

$$H_d^{(2)}(x,y)=d!\sum_{k=0}^{[d/2]}\frac{x^{d-2k}y^k}{(d-2k)!k!}. \tag{3.5}$$

Substituting $y=-1/2$ into (3.5), we arrive at the explicit formula of the (probabilist) Hermite polynomials $\mathrm{He}_d(x)$,

$$\mathrm{He}_d(x)=d!\sum_{k=0}^{[d/2]}\frac{(-1)^kx^{d-2k}}{2^k(d-2k)!k!}. \tag{3.6}$$

By using (1.1), we get the following equation:

$$e^{zw+v_1w} = \exp(-v_2w^2 - v_3w^3 - \cdots - v_mw^m) \sum_{d=0}^{\infty} \mathcal{K}(d, z, \vec{v}, m) \frac{w^d}{d!}.$$

By applying a series product to the above equation, we have

$$e^{zw+v_1w} = \exp(-v_3w^3 - \cdots - v_mw^m) \sum_{d=0}^{\infty} \sum_{j_1=0}^{[d/2]} \frac{(-v_2)^{j_1} \mathcal{K}(d-2j_1, z, \vec{v}, m)}{(d-2j_1)!j_1!} w^d.$$

The left-hand side of the above equation is a Taylor series for e^{zw+v_1w}, thus we get

$$\sum_{d=0}^{\infty} (z+v_1)^d \frac{w^d}{d!} = \exp(-v_4w^4 - \cdots - v_mw^m) \sum_{d=0}^{\infty} \sum_{j_2=0}^{[d/3]} \frac{(-v_3)^{j_2}}{j_2!} \times \sum_{j_1=0}^{[(d-3j_2)/2]} \frac{(-v_2)^{j_1} \mathcal{K}(d-2j_1-3j_2, z, \vec{v}, m)}{(d-2j_1-3j_2)!j_1!} w^d.$$

Therefore,

$$\sum_{d=0}^{\infty} (z+v_1)^d \frac{w^d}{d!} = \sum_{d=0}^{\infty} \sum_{j_{m-1}=0}^{[\frac{d}{m}]} \sum_{j_{m-2}=0}^{\left[\frac{d-mj_{m-1}}{m-1}\right]} \cdots \sum_{j_1=0}^{\left[\frac{d-3j_2-4j_3-\cdots-mj_{m-1}}{2}\right]} (-1)^{j_1+j_2+\cdots+j_{m-1}} \times \frac{v_m^{j_{m-1}} v_{m-1}^{j_{m-2}} \ldots v_2^{j_1} \mathcal{K}(d-2j_1-3j_2-\cdots-mj_{m-1}, z, \vec{v}, m)}{(d-2j_1-3j_2-\cdots-mj_{m-1})!j_1!j_2!\ldots j_{m-1}!} w^d.$$

Comparing the coefficient of w^d on both sides of the above equation, we get monomials in terms of the polynomials $\mathcal{K}(d, z, \vec{v}, m)$ by the following theorem:

THEOREM 3.1. *Let $d \in \mathbb{N}_0$, $z \in \mathbb{C}$, $m \in \mathbb{N}$, and $\vec{v} = (v_1, v_2, \ldots, v_m)$, where $v_1, v_2, \ldots, v_m \in \mathbb{R}$. Then we have*

$$(z+v_1)^d = d! \sum_{j_{m-1}=0}^{\left[\frac{d}{m}\right]} \sum_{j_{m-2}=0}^{\left[\frac{d-mj_{m-1}}{m-1}\right]} \cdots \sum_{j_1=0}^{\left[\frac{d-3j_2-4j_3-\cdots-mj_{m-1}}{2}\right]} (-1)^{j_1+j_2+\cdots+j_{m-1}} \times \frac{v_m^{j_{m-1}} v_{m-1}^{j_{m-2}} \ldots v_2^{j_1} \mathcal{K}(d-2j_1-3j_2-\cdots-mj_{m-1}, z, \vec{v}, m)}{(d-2j_1-3j_2-\cdots-mj_{m-1})!j_1!j_2!\ldots j_{m-1}!}. \tag{3.7}$$

REMARK 3.1. Proof of Theorem 3.1 can be also given by the mathematical induction method. By using the same method Kilar [**9**] gave many novel and interesting results on the m-parametric Hermite type polynomials $\mathcal{K}(d, z, \vec{v}, m)$.

