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TURÁN-TYPE INEQUALITIES OF POLYNOMIALS

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Abstract. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, V. K. Jain [Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. **59** (2016), 339–347] proved

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n\left(|a_0| + |a_n|k^{n+1}\right)}{|a_o|\,(1+k^{n+1}) + |a_n|\,(k^{n+1}+k^{2n})} \max_{|z|=1} |p(z)|.$$

We first obtain a generalization as well as improvement of the above inequality. Further, we extend our first result to a more generalized result which yields improved results of some known inequalities as particular case.

1. Introduction and preliminaries

Let $p(z)$ be a polynomial of degree n, then Bernstein's well-known inequality [**1**] is $\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|$. If we are interested to obtain a lower bound estimate of $\max_{|z|=1} |p'(z)|$ in terms of $\max_{|z|=1} |p(z)|$, such problems do not exist in literature. However, if the polynomial $p(z)$ has all its zeros in $|z| \leqslant 1$, then Turán [**14**] proved that

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

Inequality (1.1) is sharp and equality holds for $p(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. Inequality (1.1) of Turán [**14**] has been of considerable interest and applications and it would be of interest to seek its generalization for polynomials having all their zeros in $|z| \leqslant k, k > 0$. The case when $0 < k \leqslant 1$ was settled by Malik [**8**] and proved

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|.$$

While for the case $k \geqslant 1$, Govil [**4**] proved

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{1.2}$$

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Equality in (1.2) holds for $p(z) = z^n + k^n, k \geqslant 1$. Under the same hypothesis, it was Govil [**5**] who improved upon (1.2) by proving

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+k^n}\Big\{\max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)|\Big\}. \tag{1.3}$$

Equality in (1.3) holds for $p(z) = z^n + k^n, k \geqslant 1$. Recently, Jain [**7**] proved an improvement of inequality (1.2), incorporating the leading coefficient and constant term of the polynomial by using the generalized form of Schwarz's classical lemma.

THEOREM 1.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n(|a_0| + |a_n|k^{n+1})}{|a_0|(1+k^{n+1}) + |a_n|(k^{n+1}+k^{2n})} \max_{|z|=1} |p(z)|.$$

For a better insight into both Bernstein and Turán-type inequalities, one can refer to recently published monograph of Gardner et al. [**3**] (also see Marden [**9**], Milovanović et al. [**10**], Rahman and Schmeisser [**12**]).

2. Main results

In this paper, we consider the class of polynomials of degree $n \geqslant 2$ having a zero of order s, $0 \leqslant s \leqslant n-2$, at the origin. For polynomial of degree 1, the polynomial is simply $p(z) = a_0 + a_1 z$ and hence, we can easily evaluate $\max_{|z|=1}|p(z)| = |a_0| + |a_1|$ and $\max_{|z|=1}|p'(z)| = |a_1|$. Also, when $s = n-1$, the polynomial is $p(z) = a_{n-1}z^{n-1} + a_n z^n$ and hence trivially, we have $\max_{|z|=1}|p(z)| = |a_{n-1}| + |a_n|$ and $\max_{|z|=1}|p'(z)| = (n-1)|a_{n-1}| + n|a_n|$. So, in both cases, we need not to find their estimates as their exact values are known. We first prove the following generalization as well as improvement of Theorem 1.1. More precisely, we prove

THEOREM 2.1. *If* $p(z) = \sum_{\nu=s}^{n} a_\nu z^\nu$, $0 \leqslant s \leqslant n-2$, *is a polynomial of degree* $n \geqslant 2$ *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then*

$$\begin{aligned}\max_{|z|=1} |zp'(z) - sp(z)| \geqslant{}& \frac{(n-s)(|a_s| + |a_n|k^{n-s+1})}{|a_s|(1+k^{n-s+1}) + |a_n|(k^{n-s+1}+k^{2n-2s})} \max_{|z|=1} |p(z)| \\ &+ \frac{|a_s|k + |a_n|k^{n-s}}{|a_s|(1+k^{n-s+1}) + |a_n|(k^{n-s+1}+k^{2n-2s})}\zeta(k,s),\end{aligned}$$

where

$$\zeta(k,s) = \begin{cases} |a_{s+2}|k^{n-s-6}(k^4-1)\big(\sqrt{k^4+1}-1\big), & \text{if } s \leqslant n-5, \quad n \geqslant 5, \\ |a_{s+2}|(k^2-1)\big(\sqrt{k^4+k^2+1}-1\big), & \text{if } s = n-4, \quad n \geqslant 4, \\ |a_{s+2}|k\Big(\sqrt{\frac{k^4+1}{2}}-1\Big), & \text{if } s = n-3, \quad n \geqslant 3, \\ |a_{s+1}|(k^2-1), & \text{if } s = n-2, \quad n \geqslant 2. \end{cases} \tag{2.1}$$

REMARK 2.1. Setting $s = 0$ in Theorem 2.1, we get the following improvement of Theorem 1.1 recently proved by Jain [**7**] as well as inequality (1.2), for polynomials of degree $n \geqslant 2$.

COROLLARY 2.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* $n \geqslant 2$ *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n(|a_0| + |a_n|k^{n+1})}{|a_0|(1+k^{n+1}) + |a_n|(k^{n+1}+k^{2n})} \max_{|z|=1} |p(z)| + \frac{|a_0|k + |a_n|k^n}{|a_0|(1+k^{n+1}) + |a_n|(k^{n+1}+k^{2n})} \zeta(k),$$

where

$$\zeta(k) = \begin{cases} |a_2|k^{n-6}(k^4-1)(\sqrt{k^4+1}-1), & \text{if } n \geqslant 5, \\ |a_2|(k^2-1)(\sqrt{k^4+k^2+1}-1), & \text{if } n = 4, \\ |a_2|k\Big(\sqrt{\frac{k^4+1}{2}}-1\Big), & \text{if } n = 3, \\ |a_1|(k^2-1), & \text{if } n = 2. \end{cases} \tag{2.2}$$

REMARK 2.2. As $\zeta(k)$ given by (2.2) is non-negative, it follows immediately that Corollary 2.1 gives improved bound over Theorem 1.1. And to show that the bounds of Corollary 2.1 is an improvement of (1.2), it is sufficient to show that

$$\frac{|a_0| + |a_n|k^{n+1}}{|a_0|(1+k^{n+1}) + |a_n|(k^{n+1}+k^{2n})} \geqslant \frac{1}{1+k^n},$$

which is equivalent to showing $|a_n|(k^{2n+1} - k^{2n}) \geqslant |a_0|(k^{n+1} - k^n)$, that is $k^n|a_n| \geqslant |a_0|$, which clearly holds by Lemma 3.6 with $\lambda = 0$.

REMARK 2.3. In some cases the improvement can be significant and this we show by means of the following example.

EXAMPLE 2.1. Consider $p(z) = z^3 + 3z^2 + \frac{11}{4}z + \frac{3}{4}$. Clearly $p(z)$ is a polynomial of degree 3 having all its zeros in $|z| \leqslant \frac{3}{2}$. We take $k = 2$ and find that

$$\max_{|z|=1} |p(z)| = 7.5.$$
$$\min_{|z|=2} |p(z)| = 0.75.$$
$$\max_{|z|=1} |p'(z)| \geqslant 2.5, \quad \text{(by (1.2))}.$$
$$\max_{|z|=1} |p'(z)| \geqslant 4.06, \quad \text{(by Theorem 1.1)}.$$
$$\max_{|z|=1} |p'(z)| \geqslant 5.24, \quad \text{(by Corollary 2.1 for } n = 3).$$

Further, we extend Theorem 2.1 to a more generalized result which yields an improved result of some known inequalities as particular case.

THEOREM 2.2. *If* $p(z) = \sum_{\nu=s}^{n} a_\nu z^\nu$, $0 \leqslant s \leqslant n-2$, *is a polynomial of degree* $n \geqslant 2$ *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then for* $0 \leqslant l < 1$ *and* $m = \min_{|z|=k} |p(z)|$

$$\max_{|z|=1}|zp'(z)-sp(z)| \geqslant \frac{(n-s)(|a_s|k^s+lm+|a_n|k^{n+1})}{(|a_s|k^s+lm)(1+k^{n-s+1})+|a_n|(k^{n+1}+k^{2n-s})} \times \Big\{\max_{|z|=1}|p(z)|+\frac{lm}{k^s}\Big\} + \frac{(|a_s|k^s+lm)k+|a_n|k^n}{(|a_s|k^s+lm)(1+k^{n-s+1})+|a_n|(k^{n+1}+k^{2n-s})}\zeta(k,s), \tag{2.3}$$

where $\zeta(k,s)$ is as defined in (2.1).