Substituting $m = 2$ and $v_3 = v_4 = \cdots = v_m = 0$ into (3.7), we get the following result:

COROLLARY 3.1. *Let $d \in \mathbb{N}_0$ and $z \in \mathbb{C}$. Then we have*

$$(z+v_1)^d = d! \sum_{j_1=0}^{[d/2]} \frac{(-1)^{j_1} v_2^{j_1} \mathcal{K}(d-2j_1, z, (v_1, v_2, 0, \ldots, 0), 2)}{(d-2j_1)!j_1!}.$$

Substituting $m = 2$, $z = 0$, $v_1 = x$, $v_2 = y$ and $v_3 = v_4 = \cdots = v_m = 0$ into (3.7), we get the following known formulas:

$$H^{(2)}_{d-2j_1}(x, y) = \mathcal{K}(d - 2j_1, 0, (x, y, 0, \dots, 0), 2),$$

$$x^d = d! \sum_{k=0}^{[d/2]} \frac{(-y)^k H^{(2)}_{d-2k}(x, y)}{(d-2k)!k!}. \tag{3.8}$$

Substituting $y = -1/2$ into (3.8), we have the following monomials in terms of the polynomials $\mathrm{He}_d(x)$, which are also known as inverse explicit expression:

$$x^d = d! \sum_{k=0}^{[d/2]} \frac{\mathrm{He}_{d-2k}(x)}{2^k (d-2k)!k!}. \tag{3.9}$$

Setting $v_1 = x$, $v_2 = -1/2$, $v_3 = \cdots = v_m = 0$ and $z = 0$ in (1.1), we easily have the following generating function for the Hermite polynomials $\mathrm{He}_d(x)$:

$$e^{xw-\frac{1}{2}w^2} = \sum_{d=0}^{\infty} \mathrm{He}_d(x) \frac{w^d}{d!}$$

(see [**11**, **19**] and the references cited therein).

The relation between the polynomials $\mathrm{He}_d(x)$ and the physicist's Hermite polynomials $H_d(x)$ is given by $H_d(x) = 2^{d/2}\,\mathrm{He}_d(x\sqrt{2})$ (see [**19**] and the references cited therein).

Thanks to the above equation, the well-known Hermite functions can also be written as

$$\psi_d(x) = \left(d!\sqrt{\pi}\right)^{-1/2} e^{-x^2/2}\,\mathrm{He}_d\left(\sqrt{2}x\right) \tag{3.10}$$

[**19**]. The function $\psi_d(x)$ satisfies the following ODE:

$$\frac{d^2}{dx^2}\{\psi_d(x)\} + (2d + 1 - x^2)\psi_d(x) = 0$$

[**16**, **19**]. It is well-known that the above ODE is equivalent to the famous Schrödinger equation for a harmonic oscillator in the theory of quantum mechanics. Some special values of these functions are given by

$$\begin{aligned}
\psi_0(x) &= \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2},\\
\psi_1(x) &= \sqrt{2}x\psi_0(x),\\
\psi_2(x) &= \left(\sqrt{2}\right)^{-1}(2x^2 - 1)\psi_0(x),\\
\psi_3(x) &= \left(\sqrt{3}\right)^{-1}x(2x^2 - 3)\psi_0(x),\\
\psi_4(x) &= \left(2\sqrt{6}\right)^{-1}(4x^4 - 12x^2 + 3)\psi_0(x),\\
\psi_5(x) &= \left(2\sqrt{6}\right)^{-1}x(4x^4 - 20x + 15)\psi_0(x),
\end{aligned}$$

and so on. For details, see also [**3**, **16**].

By using (3.10), we present some plots of Hermite Function which visualize the seismic waves of an earthquake, see below figure.