Remark 2.4. Putting $l=0$, Theorem 2.2 reduces to Theorem 2.1.

Remark 2.5. Setting $s=0$ in Theorem 2.2, we get the following generalization of Corollary 2.1 as well as improvement of inequality (1.3) and a result recently proved by Mir [**11**, Theorem 2], for polynomials of degree $n\geqslant 2$.

Corollary 2.2. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree $n\geqslant 2$ having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then for $0\leqslant l<1$ and $m=\min_{|z|=k}|p(z)|$*

$$\max_{|z|=1}|p'(z)| \geqslant \frac{n(|a_0|+lm+|a_n|k^{n+1})}{(|a_0|+lm)(1+k^{n+1})+|a_n|(k^{n+1}+k^{2n})} \times \Big\{\max_{|z|=1}|p(z)|+lm\Big\} + \frac{(|a_0|+lm)k+|a_n|k^n}{(|a_0|+lm)(1+k^{n+1})+|a_n|(k^{n+1}+k^{2n})}\zeta(k), \tag{2.4}$$

where $\zeta(k)$ is as defined in (2.2).

Remark 2.6. Since $\zeta(k)\geqslant 0$, to verify that inequality (2.4) of Corollary 2.2 is an improvement of inequality (1.3) due to Govil [**5**], it is sufficient to show that

$$\frac{|a_0|+lm+|a_n|k^{n+1}}{(|a_0|+lm)(1+k^{n+1})+|a_n|(k^{n+1}+k^{2n})} \geqslant \frac{1}{1+k^n},$$

which is equivalent to

$$|a_n|(k^{2n+1}-k^{2n}) \geqslant (|a_0|+lm)(k^{n+1}-k^n),$$

that is

$$k^n|a_n| \geqslant |a_0|+lm,$$

which clearly holds by Lemma 3.6.

Remark 2.7. Here also, in some cases the improvement is significant and we illustrate this with the help of the previous example 2.1.

For $k=2$, we have

$$\max_{|z|=1}|p'(z)| \geqslant 2.75, \quad \text{(by (1.3))},$$

whereas

$$\max_{|z|=1}|p'(z)| \geqslant 5.3, \quad \text{(by (2.4) of Corollary 2.2 for } n=3 \text{ and } l=1).$$

3. Lemmas

We shall need the following lemmas in order to prove the above theorems and verify the claims. For a polynomial $p(z)$ of degree n, we will use $q(z) = z^n\overline{p\big(\frac{1}{\bar z}\big)}$. The first lemma is due to Frappier et. al. [**2**].

Lemma 3.1. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n and let $R \geqslant 1$. Then,*

$$\max_{|z|=R}|p(z)| \leqslant R^n \max_{|z|=1}|p(z)| - |p'(0)|(R^{n-1}-R^{n-3})\big(\sqrt{R^2+1}-1\big), \quad n \geqslant 4, \tag{3.1}$$

$$\max_{|z|=R}|p(z)| \leqslant R^n \max_{|z|=1}|p(z)| - |p'(0)|(R^2-R)\big(\sqrt{R^2+R+1}-1\big), \quad n = 3, \tag{3.2}$$

$$\max_{|z|=R}|p(z)| \leqslant R^n \max_{|z|=1}|p(z)| - |p'(0)|R\left(\sqrt{\frac{R^2+1}{2}}-1\right), \quad n = 2, \tag{3.3}$$

$$\max_{|z|=R}|p(z)| \leqslant R \max_{|z|=1}|p(z)| - (R-1)|p(0)|, \quad n = 1. \tag{3.4}$$

Lemma 3.2. *Let $f(z)$ be analytic in $|z| < 1$, with $f(0) = a$ and $|f(z)| \leqslant M$, $|z| < 1$. Then*

$$|f(z)| \leqslant M\frac{M|z|+|a|}{|a||z|+M}, \quad |z| < 1.$$

Lemma 3.2 is a well-known generalization of Schwarz's lemma [**13**, p. 212].

Lemma 3.3. *Let $f(z)$ be analytic in $|z| \leqslant 1$, with $f(0) = a$ and $|f(z)| \leqslant M$, $|z| \leqslant 1$. Then*

$$|f(z)| \leqslant M\frac{M|z|+|a|}{|a||z|+M}, \quad |z| \leqslant 1.$$

Proof. It easily follows from Lemma 3.2. □

Lemma 3.4. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then*

$$|q'(z)| \leqslant |p'(z)|.$$

The above lemma is due to Jain [**7**].

Lemma 3.5. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, then*

$$\max_{|z|=1}|p'(z)| + \max_{|z|=1}|q'(z)| \geqslant n \max_{|z|=1}|p(z)|.$$

The result is due to Govil et al. [**6**].

Lemma 3.6. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k > 0$, then for any complex number λ with $|\lambda| < 1$ and $m = \min_{|z|=k}|p(z)|$*

$$k^n|a_n| \geqslant |\lambda|m + |a_0|.$$

Proof. By hypothesis, $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k > 0$. Then, the polynomial $P(z) = e^{-i\arg a_0} p(z)$ has the same zeros as $p(z)$. Here,

$$\begin{aligned} P(z) &= e^{-i\arg a_0}\big\{|a_0|e^{i\arg a_0} + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n\big\} \\ &= |a_0| + e^{-i\arg a_0}\big\{a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n\big\}. \end{aligned}$$

Now, on $|z| = k$ for any complex number λ with $|\lambda| < 1$ and $m = \min_{|z|=k} p(z) \neq 0$, we have

$$|\lambda| m < m \leqslant |P(z)|.$$

Then by Rouche's theorem, $R(z) = P(z) + |\lambda| m$ has all its zeros in $|z| < k$ and in case $m = 0$, $R(z) = P(z)$. Thus, in any case, $R(z)$ has all its zeros in $|z| \leqslant k$. Now, applying Vieta's formula to $R(z)$, we get $\frac{|a_0| + |\lambda| m}{|a_n|} \leqslant k^n$, i.e., $k^n |a_n| \geqslant |\lambda| m + |a_0|$, which completes the proof of the Lemma 3.6. □

4. Proof of the Theorems

Proof of Theorem 2.2. Let us first assume that $p(z) = \sum_{\nu=s}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geqslant 5$ and $s \leqslant n - 5$. By hypothesis, $p(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$. Now,

$$p(z) = z^s j(z), \tag{4.1}$$

where

$$j(z) = a_s + a_{s+1} z + \cdots + a_n z^{n-s}, \tag{4.2}$$

is a polynomial of degree $n - s \geqslant 5$. Consider a polynomial

$$R(z) = p(z) + \frac{m}{k^s} \lambda z^s, \tag{4.3}$$

where λ is any complex number with $|\lambda| < 1$ and $m = \min_{|z|=k} |p(z)|$. Suppose $m \neq 0$, then for $|z| = k$

$$\left| \frac{m}{k^s} \lambda z^s \right| < m \leqslant |p(z)|.$$

Then by Rouche's theorem, it follows that $R(z)$ has all its zeros in $|z| < k$ and in case $m = 0$, $R(z) = p(z)$. Thus, in any case, $R(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$. Now,

$$R(z) = \frac{\lambda m}{k^s} z^s + a_s z^s + a_{s+1} z^{s+1} + \cdots + a_n z^n = z^s h(z), \tag{4.4}$$

where

$$h(z) = \frac{\lambda m}{k^s} + a_s + a_{s+1} z + \cdots + a_n z^{n-s}, \tag{4.5}$$

and

$$g(z) = z^{n-s} \overline{h\Big(\frac{1}{\bar{z}}\Big)}. \tag{4.6}$$

From (4.2) and (4.5), we have

$$j'(z) = h'(z). \tag{4.7}$$

We observe that

$$H(z) = h(kz), \tag{4.8}$$

is a polynomial of degree $n - s \geqslant 5$ having all its zeros in $|z| \leqslant 1$ and

$$G(z) = z^{n-s}\overline{H\Big(\frac{1}{\bar{z}}\Big)} = k^{n-s}\Big(\frac{z}{k}\Big)^{n-s}\overline{h\Big(\frac{k}{\bar{z}}\Big)} = k^{n-s}g\Big(\frac{z}{k}\Big), \quad \text{(by (4.6))}. \tag{4.9}$$