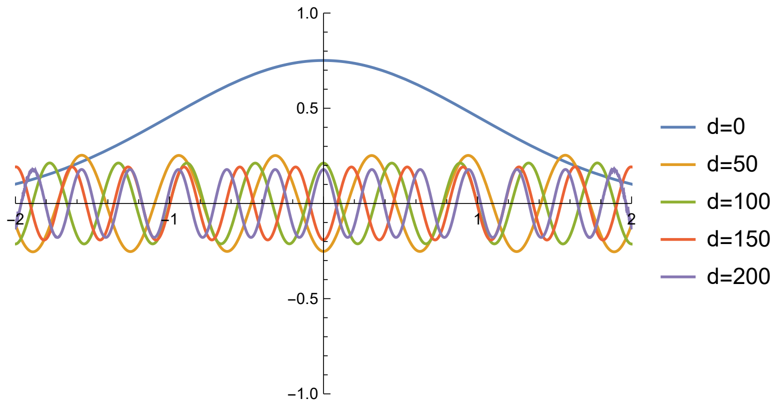

FIGURE 3. Plots of the Hermite functions $\psi_d(x)$ when $d \in \{0, 50, 100, 150, 200\}$ and $x \in [-2, 2]$

It can be easily observed from the graphs provided in Figure 3 that as the values of degree d increase, the seismic waves of earthquake can be modelled with the curves fitted by the graphs of the Hermite functions, which are expressed in terms of the Hermite polynomials of degree d.

4. Recurrence relation and derivative formula for the m-parametric Hermite type polynomials

In this section, by applying a partial derivative operator to the generating function for the m-parametric Hermite type polynomials, we derive the recurrence relation and derivative formula for these polynomials. Furthermore, recurrence relation and derivative formulas have very important applications in mathematical modeling and the calculation of numerical values of polynomials.

By applying the partial derivative operator $\frac{\partial}{\partial w}$ to (1.1), we get the following partial derivative equation:

$$\frac{\partial}{\partial w}\{\mathcal{G}(z, w, \vec{v}, m)\} = \left(z + \sum_{k=1}^{m} k v_k w^{k-1}\right)\mathcal{G}(z, w, \vec{v}, m).$$

Combining the above equation with (3.3), we get

$$\frac{\partial}{\partial w}\{\mathcal{G}(z, w, \vec{v}, m)\} = \left(z + \sum_{k=1}^{m} k v_k w^{k-1}\right)e^{zw}\sum_{d=0}^{\infty} H_d(\vec{v}; m)\frac{w^d}{d!}.$$

After some elementary calculations in the above equation, we get

$$\sum_{n=0}^{\infty}(-z)^n\frac{w^n}{n!}\sum_{d=1}^{\infty}\mathcal{K}(d, z, \vec{v}, m)\frac{w^{d-1}}{(d-1)!}$$
$$= z\sum_{d=0}^{\infty} H_d(\vec{v}; m)\frac{w^d}{d!} + \sum_{k=1}^{m} k v_k \sum_{d=0}^{\infty} H_d(\vec{v}; m)\frac{w^{d+k-1}}{d!}.$$

Therefore,

$$\sum_{d=0}^{\infty}\sum_{n=0}^{d}(-z)^{d-n}\mathcal{K}(n+1,z,\vec{v},m)\frac{w^d}{d!}$$
$$=\sum_{d=0}^{\infty}\left(zH_d(\vec{v};m)+\sum_{k=1}^{m}kd(d-1)\dots(d-k+2)v_kH_{d-k+1}(\vec{v};m)\right)\frac{w^d}{d!}.$$

Comparing the coefficient of $w^d/d!$ on both sides of the above equation, we arrive at the following theorem:

THEOREM 4.1. *Let $d\in\mathbb{N}_0$, $z\in\mathbb{C}$, $m\in\mathbb{N}$, and $\vec{v}=(v_1,v_2,\dots,v_m)$, where $v_1,v_2,\dots,v_m\in\mathbb{R}$. Then we have*

$$\sum_{n=0}^{d}(-z)^{d-n}\mathcal{K}(n+1,z,\vec{v},m) = zH_d(\vec{v};m)+\sum_{k=1}^{m}kd(d-1)\dots(d-k+2)v_kH_{d-k+1}(\vec{v};m). \tag{4.1}$$