By Lemma 3.4, we have

$$G'(z) \leqslant H'(z), \quad |z| = 1. \tag{4.10}$$

Using (4.10) we can say that a zero z_j, with $|z_j| = 1$ and multiplicity m_j, of $H'(z)$ will also be a zero, with multiplicity ($\geqslant m_j$), of $G'(z)$, thereby helping us to write

$$H'(z) = \phi(z)H_1(z), \tag{4.11}$$

$$G'(z) = \phi(z)G_1(z), \tag{4.12}$$

where

(4.13)

$$\phi(z) = \begin{cases} 1, & H'(z) \neq 0 \text{ on } |z| = 1, \\ \prod_{j=1}^{p}(z - z_j)^{m_j}; |z_j| = 1 \;\forall\; j, & H'(z) \text{ has certain zeros on } |z| = 1. \end{cases}$$

Now,

$$H_1(z) \neq 0, \quad |z| = 1. \tag{4.14}$$

By (4.10), (4.11) and (4.12), we have

$$G_1(z) \leqslant H_1(z), \quad |z| = 1. \tag{4.15}$$

Now as $H(z)$ has all its zeros in $|z| \leqslant 1$, we can say by Gauss-Lucas theorem that $H'(z)$ will also have all its zeros in $|z| \leqslant 1$. Therefore by (4.11), (4.13) and (4.14), we can say that

$$\psi(z) = \frac{G_1(z)}{H_1(z)} \tag{4.16}$$

is analytic in $|z| > r$, for certain r, with $0 < r < 1$, including ∞ and accordingly

$$f(z) = \psi\Big(\frac{1}{z}\Big), \tag{4.17}$$

with

$$\begin{aligned} f(0) = \psi(\infty) &= \lim_{z\to\infty} \psi(z), \\ &= \lim_{z\to\infty} \frac{G'(z)}{H'(z)}, \quad \text{(by (4.16), (4.11) and (4.12))}, \\ &= \frac{\overline{\frac{\lambda m}{k^s} + a_s}}{a_n k^{n-s}} \end{aligned} \tag{4.18}$$

is analytic in $|z| < \frac{1}{r}$, $\frac{1}{r} \geqslant 1$. Further $|\psi(z)| \leqslant 1$, $|z| = 1$ by (4.15) and therefore

$$|f(z)| \leqslant 1, \quad |z| = 1, \quad \text{(by (4.17))},$$

which by (4.18) and Lemma 3.3, helps us to write

$$|f(z)| \leqslant \frac{|z| + \left|\frac{a_s + \frac{\lambda m}{k^s}}{a_n k^{n-s}}\right|}{1 + \left|\frac{a_s + \frac{\lambda m}{k^s}}{a_n k^{n-s}}\right||z|}, \quad |z| \leqslant 1,$$

i.e.

$$|f(re^{i\theta})| \leqslant \frac{|a_n|k^n r + |a_s k^s + \lambda m|}{|a_s k^s + \lambda m| r + |a_n| k^n}, \quad r \leqslant 1 \text{ and } 0 \leqslant \theta \leqslant 2\pi,$$

i.e.

$$\left|\psi\Big(\frac{1}{r}e^{-i\theta}\Big)\right| \leqslant \frac{|a_n|k^n r + |a_s k^s + \lambda m|}{|a_s k^s + \lambda m| r + |a_n| k^n}, \quad 0 < r \leqslant 1 \text{ and } 0 \leqslant \theta \leqslant 2\pi, \quad \text{(by (4.17))},$$

i.e.

$$|\psi(Re^{-i\theta})| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m| R}{|a_s k^s + \lambda m| + |a_n| k^n R}, \quad R \geqslant 1 \text{ and } 0 \leqslant \theta \leqslant 2\pi,$$

i.e.

$$|G_1(Re^{-i\theta})| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m| R}{|a_s k^s + \lambda m| + |a_n| k^n R} |H_1(Re^{-i\theta})|, \quad R \geqslant 1, \quad \text{(by (4.16))},$$

i.e.

$$|G'(Re^{-i\theta})| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m| R}{|a_s k^s + \lambda m| + |a_n| k^n R} |H'(Re^{-i\theta})|, \quad R \geqslant 1, \ \text{(by (4.11) and (4.12))},$$

i.e.

$$|G'(z)| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m||z|}{|a_s k^s + \lambda m| + |a_n| k^n |z|} |H'(z)|, \quad |z| \geqslant 1,$$

i.e.

$$k^{n-s-2}\left|g'\Big(\frac{z}{k}\Big)\right| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m||z|}{|a_s k^s + \lambda m| + |a_n| k^n |z|} |h'(kz)|, \quad |z| \geqslant 1, \quad \text{(by (4.8) and (4.9))}. \tag{4.19}$$

By taking $z = ke^{i\theta}$ in (4.19), we get

$$k^{n-s-2}|g'(e^{i\theta})| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m| k}{|a_s k^s + \lambda m| + |a_n| k^{n+1}} |h'(k^2 e^{i\theta})|, \quad 0 \leqslant \theta \leqslant 2\pi,$$

which implies

$$k^{n-s-2} \max_{|z|=1} |g'(z)| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m| k}{|a_s k^s + \lambda m| + |a_n| k^{n+1}} \max_{|z|=k^2} |h'(z)|. \tag{4.20}$$

Applying (3.1) of Lemma 3.1 to (4.20), we have

$$k^{n-s-2} \max_{|z|=1} |g'(z)| \leqslant \frac{|a_n|k^n + |a_s k^s + \lambda m| k}{|a_s k^s + \lambda m| + |a_n| k^{n+1}} \Big\{ k^{2n-2s-2} \max_{|z|=1} |h'(z)| - |a_{s+2}| (k^{2n-2s-4} - k^{2n-2s-8}) \big(\sqrt{k^4+1} - 1\big) \Big\}$$

i.e.

$$\max_{|z|=1}|g'(z)| \leqslant \frac{|a_n|k^n + |a_sk^s + \lambda m|k}{|a_sk^s + \lambda m| + |a_n|k^{n+1}}\Big\{k^{n-s}\max_{|z|=1}|h'(z)| - |a_{s+2}|(k^{n-s-2} - k^{n-s-6})\big(\sqrt{k^4+1}-1\big)\Big\}. \tag{4.21}$$

By Lemma 3.5, we have

$$\max_{|z|=1}|g'(z)| + \max_{|z|=1}|h'(z)| \geqslant (n-s)\max_{|z|=1}|h(z)|,$$

and using (4.21), we get

$$\max_{|z|=1}|h'(z)| \geqslant \frac{(n-s)(|a_sk^s + \lambda m| + |a_n|k^{n+1})}{|a_sk^s + \lambda m|(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})}\max_{|z|=1}|h(z)| + \frac{|a_sk^s + \lambda m|k + |a_n|k^n}{|a_sk^s + \lambda m|(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})} \times |a_{s+2}|k^{n-s-6}(k^4-1)\big(\sqrt{k^4+1}-1\big).$$

By (4.4) and (4.7), we have

$$\max_{|z|=1}|j'(z)| \geqslant \frac{(n-s)(|a_sk^s + \lambda m| + |a_n|k^{n+1})}{|a_sk^s + \lambda m|(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})}\max_{|z|=1}|R(z)| + \frac{|a_sk^s + \lambda m|k + |a_n|k^n}{|a_sk^s + \lambda m|(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})} \times |a_{s+2}|k^{n-s-6}(k^4-1)\big(\sqrt{k^4+1}-1\big).$$

Again, by (4.1) and (4.3), we have

$$\max_{|z|=1}|zp'(z) - sp(z)| \geqslant \frac{(n-s)(|a_sk^s + \lambda m| + |a_n|k^{n+1})}{|a_sk^s + \lambda m|(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})} \times \max_{|z|=1}\Big|p(z) + \frac{\lambda m}{k^s}z^s\Big| + \frac{|a_sk^s + \lambda m|k + |a_n|k^n}{|a_sk^s + \lambda m|(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})} \times |a_{s+2}|k^{n-s-6}(k^4-1)\big(\sqrt{k^4+1}-1\big). \tag{4.22}$$