By applying the partial derivative operator $\frac{\partial^k}{\partial z^k}$ to (1.1), we obtain the following partial derivative equation:

$$\frac{\partial^k}{\partial z^k}\{\mathcal{G}(z,w,\vec{v},m)\}=w^k\mathcal{G}(z,w,\vec{v},m). \tag{4.2}$$

Combining (4.2) with (1.1), we get

$$\sum_{d=0}^{\infty}\frac{\partial^k}{\partial z^k}\{\mathcal{K}(d,z,\vec{v},m)\}\frac{w^d}{d!}=\sum_{d=0}^{\infty}\mathcal{K}(d,z,\vec{v},m\frac{w^{d+k}}{d!}.$$

After some elementary calculations in the above equation, we obtain

$$\sum_{d=0}^{\infty}\frac{\partial^k}{\partial z^k}\{\mathcal{K}(d,z,\vec{v},m)\}\frac{w^d}{d!}=\sum_{d=0}^{\infty}d(d-1)\dots(d-k+1)\mathcal{K}(d-k,z,\vec{v},m)\frac{w^d}{d!}.$$

Comparing the coefficient of $w^d/d!$ on both sides of the above equation, we arrive at the following theorem:

THEOREM 4.2. *Let $d,k\in\mathbb{N}_0$ with $d\geqslant k$, $z\in\mathbb{C}$, $m\in\mathbb{N}$, $\vec{v}=(v_1,v_2,\dots,v_m)$, $v_1,v_2,\dots,v_m\in\mathbb{R}$. Then we have*

$$\frac{\partial^k}{\partial z^k}\{\mathcal{K}(d,z,\vec{v},m)\}=d(d-1)\dots(d-k+1)\mathcal{K}(d-k,z,\vec{v},m). \tag{4.3}$$

A special value of (4.3) is given as follows. Substituting $k=1$ into (4.3), we get the following corollary:

COROLLARY 4.1. *Let $d\in\mathbb{N}$, $z\in\mathbb{C}$, $m\in\mathbb{N}$, and $\vec{v}=(v_1,v_2,\dots,v_m)$, where $v_1,v_2,\dots,v_m\in\mathbb{R}$. Then we have $\frac{\partial}{\partial z}\{\mathcal{K}(d,z,\vec{v},m)\}=d\mathcal{K}(d-1,z,\vec{v},m)$.*

REMARK 4.1. Substituting $\bar{z}$ which is conjugated of z, into (1.1), all polynomials concerning z and $\bar{z}$ can be modified. These modifications can also affect the shape of the graph of the polynomials and also seismic waves. By using a component of variable z, concerning x and y, derivative formulas for the m-parametric Hermite type polynomials were given [**12**, **13**].

5. Generating functions and some relations for the polynomial type Rocking Bearing to the waves of rock and soil

The polynomial Rocking Bearing (PRB) related to earthquake waves and their application by applying polynomial Rocking Bearings, seismic isolation systems on irregular bridges, and seismic performance of rocking base-isolated structures related to earthquake loads can be analyzed [**17**, **18**].

The PRB is defined by

$$G(x) = a_1x^6 + a_2x^4 + a_3x^2, \tag{5.1}$$

where a_1, a_2, a_3 are arbitrary constant [**18**].

Substituting (3.9) into the polynomials $G(x)$, we give a relation between the PRB and the Hermite polynomials as follows:

$$G(x) = 720a_1 \sum_{j=0}^{3} \frac{\mathrm{He}_{6-2j}(x)}{2^j j!(6-2j)!} + 24a_2 \sum_{j=0}^{2} \frac{\mathrm{He}_{4-2j}(x)}{2^j j!(4-2j)!} + 2a_3 \sum_{j=0}^{1} \frac{\mathrm{He}_{2-2j}(x)}{2^j j!(2-2j)!}.$$

A variable of this polynomial has positive even integers; thus, the polynomial $G(x)$ is an even polynomial. The graph of this type of polynomial is symmetric concerning the $0y$-axis. The shape of this type of polynomial is similar to the shape of the bearing.

We also observe that graph of the polynomials PRB is also related to the seismic waves propagating in Figure 1 and Figure 2.