For every complex number λ, we have $|a_sk^s + \lambda m| \leqslant |a_s|k^s + |\lambda|m$, and since both

$$\Big(\frac{x+|a_n|k^{n+1}}{x(1+k^{n-s+1})+|a_n|(k^{n+1}+k^{2n-s})}\Big) \text{ and } \Big(\frac{xk+|a_n|k^{n}}{x(1+k^{n-s+1})+|a_n|(k^{n+1}+k^{2n-s})}\Big)$$

are decreasing functions of x for $k \geqslant 1$, it follows from (4.22) that for every λ with $|\lambda| < 1$ and $|z| = 1$,

$$\begin{aligned}\max_{|z|=1} |zp'(z) - sp(z)| \geqslant{}& \frac{(n-s)(|a_s|k^s + |\lambda|m + |a_n|k^{n+1})}{(|a_s|k^s + |\lambda|m)(1 + k^{n-s+1}) + |a_n|(k^{n+1} + k^{2n-s})} \\ &\times \max_{|z|=1} \left| p(z) + \frac{\lambda m}{k^s} z^s \right| \\ &+ \frac{(|a_s|k^s + |\lambda|m)k + |a_n|k^n}{(|a_s|k^s + |\lambda|m)(1 + k^{n-s+1}) + |a_n|(k^{n+1} + k^{2n-s})} \\ &\times |a_{s+2}|k^{n-s-6}(k^4 - 1)\big(\sqrt{k^4+1} - 1\big).\end{aligned} \tag{4.23}$$

Suppose z_0 on $|z| = 1$ is such that

$$\max_{|z|=1} |p(z)| = |p(z_0)|. \tag{4.24}$$

Now,

$$\left| p(z_0) + \frac{\lambda m z_0^s}{k^s} \right| \leqslant \max_{|z|=1} \left| p(z) + \frac{\lambda m z^s}{k^s} \right|. \tag{4.25}$$

On the left hand side of inequality (4.25), for suitable choice of the argument of λ, we have

$$\left| p(z_0) + \frac{\lambda m z_0^s}{k^s} \right| = |p(z_0)| + \frac{|\lambda| m}{k^s}. \tag{4.26}$$

Applying (4.24) and (4.26) to (4.25), we have

$$\max_{|z|=1} |p(z)| + \frac{|\lambda| m}{k^s} \leqslant \max_{|z|=1} \left| p(z) + \frac{\lambda m}{k^s} z^s \right|. \tag{4.27}$$

Applying (4.27) to (4.23), we get

$$\begin{aligned}\max_{|z|=1} |zp'(z) - sp(z)| \geqslant{}& \frac{(n-s)(|a_s|k^s + |\lambda|m + |a_n|k^{n+1})}{(|a_s|k^s + |\lambda|m)(1 + k^{n-s+1}) + |a_n|(k^{n+1} + k^{2n-s})} \\ &\times \left\{ \max_{|z|=1} |p(z)| + \frac{|\lambda| m}{k^s} \right\} \\ &+ \frac{|a_s|k^s + |\lambda|mk + |a_n|k^n}{(|a_s|k^s + |\lambda|m)(1 + k^{n-s+1}) + |a_n|(k^{n+1} + k^{2n-s})} \\ &\times |a_{s+2}|k^{n-s-6}(k^4 - 1)\big(\sqrt{k^4+1} - 1\big).\end{aligned} \tag{4.28}$$

Setting $|\lambda| = l$, $0 \leqslant l < 1$ in (4.28) gives inequality (2.3) for $n \geqslant 5$ and $s \leqslant n-5$. For the case $n \geqslant 4$ and $s = n-4$, $n \geqslant 3$ and $s = n-3$, $n \geqslant 2$ and $s = n-2$, the proof follows in a similar way as above but applying inequalities (3.2), (3.3) and (3.4) respectively instead of (3.1) of Lemma 3.1 to inequality (4.20). □

Proof of Theorem 2.1. The proof of Theorem 2.1 follows on the same lines as that of Theorem 2.2, just by taking $\lambda = 0$. □

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