Is it possible to generalize the PRB? That is, we set

$$P_n(x) = \sum_{j=1}^{n} c_j x^{2j}. \tag{5.2}$$

Substituting $n = 3$ into (5.2), we get to arrive at the following well-known PRB, that is $P_3(x) = c_1x^2 + c_2x^4 + c_3x^6 = G(x)$.

Putting (3.9) in (5.2), we get the following result:

THEOREM 5.1. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then we have*

$$P_n(x) = \sum_{j=1}^{n} (2j)! c_j \sum_{k=0}^{j} \frac{\mathrm{He}_{2j-2k}(x)}{2^k k!(2j-2k)!}. \tag{5.3}$$

This gives us the polynomial type Rocking Bearing presented a linear combination of the Hermite polynomials.

Thus, we pose the following open problem associated with generating function for the polynomials $P_n(x)$.

How can we construct generating function for $P_n(x)$ polynomials? That is, if $f(t,x) = \sum_{n=0}^{\infty} P_n(x)t^n$, then find $f(t,x) =?$

6. Integral representations of the Hermite polynomials and the polynomial type Rocking Bearing

In this section, we give not only p-adic integral representation but also the Riemann integral representation of the Hermite polynomials and the polynomial type Rocking Bearing. The p-adic integral and the Riemann integral have many applications in many branches of mathematics, physics, engineering, and other sciences.

6.1. p-adic integral representation of the Hermite polynomials and the polynomial type Rocking Bearing. Let $\mathbb{K}$ be a field with a complete valuation and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be a set of the uniformly differential function f on $\mathbb{Z}_p$.

Let $f \in (\mathbb{Z}_p \to \mathbb{K})$. The Volkenborn integral (or the bosonic p-adic integral) of the uniformly differential function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{6.1}$$

where $\mu_1(x)$ is given by $\mu_1(x) = 1/p^N$ (*cf.* [**14**, **20**, **21**]).

Schikhof [**20**] gave the following integral equation for the Volkenborn integral:

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_1(x) - \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = f(0), \tag{6.2}$$

where $f'(x) = \frac{d}{dx}\{f(x)\}$. By applying (6.2) to (3.5), we get

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} H_d^{(2)}(x,y)d\mu_1(x)d\mu_1(y) = d! \sum_{k=0}^{[d/2]} \frac{B_{d-2k}B_k}{(d-2k)!k!},$$

where B_d denote the Bernoulli numbers, which are defined by the following generating function:

$$\frac{w}{e^w - 1} = \sum_{d=0}^{\infty} B_d \frac{w^d}{d!}$$

(see [**1**, **21**] and the references cited therein).

By applying (6.2) to (3.6), we obtain

$$\int_{\mathbb{Z}_p} \mathrm{He}_d(x)d\mu_1(x) = d! \sum_{k=0}^{[d/2]} \frac{(-1)^k B_{d-2k}}{2^k(d-2k)!k!}. \tag{6.3}$$

By applying the Volkenborn integral to (3.9), we get

$$B_d = d! \sum_{k=0}^{[d/2]} \frac{1}{2^k k!} \sum_{j=0}^{[(d-2k)/2]} \frac{(-1)^j B_{d-2k-2j}}{2^j(d-2k-2j)!j!}. \tag{6.4}$$

By applying the Volkenborn integral to (5.1), we have

$$\int_{\mathbb{Z}_p} G(x)d\mu_1(x) = a_1B_6 + a_2B_4 + a_3B_2 = \frac{a_1}{42} - \frac{a_2}{30} + \frac{a_3}{6}.$$

Remark 6.1. Observe that the coefficients of a_1, a_2 and a_3 of the PRB may be also calculated with the help of the above equation as well as the field and laboratory studies.

By applying the Volkenborn integral to (5.2) and (5.3), respectively, we get

$$\int_{\mathbb{Z}_p} P_n(x)d\mu_1(x) = \sum_{j=1}^{n} c_j B_{2j},$$

$$\int_{\mathbb{Z}_p} P_n(x)d\mu_1(x) = \sum_{j=1}^{n}(2j)!c_j \sum_{k=0}^{j} \frac{1}{2^k k!(2j-2k)!} \int_{\mathbb{Z}_p} \mathrm{He}_{2j-2k}(x)d\mu_1(x). \tag{6.5}$$

Combining the above equation with (6.3), we obtain

$$\int_{\mathbb{Z}_p} P_n(x)d\mu_1(x) = \sum_{j=1}^{n}(2j)!c_j \sum_{k=0}^{j} \frac{1}{2^k k!} \sum_{v=0}^{j-k} \frac{(-1)^v B_{2j-2k-2v}}{2^v(2j-2k-2v)!v!}. \tag{6.6}$$

Combining (6.5) with (6.6), we arrive at the following theorem:

Theorem 6.1. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n} c_j B_{2j} = \sum_{j=1}^{n}(2j)!c_j \sum_{k=0}^{j} \frac{1}{2^k k!} \sum_{v=0}^{j-k} \frac{(-1)^v B_{2j-2k-2v}}{2^v(2j-2k-2v)!v!}. \tag{6.7}$$

6.2. Riemann integral representation of the Hermite polynomials and the polynomial type Rocking Bearing. By applying the Riemann integral to (5.2) and (5.3), respectively, we have

$$\int_0^1 P_n(y)d(y) = \sum_{j=1}^{n} \frac{c_j}{2j+1}, \tag{6.8}$$

$$\int_0^1 P_n(y)d(y) = \sum_{j=1}^{n}(2j)!c_j \sum_{k=0}^{j} \frac{1}{2^k k!(2j-2k)!} \int_0^1 \mathrm{He}_{2j-2k}(y)d(y).$$

Thus, we obtain

$$\int_0^1 P_n(y)d(y) = \sum_{j=1}^{n}(2j)!c_j \sum_{k=0}^{j} \frac{1}{2^k k!} \sum_{v=0}^{j-k} \frac{(-1)^v}{2^v(2j-2k-2v+1)!v!}. \tag{6.9}$$

Combining (6.8) with (6.9), we arrive at the following finite sums:

Theorem 6.2. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n} \frac{c_j}{2j+1} = \sum_{j=1}^{n}(2j)!c_j \sum_{k=0}^{j} \frac{1}{2^k k!} \sum_{v=0}^{j-k} \frac{(-1)^v}{2^v(2j-2k-2v+1)!v!}.$$

7. Conclusions

In this paper, by applying generating functions and their functional and derivative equations, some novel formulas and relations are given. By using generating functions for the m-parametric Hermite type polynomials, some special values of these polynomials are also given. By applying a partial derivative operator to the generating functions for the m-parametric Hermite type polynomials, a novel recurrence relation and derivative formulas for these polynomials was given. p-adic integral representation and the Riemann integral representation for the Hermite polynomials and the polynomial type Rocking Bearing are also given. In order to visualize curve fitting approach to the seismic waves, many plots of the Hermite functions with MATHEMATICA (Version 12.0.0) with their codes are drawn. By using these graphs, some observations with their evaluations on the shapes of the seismic waves propagated in the ground (rock and/or soil) are also given. These evaluations give us not only the curve fitting of seismic waves in earthquake to Hermite polynomials, but also the shape of the seismic waves propagating in the ground (rock and/or soil) compared with the graphs of the Hermite polynomials. All of these graphs are given in the figures. These graphs will contribute to the understanding and interpretation of more in-depth theories about seismic wave propagation in earthquakes.

Consequently, it can be observed that the red curve and green curve in Figure 1 (left) coincide with the graphs of the Hermite polynomials and functions in Figure 2. We observe that the curve in the rock and the curve in the soil of Figure 1 (right) are also represented by the curve of PRB-type polynomials associated with $P_n(x)$ polynomials.

In our future studies or investigations, we look for the applications of $P_n(x)$ polynomials to not only the curve in rock, but also the curve in soil; and real-world problems. We will also investigate the solution to the open problems, which are given in this paper.

Acknowledgments. The authors dedicate this article to the souls of our citizens who died in the earthquake disaster that occurred in Pazarcık and Elbistan districts of Kahramanmaraş city and also in Defne and Samandağ districts of Hatay city of Turkey from February 6 to February 20, 2023.

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